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Is Choice the Correct Primitive? On Using Certainty Equivalents and Reference Levels to Predict Choices Among Gambles

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Abstract

Choice is viewed as a derived, not a primitive, concept. Individual gambles are assigned subjective certainty equivalents (CE_1); the choice set X has an associated reference level [$RL(X)$] based on the CE_1s of its members; the outcomes of each gamble are recoded as deviations from the $RL(X)$; and new CE_2s are constructed. The gamble having the largest CE_2 is chosen. The CEs are described by the rank- and sign-dependent theory of Luce (1992b). The concept of RL is studied axiomatically. The model predicts many behavioral anomalies and is tested with data sets of Mellers, Chang, Birnbaum, and Ordóñez (1992).

Key words: certainty equivalents, reference levels, rank- and sign-dependent utility

Since World War II, many of us trying to understand decision making in contexts of risk and uncertainty have assumed that the basic primitive should be choice. Choices are said to reveal our preferences; they are viewed as objective; and we have theorized about them at considerable length. This approach has been followed almost without exception among economists and other mathematically inclined theorists working on such problems. It has been less universal among psychologists—in particular, those of a strong experimental disposition. To them it has seemed equally natural and appropriate to ask subjects to offer direct evaluations of either risky or uncertain alternatives or both—which can be lumped under the single term *gambles*. Subjects have been asked to assign selling prices, buying prices, attractiveness judgments, and certainty equivalents in addition to making choices (Mellers, Chang, Birnbaum, and Ordóñez, 1992; Mellers, Weiss,

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and Birnbaum, 1992; Sarin, 1982). Of course, economists are interested in buying prices, selling prices, and certainty equivalents, but they have tended to view them as derivative from theories of choice. For example, almost without comment or question, choice theorists have assumed that a judged certainty equivalent is the same as the sum of money for which, in a choice between the money and the gamble, the decision maker is indifferent between the two.[1]

Several recent empirical studies lead one to reject the latter hypothesis about certainty equivalents. The most vivid is the case of so-called preference reversals, in which subjects provide both judged[2] CEs to gambles and choose between pairs of them. The unpleasant finding is that for many subjects, gamble pairs exist for which the order established by the judged CEs (e.g., prices) is opposite to the choice made by the same subject (Birnbaum and Sutton, 1992; Bostic, Herrnstein, and Luce, 1990; Goldstein and Einhorn, 1987; Grether and Plott, 1979; Hamm, 1980; Holt, 1986; Lichtenstein and Slovic, 1971, 1973; Loomes, 1990; Loomes and Sugden, 1983; Mowen and Gentry, 1980; Pommerehne, Scheider, and Zweifel, 1982; Reilly, 1982; Schkade and Johnson, 1989; Segal, 1988; Slovic and Lichtenstein, 1983; Tversky, Slovic, and Kahneman, 1990). Moreover, as Mellers, Ordóñez, and Chang (1992) have shown, some of these reversals do not disappear even if CEs and choices are made at the same time. However, these reversals are much reduced when choice indifference procedures are used. Indeed, some of the judged CEs are rather far removed from the choice indifference points (Bostic et al., 1990). Thus, judged CEs (prices) simply cannot be viewed as special cases of choices.

A second disturbing fact about choice data is that sets of three or more gambles can be selected for which subjects exhibit intransitive choices (Lindman and Lyons, 1978; Montgomery, 1977; Ranyard, 1977; Tversky, 1969). If correct, such behavior is not consistent with typical choice theories, although a few attempts have been made to accommodate these data (Fishburn and LaValle, 1988; Mellers, Ordóñez, and Birnbaum, 1992).

A third disturbing fact is that the choice context affects the choices made. In one demonstration (MacCrimmon, Stanbury, and Wehrung, 1980), the same pair of gambles, g and h, was embedded in two different contexts or sets of alternatives. The alternatives were given a stock-market interpretation. Business executives were asked, on separate occasions, to rank order their preferences for the risky alternatives in each set. The surprising fact was that for nearly 25% of the executives, the preference ordering of the pair was context dependent: in one context, alternative g was ranked higher than h, and in the other, h higher than g. This result is not something predicted by any choice theory of which we are aware. Other related material on context effects can be found in Birnbaum (1992), Birnbaum, Parducci, and Gifford (1971), Keller (1985), Lindman (1971), Mellers and Birnbaum (1982), Mellers, Ordóñez, and Birnbaum (1991), Parducci (1965, 1974, 1982), Parducci, Calfee, Marshall, and Davidson (1960), and Parducci and Perett (1971).

Over the years, various psychologists have argued that one can understand these apparent anomalies by recognizing that the context somehow establishes a temporary status quo or, as we shall call it, *reference level,* and that this quantity influences choice (Helson, 1964; Hershey and Schoemaker, 1985; Johnson, Payne, and Bettman, 1988; Loomes and Sugden, 1983, 1986; Tversky and Kahneman, 1991). This idea, while intuitively reasonable, typically lacks much precision, and, so far we know, it has not been developed as a formal theory on a par with the highly axiomatic choice theories. The purpose of this article is to explore one way to make the concept sufficiently precise for testing and to test it against an existing body of data.

This article explores, in a highly preliminary fashion, an approach in which certainty equivalents are the basic primitive and choices are a derived concept. The basic steps involved are easily understood without any mathematical details. We assume that a decision maker engages in a four-stage process when confronting a choice set.

In the first stage, each monetary gamble is evaluated separately, yielding a subjective, but monetary, measure of worth—a so-called certainty equivalent. Even in the case of risk, the certainty equivalent is not, in general, the expected value. For the gamble g, denote this value by $\mathrm{CE}_1(g)$.

Stage 2 involves the formation of a reference level $\mathrm{RL}(X)$ for the entire choice set X that is based on these CE_1s. In section 1 we describe some properties that we believe such a reference level should exhibit, but they are not sufficient to determine it uniquely. Among the several possibilities outlined, we choose one on empirical grounds that seems to work satisfactorily in the data in section 6. It is this: If any CE_1 is a gain (i.e., positive), then $\mathrm{RL}(X)$ is the smallest of the gains; if all CE_1s are losses (i.e., negative), then $\mathrm{RL}(X)$ is the smallest loss. The highly asymmetric rule is very conservative for gains and very risky for losses.

Stage 3 recasts all of the gambles in terms of gain and losses relative to the RL, i.e., the value $\mathrm{RL}(X)$ is subtracted from each monetary outcome.

In the last stage, a certainty equivalent called CE_2 is recalculated for each gamble, and the gamble with the largest CE_2 is chosen, with ties being resolved randomly. This is described formally in section 2.

Recasting all of the gambles in terms of gains and losses relative to $\mathrm{RL}(X)$ makes the choice situation highly context dependent. This context dependence is especially true if the theory for CE_2 is asymmetric in the weights assigned to an event depending on whether it is associated with a gain or a loss.

Left open is the question of exactly what theory to use for CE_1 and for CE_2. We have used the one developed by Luce (1992b), which is very close to what Luce and Fishburn (1991) called rank- and sign-dependent utility and Tversky and Kahneman (1992) called a cumulative representation. This theory is outlined in section 3 and described more fully in appendix B. For applications to binary gambles, it amounts to weighted utility in which the weights assigned to events depend on the sign of the outcome attached to the weight. As in prospect theory (Kahneman and Tversky, 1979), the weights do not add to 1 for gambles of gains and losses.

In order to reduce the number of free parameters to a manageable number, section 4 describes functional forms (power functions) for both the utility and weighting functions. We assume that both CE_1 and CE_2 are of the same general form, but we allow for the possibility of different parameter values. Indeed, the data lead to radically different values, which turns out to be of some interest.

Section 5 gives a series of examples designed to convince the reader that such a theory naturally exhibits a number of empirical anomalies: the Allais paradox, context effects, preference reversals, and choice intransitivities. Section 6 fits the model to a body of data involving judged prices of gambles and choices between pairs of them. The fits are reasonably good. For CE_1 the utility functions grow only slightly less fast than linear, whereas the exponents for the weights are roughly 3/4 for gains and 2/3 for losses. These values are fairly consistent with previous work. The surprise is the parameter values for CE_2. The utility functions are very nearly constant. The pattern for the weights depends markedly on whether the original gambles are all gains or all losses. In the former case, the weight of an event that leads to a loss relative to the RL varies considerably with the probability of that event, whereas the weight of an event leading to a gain relative to RL is nearly constant. For loss gambles, exactly the opposite pattern is found. These findings may mean that a rather different theory for CE_2 needs to be developed.

The present article demonstrates that by starting with certainty equivalents, establishing a reference level, recoding the gambles relative to the reference level, and then reevaluating them to make the choice, one can predict a number of behavioral anomalies and accommodate a body of data. It is a first attempt, and almost certainly variants on it will prove to be better. We are struck by the markedly different parameter values in the two CE calculations, and we suspect that different models for the CEs should be used. Furthermore, it would be desirable to achieve a deeper understanding of reference levels, in particular to arrive at some natural axioms that single out one or a very few possibilities.

1. Axioms about reference levels

Let X denote a finite set of gambles, including pure sums of money among the possibilities. Assume that the reference level of the choice set X is a pure sum of money, denoted RL(X). The problem is to characterize this set function. To do so, we list some properties that seem plausible for reference levels and discuss several functions that satisfy these conditions.

One can question whether, in practice, such a set function is well defined. Certainly, when X is very large—e.g., the choice of a stock from those listed on some stock exchange—one can question whether there is a stable RL for the entire set and, even more so, whether it is determined by examining the RLs of each component gamble. It is likely, therefore, that the theory presented is limited to comparatively small choice sets with which the decision maker is not overly familiar. Fortunately, the axiomatization does not really depend on there being very large choice sets, and it is easy to modify it to apply to any uniformly imposed size restriction.

So, we assume that there exists a family $\mathscr{G}$ of finite sets of finite gambles and a function RL from $\mathscr{G}$ into the real numbers.[3] First, there are two structural axioms on the domain $\mathscr{G}$:

Axiom 1. If $X, Y \in \mathscr{G}$, then $X \cup Y \in \mathscr{G}$.

In words, if X and Y are possible choice sets, then so is their union.

Axiom 2. For any integer n, if $\alpha_1, \alpha_2, \ldots \alpha_n$ are real numbers, then $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \in \mathscr{G}$.

In words, any finite set of sums of money is a possible choice set. Of course, such choice sets are hypothetical because they are never actually offered, the reason being that the choice is known without asking the decision maker.

The next two assumptions concern the function RL. The first simply says that the reference level of a single sum of money is that value.

Axiom 3. For any real α, $\mathrm{RL}(\alpha) = \alpha$.

The next axiom is more controversial. It says, in essence, that when X and Y are disjoint choice sets, the reference level of $X \cup Y$ depends only on the reference levels of X and of Y and not on anything else about X and Y. Formally,

Axiom 4. Suppose that $X, Y, X', Y' \in \mathcal{G}$ with $X \cap Y = X' \cap Y' = \varnothing$. If $\mathrm{RL}(X) = \mathrm{RL}(X')$ and $\mathrm{RL}(Y) = \mathrm{RL}(Y')$, then $\mathrm{RL}(X \cup Y) = RL(X' \cup Y')$.

The following characterization of reference level is an easy consequence of these assumptions:

Theorem. If axioms 1–4 hold, then for any $X = \{g_1, \ldots, g_n) \cup \mathcal{G}$,

$$\mathrm{RL}(X) = \mathrm{RL}[\mathrm{RL}(\{g_1\}), \ldots, \mathrm{RL}(\{g_n\})].$$

To gain more detailed information, we need two things. The first is, what is the RL of a single gamble? We take it to be the CE of that gamble:

Axiom 5. If $\{g\} \in \mathcal{G}$, then $\mathrm{RL}(\{g\}) = \mathrm{CE}(g)$.

Next, we must consider what the RL is when the choice is among pure sums of money. One's intuition is not strong because such choices, being obvious, are never encountered. Nonetheless, the theorem makes clear that such reference levels must be contemplated. Various possibilities come to mind, but a number of them do not seem to work.

For example, consider the following averaging property: for $X, Y \in \mathcal{G}$ with $X \cap Y = \varnothing$, then[4] $\mathrm{RL}(X \cup Y) = [\mathrm{RL}(X)|X| + \mathrm{RL}(Y)|Y]/|X \cup Y|$. Together with axiom 5, this property readily implies that $\mathrm{RL}(X)$ is the average of the CEs of the gambles in X. It is not difficult to show by example that such monetary averages violate axiom 4.

Another property that probably should not be invoked is monotonicity: for $X, Y, Z \in \mathcal{G}$ with $X \cap Z = Y \cap Z = \varnothing$, $\mathrm{RL}(X) \geq \mathrm{RL}(Y)$ if and only if $\mathrm{RL}(X \cup Z) \geq \mathrm{RL}(Y \cup Z)$. This property is, at best, plausible as an "if…, then…" statement because it is easy to imagine that Z may be so dominant that equality on the right says nothing whatsoever about the inequality on the left.

An additional property that seems highly desirable is that if a choice set (of numbers) is partitioned into a collection of choice subsets, then RL is independent of the partitioning:

Axiom 6. Suppose that $\{X_1, \ldots, X_n\}$ partitions that set X into nonempty subsets. Then

$$\mathrm{RL}(X) = \mathrm{RL}\{\mathrm{RL}(X_1), \ldots \mathrm{RL}(X_n)\}.$$

Axioms 1-6 do not uniquely determine RL(X). It is easy to demonstrate that each of the following functions satisfy them (only axiom 6 need be checked):

$$\mathrm{RL}(X) = \max_{g \in x} \mathrm{CE}(g). \tag{1a}$$

$$\mathrm{RL}(X) = \min_{g \in x} \mathrm{CE}(g). \tag{1b}$$

$$\mathrm{RL}(X) = \begin{cases} \min_{g \in x} \mathrm{CE}\,(g) & \text{if some } \mathrm{CE}(g) > 0. \\ \max_{g \in x} \mathrm{CE}(g) & \text{if all } \quad \mathrm{CE}(g) \le 0. \end{cases} \tag{1c}$$

Let f be some strictly increasing function on the real numbers. A.A.J. Marley (personal communication) has suggested

$$\mathrm{RL(X)} = \mathrm{f}^{-1}\{\Sigma_{g \in x} \mathrm{f[CE(g)]}\}. \tag{1d}$$

The data described in section 6 were analyzed using the first three equations, and equation (1c) was clearly the best. A further assumption that leads to it is the following:

Axiom 7. For real numbers α and β with $\alpha \ge \beta$,

$$\mathrm{RL}(\{\alpha,\beta\}) = \begin{cases} \beta, & \text{if } \beta > 0 \\ \alpha & \text{if } \beta \le 0. \end{cases}$$

Corollary. If axioms 1-7 hold, then equation (1c) holds.

2. A principle of choice

Having established a reference level for a choice set, we postulate that all consequences are reevaluated as gains and losses relative to that reference level. For example, a gamble with two positive outcomes ends up being evaluated as a gain and a loss when the RL lies between the outcomes. To formulate this precisely, the following definition is useful.

Definition. Suppose $g = (x_1,A_1; \ldots ;x_k,A_k)$ is a gamble and λ is a sum of money. Then $g \ominus \lambda$ is defined to be the gamble $(x_1 - \lambda, A_1; \ldots ; x_k - \lambda, A_k)$.

Now our assumption is that if X is the choice set and its reference level is RL(X), then the choice is based on modifying X to the set of gambles $\{g \ominus \mathrm{RL}(X) \mid g \in X\}$. It is this modified set of gambles, in which the concept of gain and loss has been shifted to deviations from the reference level, that is then revalued according to some CE function, and the alternative with the highest CE is selected. In summary:

Principle of Choice. Suppose X is a set of choice alternatives with the reference level RL(X). Then g in X is the choice from X if and only if

$$\mathrm{CE}[g \ominus \mathrm{RL}(X)] = \max_{h \in x} \mathrm{CE}[h \ominus \mathrm{RL}(X)]. \tag{2}$$

Ties are resolved at random.

We cannot give any strong defense of this principle other than the widespread intuition that choices are affected by reference levels that, in turn, depend upon the choice context. In sections 5 and 6, we explore some of the consequences of this entire package of assumptions, and it will be seen that some of the anomalies of choice data are natural consequences of the formulation.

3. A theory of certainty equivalents

As was mentioned earlier, a way to construct a theory of CEs has been offered[5] (Luce, 1992b). The model assumes a singular point—the status quo—with respect to which consequences and gambles are partitioned into gains and losses. Any mixed gamble of gains and losses is assumed to be recast as the corresponding binary gamble whose outcomes are the subgamble of just the gains, conditional on a gain occurring, and the subgamble of just the losses, conditional on a loss occurring. Moreover, this binary gamble is assumed to be evaluated by separately evaluating two simpler and closely related gambles, namely, the pure gain one in which the loss term is replaced by the status quo and the pure loss one in which the gain term is replaced by the status quo, and then adding these two evaluations. This "equivalance," which is implicit in Kahneman and Tversky's (1979) prospect theory and in its generalization to arbitrary, finite, risky alternatives (Luce and Fishburn, 1991; Tversky and Kahneman, 1992) is the single nonrationality of the theory. Few data exist on it, but those that do support the idea (Slovic and Lichtenstein, 1968).

This equivalence assumption reduces the problem to evaluating gambles of just gains and of just losses, which is done inductively by subtracting away the consequence that is closest to the status quo—the smallest gain or loss, as the case may be. This reformulation reduces the gamble to one having a zero consequnce and a gamble of all gains or of all losses but with one fewer consequence. Utility of this reduced gamble is simply the utility of the nonzero subgamble times the appropriate weight for the corresponding event. Thus the smallest gain or loss of the subgamble can be subtracted again, thereby reducing the size of the gamble once more. This recursive reduction leads to a rank-dependent representation.[6] Appendix B provides a more detailed statement of the theory.

In summary, the derived representation has the following character. There are three numerical functions, one a utility, U, over (incremental) consequences, and the other two, weighting functions, W_+ and W_-, over events. In the case of a binary gamble $g = (x,A;y,B)$, where x is a gain and y is a loss, and A and B form a partition of the event $A \cup B$ on which g is conditioned, then

$$U[\mathrm{CE}(g)] = U(x)W_+(A)/W_+(A \cup B) + U(y)W_-(B)/W_-(A \cup B). \tag{3}$$

The most notable feature of this expression is the fact that the weights do not in general add to 1. As was noted above, this feature is the key source of nonrationality in the theory and is also the reason that the utility function is not an interval scale, but rather a ratio scale function in which $U(\text{status quo}) = 0$. For gambles of gains and losses, the following rank-dependent form arises. Suppose that the event underlying a gamble g of just gains is A and that it is partitioned into subevents π_i with associated consequences. x_i, which have been labeled so the order is $x_1 \geq \ldots \geq x_n >$ status quo. Then

$$U(g) = \Sigma_i U(x_i) W_{+,n}(\pi_i), \quad \Sigma_i W_{+,n}(\pi_i) = 1, \tag{4}$$

where, with

$$\pi(j) = \bigcup_{i=1}^{j} \pi_i, \tag{5}$$

the weights are defined by the cumulative form:

$$W_{+,n}(\pi_i) = (W_+[\pi(i)] - W_+[\pi(i-1)])/W_+[\pi(n)]. \tag{6}$$

For a set of all losses, exactly the same form is arrived at, except that the subscript $+$ is replaced by $-$. Thus, the weights differ depending on whether we are dealing with gains or losses.

Note that this representation is rank dependent in the sense that the weight assigned to an event depends upon the order of the events that is established by the preference ordering of the associated consequences. In the analysis of data in section 6, the rank dependence plays no role because the gambles are all binary.

4. Special forms for U and W_s

To simplify the examples we provide of the theory, we shall study only binary gambles of the form $g_i = (x_i, p_i; y_i, 1-p_i)$, where $x_i \geq 0 \geq y_i$. We abbreviate this notion to $(x_i, p_i; y_i)$ and denote $\mathrm{CE}(g_i)$ by C_i.

Luce and Fishburn (1991) provide an argument for assuming that the utility function is a power function of money of the following form:

$$U(x) = \begin{cases} x^{\beta(+)}, & x \geq 0 \\ -k|x|^{\beta(-)}, & x < 0 \end{cases} \tag{7}$$

Note that there is a discontinuity in slope at 0. By equations (3) and (7),

$$(C_i) = x_i^{\beta(+)} W_+(p_i) - k|y_i|^{\beta(-)} W_-(1 - p_i). \tag{8}$$

Next we consider the dependence of W_s, where $s = +,-$, on p. The following condition, known as the reduction of compound gambles, is certainly rational and somewhat plausible:

$$((x,p;0),q;0) \sim (x,pq;0). \tag{9}$$

It follows from equations (8) and (9) and a well-known functional equation that

$$W_s(p) = p^{\gamma(s)}, \qquad \gamma(s) > 0. \tag{10}$$

Some data suggest that equation (10) may be too simple. Using a power function for utility with an exponent of 0.88, Tversky and Kahneman (1992) use $U[\mathrm{CE}(x,p,0)]/U(x)$ as an estimate of the weights in equation (8). They found the weights were larger than p for small p and smaller than p for the rest of the range. In fact, they fit these data by the following functional the form:

$$W_s(p) = p^{\gamma(s)}/[p^{\gamma(s)} + (1 - p)^{\gamma(s)}]^{1/\gamma(s)}, \tag{11}$$

with the exponents estimated to be $\gamma(+) = 0.61$, and $\gamma(-) = 0.69$.

For the data discussed in section 6, the results are more satisfactory using the simpler equation (10) than equation (11), and so we use it in the remainder of this article.

5. Some illustrative calculations

The purpose of this section is to demonstrate by simple examples that the present theory can exhibit many of the anomalies found with choice data. These are not attempts to fit actual data, because usually not enough data are reported to permit serious estimation. The calculations are merely demonstrations that, with certain choices for the parameters, the reference-level theory can mimic observed patterns of choices. It should be noted that we have used different sets of parameters to account for different anomalies. We do not know whether there exists a single set of parameters that can account for all of the examples, nor do we know of any evidence of individual subjects exhibiting all of the anomalies.

5.1. Reference levels affect choices

Table 1 take two gambles used in some of the preference reversal studies and shows that choice can depend on whether the reference level is the minimum or the maximum CE. The two columns on the far left of table 1 show that when the reference level is assumed to be the minimum CE, gamble b is preferred to gamble a, and when the reference level is the maximum CE, gamble a is preferred to gamble b.

Table 1. An example of how changing RLs can influence the order between two gambles

Gamble	x	p	y	CE(g)	CE($g\ominus$RL) RL = min CE (g)	CE($g\ominus$RL) RL = max CE(g)
a	16	0.3056	−1.5	1.60	0.37	−0.01[a]
b	4	0.9722	−1	2.71	0.74[a]	−0.02

[a]Predicted preference.
Note: The values of the parameters used are $\beta(+) = 0.8$, $\beta(-) = 0.5$, $\gamma(+) = 0.9$, $\gamma(-) = 0.2$, and $k = 1.5$.

5.2. *The Allais paradox*

Table 2 exhibits the gambles of the standard Allais paradox. Using minimum CE as the reference level, we see that the model can exhibit the reversal so often found with human subjects. In particular, gamble a is preferred to gamble b, and gamble b* is preferred to gamble a*. The reference-level theory predicts this pattern with the parameters shown in the caption.

5.3. *Context effects in choices*

In the MacCrimmon et al. (1980) experiment, the same group of subjects was presented with two contexts, each consisting of five gambles. Gambles were of the form (x,p,y). Parameters are shown for two contexts, labeled B and C in table 3. With the exception of gamble B1 = C1, context B has the feature that x is fixed at 20, and context C has the feature that p is fixed at 0.6815. Notice that gambles B4 and C4 are the same in both contexts. For 22.5% of the subjects, gamble B4 was preferred to gamble B1 in context B, but gamble C1 = B1 was preferred to gamble C4 = B4 in context C. If the reference level is assumed to be the minimum CE within a context (a reasonable assumption given that subjects were asked to rank their preferences for the five gambles simultaneously), the theory can describe the observed preference orders with the parameter values shown in the caption.

Table 2. The standard Allais paradox

Gamble	Probability 0.01	Probability 0.10	Probability 0.89	CE(g)	CE($g\ominus$RL)
a	\$1M	\$1M	\$1M	\$1M	\$0[a]
b	\$0	\$5M	\$1M	\$1.36M	−\$0.08M
a*	\$1M	\$1M	\$0	\$0.11M	−\$0.04M
b*	\$0	\$5M	\$0	\$0.5M	\$0.29M[a]

[a]Predicted preference.
Note: The values of the parameters used are $\beta(+) = 0.9$, $\beta(-) = 0.9$, $\gamma(+) = 0.9$, $\gamma(-) = 0.2$, and $k = 1.5$.

Table 3. The two families of gambles, each of the form $(x,p;y)$

Gamble	x	p	y	CE(g)	CE($g\ominus$RL)
B1	5	1	5	5	70.9
B2	20	0.0692	3.90	5.36	
B3	20	0.2752	−0.70	4.18	
B4	20	0.6815	−19.30	−5.22	77.7[a]
B5	20	0.9046	−137	−65.9	

Gamble	x	p	y	CE(g)	CE($g\ominus$RL)
C1	5	1	5	5	13.8[a]
C2	10	0.6815	−3.10	2.55	
C3	15	0.6815	−11.20	−1.68	
C4	20	0.6815	−19.30	−5.22	5.59
C5	25	0.6815	−27.40	−8.33	

[a]Two gambles in common that were used by MacCrimmon et al. (1980).
Note: The values of the parameters used are $\beta(+) = 0.9$, $\beta(-) = 0.9$, $\gamma(+) = 0.9$, $\gamma(-) = 0.2$, and $k = 1.5$.

5.4. *Preference reversals*

Predictions of the theory for the four gambles of Lichtenstein and Slovic (1971) that were also used by Bostic et al. (1990) are shown in table 4. Bostic et al. (1990) found significant preference reversals for gamble pairs 1 and 4, but consistent preference orders for gamble pairs 2 and 3. We see that the reference-level theory can account for this pattern of preference reversals using the parameter values shown in the caption.

Table 4. Predicted preference reversals applied to gambles used by Lichtenstein and Slovic (1971)

Case	x	p	y	CE(g)	CE($g\ominus$RL)	Choice	χ^2
1	16	.3056	−1.5	6.30	−0.32		10.5[a]
	4	.9722	−1	3.22	−0.28	c	
2	9	.1944	−0.5	3.95	1.88	c	1.5
	2	.8056	−1	0.70	−0.05		
3	6.5	.5000	−1	2.87	0.14	c	1.0
	3	.9444	−2	1.38	−0.01		
4	8.5	.3889	−1.5	2.56	−0.45		17.2[b]
	2.5	.9444	−0.5	2.00	−0.27	c	

[a]$p < 0.01$.
[b]$p < 0.001$.
[c]Which gamble is chosen according to the theory under discussion using the minimum CE to establish the RL in each pair. The predictions agree with the observed data.
Note: The χ^2 values are means of two independent estimates given by Bostic et al. (1990). The parameters used are $\beta(+) = 0.7$, $\beta(-) = 0.9$, $\gamma(+) = 0.2$, $\gamma(-) = 0.4$, and $k = 1.5$.

5.5. *Intransitivities*

The reference-level theory can also produce intransitive preferences. Table 5 provides an example of an intransitive triple. Suppose subjects prefer gamble a to gamble b, gamble b to gamble c, but gamble c to gamble a. Using the parameter values shown in the caption, the reference-level theory can describe the pattern of intransitive preferences.

Table 6 summarizes the parameters that were used to induce the anomalies. Note that the same values were used in tables 1 and 5 and the same values in tables 2 and 3.

6. Application to data

6.1. *The experiment*

To examine the reference-level theory more deeply and systematically, one needs a large data set in which subjects both report certainty equivalents and make choices. Mellers, Chang, et al. (1992) reported three such data sets to test their theory of preference reversals for risky options. Table 7 shows the stimuli of the form $(x,p;0), x > 0$. Rows are levels of p, columns are levels of x, and entries are expected values. Note that the values have been selected so the entries along the negative diagonals are approximately equal. A corresponding set of gambles of losses was formed by changing x to $-x$ everywhere.

Each subject assigned prices, made choices, and gave attractiveness ratings. Attractiveness ratings will not be discussed because the reference-level theory makes no predictions for that task. In the pricing task, subjects were presented with the 36 gambles one at a time, and they assigned minimum selling prices or avoidance prices (i.e., the maximum amount one is willing to pay to avoid playing the gambles). In the choice task, each possible pair of nondominated[7] alternatives was presented, and the subjects selected the gamble they preferred to play. Pricing data are median prices, and choice data are the proportion of choices over subjects.

Table 5. An intransitivity predicted by the model

					CE($g\ominus$RL)		
Case	x	p	y	CE(g)	$\{a,b\}$	$\{b,c\}$	$\{a,c\}$
a	9	0.4	1.5	4.64	-0.43^a		-0.28
b	5.5	0.9	1	5.10	-0.45	-0.21^a	
c	9.5	0.5	0	4.36		-0.24	-0.24^a

[a]The gamble chosen in the indicated choice set.
Note: The parameters used are $\beta(+) = 0.8, \beta(-) = 0.5, \gamma(+) = 0.9, \gamma(-) = 0.2$, and $k = 1.5$.

Table 6. Summary of the parameters used in generating the anomalies of table 1-5

Anomaly	$\beta(+)$	$\beta(-)$	$\gamma(+)$	$\gamma(-)$	k
max-min	0.8	0.5	0.9	0.2	1.5
Allais	0.9	0.9	0.9	0.2	1.5
Context	0.9	0.9	0.9	0.2	1.5
Pref. Rev.	0.7	0.9	0.2	0.4	1.5
Intrans.	0.8	0.5	0.9	0.2	1.5

Table 7. The stimuli $(x,p;0)$ used by Mellers, Chang, et al. (1992)

	Amount x					
Prob p	\$3.00	\$5.40	\$9.70	\$17.50	\$31.50	\$56.70
0.05	0.15	0.27	0.49	0.88	1.58	2.84
0.09	0.27	0.49	0.87	1.58	2.84	5.10
0.17	0.48	0.86	1.55	2.80	5.04	9.07
0.29	0.87	1.57	2.81	5.08	9.14	16.44
0.52	1.56	2.81	5.04	9.10	16.38	29.48
0.94	2.82	5.08	9.12	16.45	29.61	53.30

Note: Entries are expected values, and are close to equal along negative diagonals.

6.2. *The model fit to the data*

Consider the gamble pair $g = (x,p;0)$ and $h = (u,q;0)$, with $x > 0$ and $u > 0$. By equations (8) and (10),

$$U[\mathrm{CE}(g)] = x^{\beta(+)}p^{\gamma(+)} \text{ and } U[\mathrm{CE}(h)] = u^{\beta(+)}q^{\gamma(+)}. \tag{12}$$

Suppose, for the sake of argument, $x < u$, $\mathrm{CE}(g) < \mathrm{CE}(h)$, $\mathrm{CE}(g) < x$, and that the minimum reference level is used. Then the subject will choose g over h if

$$\begin{aligned}&[x - \mathrm{CE}(g)]^{\beta(+)}p^{\gamma(+)} - k[\mathrm{CE}(g)]^{\beta(-)}(1-p)^{\gamma(-)}\\ &> [u - \mathrm{CE}(g)]^{\beta(+)}q^{\gamma(+)} - k[\mathrm{CE}(g)^{\beta(-)}(1-q)^{\gamma(-)}.\end{aligned} \tag{13}$$

As the assumed conditions change, the equation must be altered. For example, if $x < \mathrm{CE}(g)$, then the first term to the left of the inequality becomes $-k[\mathrm{CE}(g)-x]^{\beta(-)}p^{\gamma(-)}$. For the present data, we must work with choice proportions, so equation (13) cannot be used directly. Our assumption is that choice proportions are a function of the difference between the two components of the above (and similar) inequalities. The monotonic function will be approximated by a cubic polynomial.

This version of the theory was fit to the data of Mellers, Chang, et al. (1992) using a Fortran program that relied on Chandler's (1969) STEPIT subroutine. For each task, the fit of the model was evaluated using the following loss index:

$$L_T = \Sigma(X - \hat{X})^2/(X - \overline{X})^2,$$

where X is the response (i.e., median selling price of choice proportion), $\hat{X}$ is the predicted value for the response, $\overline{X}$ is the grand mean within a task. The summation is over all gambles in the pricing task or pairs of gambles in the choice task. L_T represents the percentage of residual variance in a task. The index to be minimized was the sum of the two L_Ts, one for each task. Separate sets of parameters were determined for each task.

6.3. *The parameters*

After some exploration of the parameter space, we concluded that the fit to these data was relatively insensitive to the choice of k, so we set it to 1. In each domain, six parameters, shown in table 8, were estimated, and an additional four were found for the cubic polynomial in the choice task. Two versions of the reference-level theory were fit to each set of data—one in which the RL was assumed to be the maximum of the CEs in the choice set and one in which it was the minimum. As will be made clear below, the fit is best if the RL is taken to be the minimum in the domain of gains, i.e., the smaller gain, and the maximum in the domain of losses, i.e., the smaller loss.

Before discussing the fit of the data, the pattern of these parameters bears some comment. First, the values differ radically in the pricing and the choice situations. Initially we attempted to fit a version of the theory in which the parameters were constrained to be the same, but the fit was very poor. Second, the pattern of price estimates for gains and losses is quite similar, with the utility function virtually proportional to money. Third, values of the choice parameters are quite startling. In each case, 3 of the 4 are quite close to 0, which means that the corresponding function is close to a constant. For example, if the weight parameter is 0.01, then the weight associated to the smallest probability, 0.05, is 0.970, which of course is little different from its weight 0.999 for the largest probability, 0.94. And for a utility parameter of 0.11, the utility function varies by a factor of 1.38 when going from the smallest outcome, \$3.00, to the largest, \$56.70.

Thus, within the domain of gains, the effect of a "gain" relative to the RL is largely independent of either the amount involved or its probability of occurring. What does seem to have an impact is the probability of a "loss." The pattern for the loss domain is

Table 8. The best-fitting reference-level parameters with $k = 1$ for the Mellers, Chang, et al. (1992) data

		$\beta(+)$	$\beta(-)$	$\gamma(+)$	$\gamma(-)$
Gains	Price	0.97		0.78	
	choice	0.14	0.11	0.06	1.32
Losses	Price		0.93		0.67
	choice	0.04	0.13	0.92	0.01

reversed: it varies mostly with the probability of a "gain" relative to the RL. This finding was not anticipated, and it strikes us as rather counterintuitive.

6.4. The quality of the fit

Values of L_T summed over tasks were for gains and 10% for losses. Figure 1 shows the scatter diagram for observed and predicted values for the domain of gains when the RL is taken to be the minimum, and figure 2 for the domain of loss when RL is taken to be the maximum. When the RL were reversed (i.e., RL is the maximum CE in the domain of gains and the minimum CE in the domain of losses, summed values of L_T increased to 31% and 39%, respectively, and the corresponding scatter diagrams are those of figures 3 and 4. Clearly, the scatter is appreciably worse than in figures 1 and 2.

The shift in the RL from the minimum CE in the domain of gains to the maximum CE in the domain of losses presumably reflects, once again, the fact that people may be pessimistic with respect to gains and optimistic with respect to losses. This change, rather than in the shape of the utility function, is the way in which this model accommodate risk aversion for gains and risk-seeking behavior for losses.

We also attempted to fit the data using RL set to 0, which is usually thought to be the status quo. In this case, the fit was reasonable in the domain of gains but very poor in the domain of losses. Clearly it is going to take some carefully designed experiments to gain a better understanding of what controls the RL, but these data suggest that it may be a

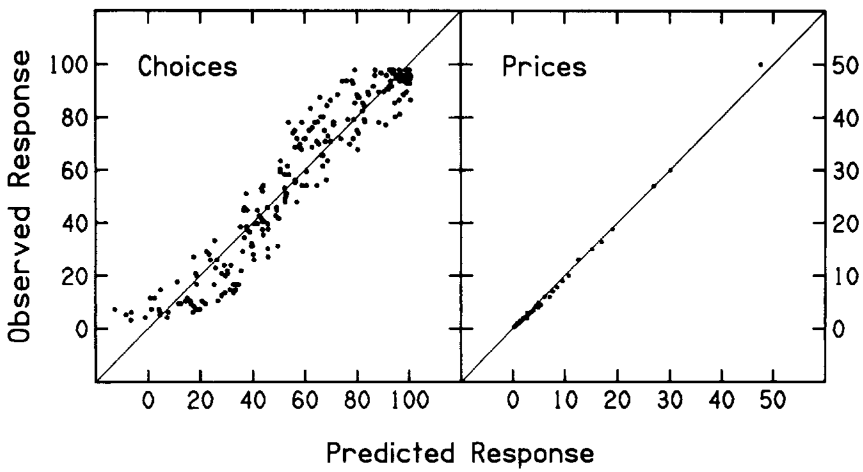

Figure 1. Scatterplots showing observed vs. predicted choice proportions on the left and observed vs. predicted selling prices on the right for gambles in the domain of gains. Predictions are based on the assumption that the reference level is the minimum CE (smallest gain).

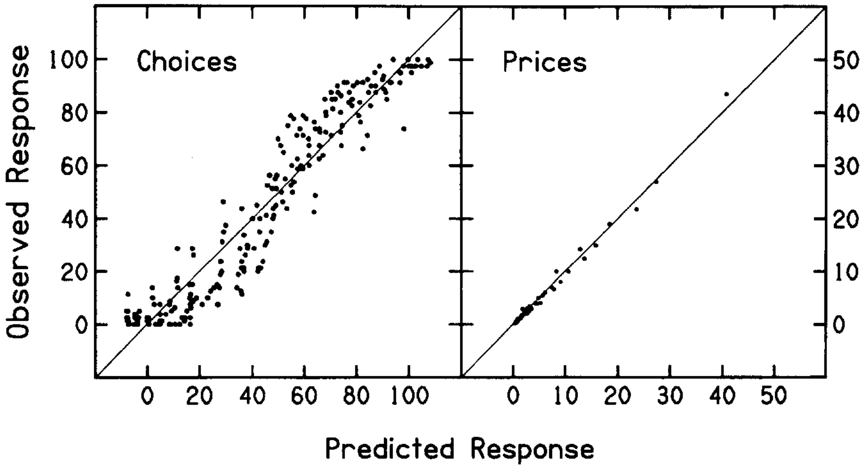

Figure 2. Scatterplots showing observed vs. predicted choice proportions on the left and observed vs. predicted avoidance prices on the right for gambles in the domain of losses. Predictions are based on the assumption that the reference level is the maximum CE (smallest loss).

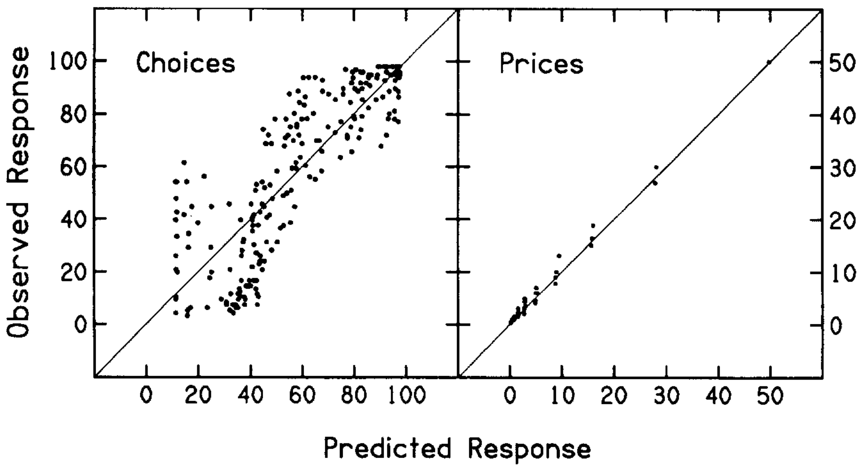

Figure 3. Data versus predictions, plotted as in figure 1. Predictions differ from those of figure 1 in that the reference level is assumed to be the maximum CE (largest gain).

useful concept. Presumably, if our parameter estimates are to be taken seriously, one needs larger ranges of outcomes and more elaborate gambles involving both gains and losses to test the model adequately.

Returning to the best fits using the parameters of table 8 and the minimum RL for gains and the maximum for losses, it is evident that we are not doing as well for choices as for prices, and some of the deviations may be slightly systematic. Nonetheless, these overall fits appear fairly reasonable. Now it is important to ask whether the theory can

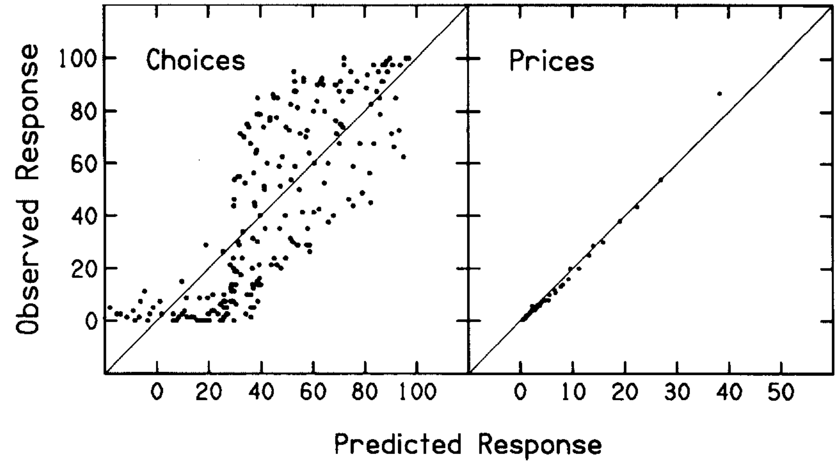

Figure 4. Data versus predictions, plotted as in figure 2. Predictions differ from those of figure 2 in that the reference level is assumed to be the minimum CE (largest loss).

describe two critical features of the data: violations of transitivity and preference reversals. The next two subsections address each of these questions.

6.5. *Stochastic transitivity*

Table 9 shows the percentage of violations of weak, moderate, and strong stochastic transitivity in the data and those predicted by the reference-level theory. Correlations between predicted and observed violations of strong stochastic transitivity were statistically significant at the 0.01 level in the domain of gains and not significant for losses.

6.6. *Preference reversals*

As was mentioned earlier, several recent studies have shown that for certain pairs of gambles, the preference order established by judged CEs is the opposite of that established by direct choice. Gamble pairs are usually matched on expected value and constructed so that one, often called the P-bet, has a large probability of winning a relatively

Table 9. Observed and predicted percentages of violations of weak, moderate, and strong stochastic transitivity

		Weak	Moderate	Strong
Gains	Observed	3	11	44
	predicted	3	5	42
Losses	Observed	0	6	34
	predicted	0	0	18

small amount, and the other, called the \$-bet, has a small probability of winning a relatively large amount.

In the two data sets of Mellers, Chang, et al. (1992), there were 55 pairs of gambles with equal expected value. Table 10 shows the observed and predicted frequencies of preferences orders for these gamble pairs in the domains of gains and losses. Columns in each panel refer to the choice preference, P-bet or \$-bet, and the rows refer to the preference as reflected in selling prices. Cells on the main diagonal are consistent preference orders, and those off the major diagonal are preference reversals. The left panels show the observed data. In the domain of gains, 37 out of 55 pairs of gambles had larger selling prices for the \$-bet, but choice proportions favoring the P-bet. In the domain of losses, 30 out of 55 pairs of gambles had smaller avoidance prices for the P-bet, but choice proportions favoring the \$-bet over the P-bet. The reference-level theory predicts these patterns.

The theory not only predicts preference reversals for gambles with equal expected values, but it is consistent with the entire set of preference orders for both tasks. Figure 5 presents the observed and predicted preference rank orders in the domain of gains, with rank 1 being the lowest-ranked and 36 the highest-ranked gamble. Recall from table 7 that gambles with equal expected value lie along the negative diagonals. Arrows on the negative diagonals indicate the direction of preference for these gambles. Those pointing up to the right correspond to the gamble pair for which the \$-bet was ranked higher than the P-bet, and those pointing down to the left represent the opposite preference. Single lines indicate ties. Upper panels are the observed orders, and lower ones the predicted ones.

Notice that the two tasks have quite different preference orders. In pricing, \$-bets are almost uniformly preferred to P-bets of equal expected value. In choice, \$-bets are more preferred for small probabilities and P-bets are preferred for larger probabilities. The reference-level theory captures this pattern. As a measure of the correlation between the observed and predicted rank orders, we have the gamma statistic, which has the value of 0.98 for prices and of 0.98 for choices in the domain of gains.

The comparable observed and predicted rank orders in the domain of losses is shown in figure 6. Here the meaning of the ranks is changed: 1 is the most preferred and 36 the least preferred. The gamma measures for losses are 0.96 for prices and 0.97 for choices. Again, the reference-level theory is consistent with these data.

Table 10. Observed and predicted preference reversals in the choice data of Mellers, Chang, et al. (1992)

	Domain of gains				Domain of losses			
	Observed choice		Predicted choice		Observed choice		Predicted choice	
Price	P	\$	P	\$	P	\$	P	\$
P	0	1	0	0	25	30	25	30
\$	37	17	36	16	0	0	0	0

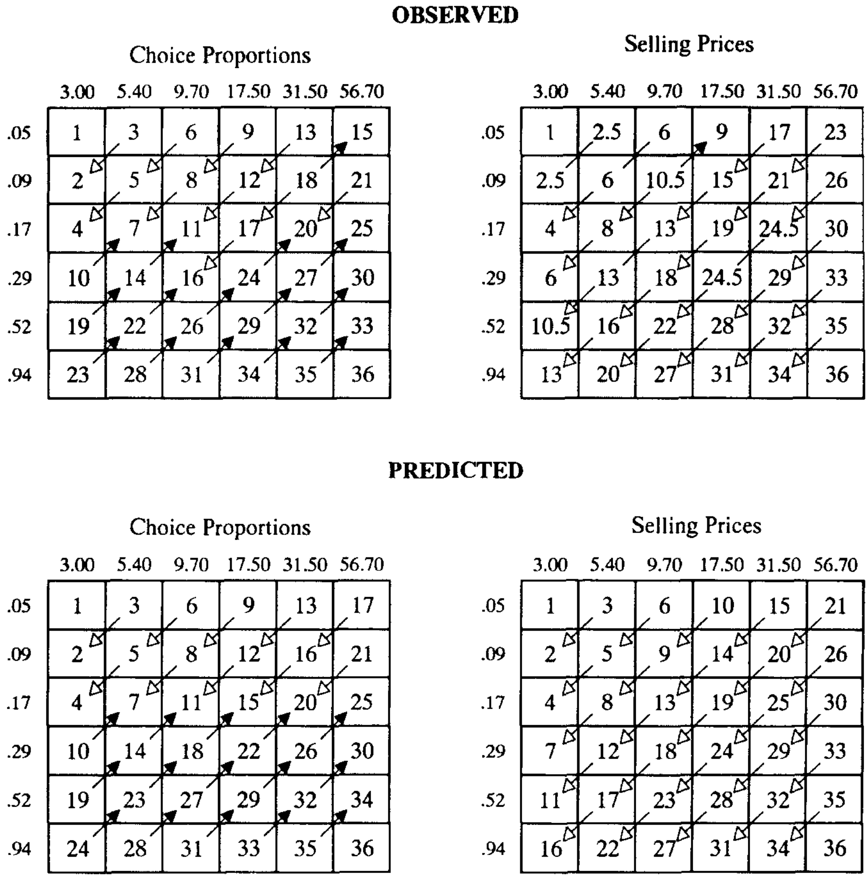

Figure 5. Observed and predicted preference orders for choice proportions and selling prices in the domain of gains. Orders for choices are derived, and orders for prices are obtained directly from median responses. Larger numbers refer to higher-ranked gambles. Arrows show the direction of preference for sequential gamble pairs with the same expected values. Arrows pointing up indicate a preference for the P-Bet, and arrows pointing down indicate a preference for the $-Bet. Single lines represent ties.

7. Comparison with change-of-process, disappointment, and PE reframing theories

Mellers, Chang, et. al (1992) proposed a quite different account of preference reversals, referred to as change-of-process theory, which makes predictions about prices, strength of preference judgments, and attractiveness ratings. According to that theory, subjects use different processing strategies depending on the task, but the utilities and decision weights remain constant. For the simple gambles shown in table 7, attractiveness ratings were described by an additive combination of utilities and weights. Judged prices were predicted by a multiplicative model of utilities and decision weights, and strength-of-preference judgments were consistent with a contrast weighting model in which the weight of a dimension (either probability or amount) depends on the contrast between

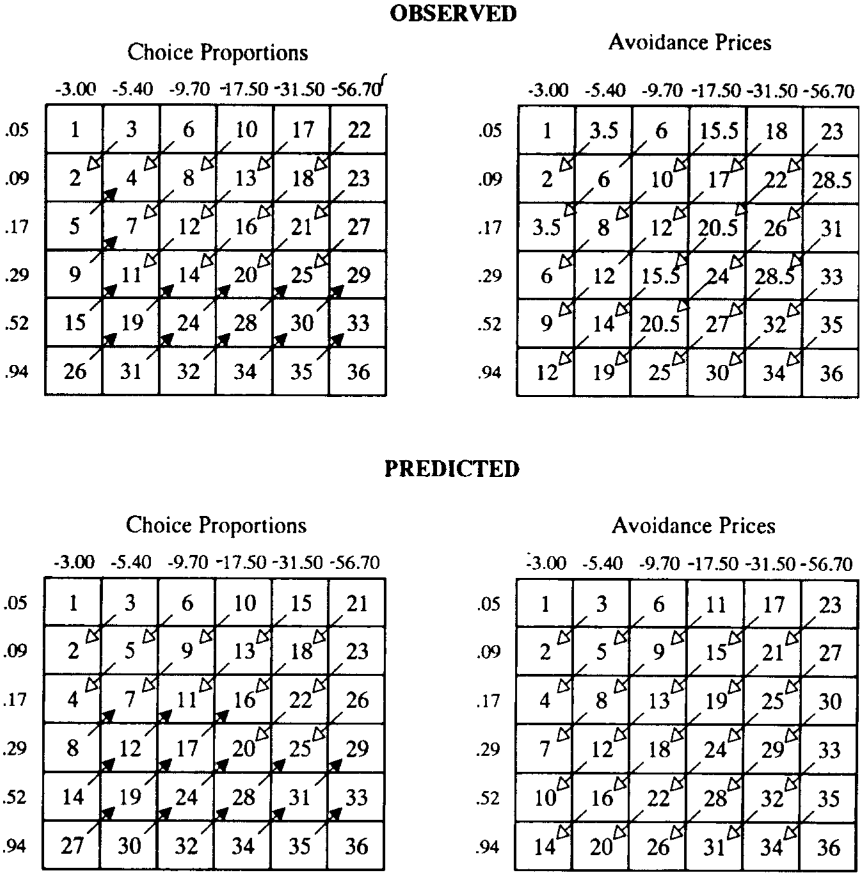

Figure 6. Observed and predicted preference orders for choice proportions and avoidance prices in the domain of losses. Orders for choices are derived, and orders for prices are obtained directly from median responses. Smaller numbers refer to higher-ranked gambles. Arrows show the direction of preference for sequential gamble pairs with the same expected values. Arrows pointing up indicate a preference for the $-Bet, and arrows pointing down indicate a preference for the P-Bet. Single lines represent ties.

the two gambles along that dimension. Smaller contrasts receive less weight than larger contrasts. These three models, together with the assumption of scale convergence, reproduce the preference orders in figures 5 and 6 and the preference order for attractiveness ratings of gambles.

Change-of-process theory differs from the reference-level theory in that subjects are assumed to use different strategies in different tasks with the same scales. The reference-level theory implies that subjects use the same strategy in the pricing and choice tasks, but reference levels and scales (utilities and decision weights) are allowed to vary across tasks. The parameters in table 8 show that for the reference-level theory, weights and utilities vary dramatically across tasks. The experiment of Mellers, Chang, et al. (1992) does not provide a good test between the theories. Further work is needed to investigate and capitalize on differences between these critical assumptions of changing decision strategies and invariant scales versus changing scales and reference levels and invariant decision strategies.

Some parallels exist between the reference-level theory and disappointment theory, a generalization of expected utility theory proposed by Loomes and Sugden (1986). Disappointment theory asserts that people form expectations about gambles and modify the consequences according to those expectations. By incorporating a disappointment-elation function, Loomes and Sugden (1986) proposed that utilities are modified to reflect anticipated feelings that individuals might have if outcomes are better or worse than expected.

The theories are similar in that they both assume individuals modify the consequences of a gamble based on a reference level. They differ in several ways, however. Perhaps the most fundamental difference is the assumption about the reference level. In disappointment theory, it is the expected utility of the gamble, which means that the reference level for a gamble is independent of other gambles with which it is compared. In our theory, the reference level is an extremum of the certainty equivalents of the gambles in the choice set. Thus, a gamble alone does not have a reference level; it depends upon the other gambles with which it is compared. This assumption allows the reference-level theory to account for preference reversals and violations of strong stochastic transitivity, whereas disappointment theory is unable to describe these effects.

Hershey and Schoemaker (1985) showed the following inconsistency in behavior. Suppose a subject assigns the value S as the CE of the gamble $(200,\frac{1}{2},0)$ and that at a later time the same subject is asked to adjust p so as to be indifferent between S and $(200,p,0)$. Then, in general, $p \neq \frac{1}{2}$. The same inconsistency is seen if one first asks for the probability equivalent (PE) for S and then later asks for the CE of the resulting gamble. To explain this inconsistency, they proposed that in determining the PE the subject compares not S and $(200,p,0)$ but rather $S\text{-}S = 0$ and $(200\text{-}S,p,\text{-}S)$. The choice is then made by applying a sign-dependent model such as prospect theory. This is very close to what we have proposed, except that they did not distinguish between gains and losses, as we have found it necessary to do, and the reference level is not chosen in terms of the relation between S and $CE(200,p,0)$.

8. Conclusions

The major conclusion is that it is entirely feasible to develop a theory of choice in terms of a theory of certainty equivalents and one of reference levels. In a sense, one can view this article as a demonstration of that thesis and of the fact that such a theory exhibits some of the anomalies of the choice data such as preference reversals, intransitivities, and context-induced reversals. Basically it has three theoretical components: the theory of RLs of individual gambles, the theory of RL for sets of gambles, and the theory of how the gambles are evaluated after being modified as gains and losses relative to the RL. For simplicity we have assumed here that the first and third of these component theories are indentical in form although, judging by the data analyzed, with very different parameter values. This identity of form need not be the case. We also assumed that they have a rank-and sign-dependent form described in several recent papers. The axiomatic theory of RL apparently is new, and it has not been subject to any direct evaluation. So all components of the theory need further study.

The analysis of the Mellers, Chang, et al. (1992) data not only gives some encouragement to pursue the model, but resulted in some surprises. First, it is clear that for gains the RL is better taken as the minimum of the CEs of the alternatives, whereas for losses it is better taken as the maximum. Second, the parameters for the CE model for prices and for choices are quite different. For prices they are about what one would expect: the utility function is somewhat more convex than linear, and the weighting functions have exponents that tend to overweight probabilities. For choice, the situation is different. The utility functions are essentially constant. In the domain of gains, the only parameter that exhibits significant variations with the value of its argument is the weight assigned to the probability of a "loss" relative to the RL; in the domain of losses, only the weight assigned to the probability of a "gain" relative to the RL matters. This finding, if sustained in additional research, is striking and should be of significance in applications.

Appendix A: Proofs of theorem and corollary

Proof of theorem

Suppose $X = \{g_1, g_2, \ldots, g_n\}$. Define $Y = \{g_1\}$, $Z = \{g_2, \ldots, g_n\}$, $Y' = \{\mathrm{RL}(Y)\}$, and $Z' = Z$. Clearly, $Y \cap Z = Y' \cap Z' = \varnothing$. By axiom 3, $\mathrm{RL}(Y') = \mathrm{RL}(\{\mathrm{RL}(Y)\}] = \mathrm{RL}(Y)$, and because $Z' = Z$, $\mathrm{RL}(Z') = \mathrm{RL}(Z)$. Thus, by axiom 4,

$$\begin{aligned}\mathrm{RL}(X) &= \mathrm{RL}(Y \cup Z) = \mathrm{RL}(Y' \cup Z') = \mathrm{RL}[\{\mathrm{RL}(g_1)\} \cup Z] \\ &= \mathrm{RL}[\{\mathrm{RL}(g_1)\} \cup \{g_2, \ldots, g_n\}].\end{aligned}$$

The argument may be repeated on each of the other g_i permitting its replacement by $\mathrm{RL}(\{g_i\})$. Thus,

$$\mathrm{RL}(X) = \mathrm{RL}[\{\mathrm{RL}(\{g_1\}), \ldots, \mathrm{RL}(\{g_n\})].$$

Proof of corollary

Define

$$X^+ = \{g \in X \mid \mathrm{CE}(g) > 0\}, X^- = \{g \in X \mid \mathrm{CE}(g) \leq 0\}.$$

By axioms 6 and 7,

$$\mathrm{RL}(X) = \mathrm{RL}\{\mathrm{RL}(X^+), \mathrm{RL}(X^-)\}$$
$$= \begin{cases} \mathrm{RL}(X^+) & \text{if } X^+ \neq \emptyset \\ \mathrm{RL}(X^-) & \text{if } X^+ = \emptyset. \end{cases}$$

Suppose $X^+ \neq \emptyset$. Let $\alpha_i = \mathrm{CE}(g_i)$, where the indices are chosen so that $\alpha_i \leq \alpha_{i+1}$. Note that $\alpha_i > 0$. By axiom 7,

$$\mathrm{RL}\{\alpha_i, \alpha_{i+1}\} = \alpha_i.$$

Thus, using axiom 6 to partition X^+ into a set of pairs and a singleton if the total number of gambles is odd,

$$\mathrm{RL}(X^+) = \begin{cases} \mathrm{RL}\{\alpha_i, \alpha_3, \ldots, \alpha_{k-1}\} & \text{if } k \text{ is even} \\ \mathrm{RL}\{\alpha_1, \alpha_3, \ldots, \alpha_k\} & \text{if } k \text{ is odd.} \end{cases}$$

Continuing inductively, we see that

$$\mathrm{RL}(X) = \mathrm{RL}(X^+) = \alpha_1 = \min_{g \in x^+} \mathrm{CE}(g).$$

If $X^+ = \emptyset$, then by a similar induction

$$\mathrm{RL}(X) = \mathrm{RL}(X^-) = \max_{g \in x^+} \mathrm{CE}(g).$$

Thus, equation (1c) is proved.

Appendix B: Additional details on the CE theory

The theory of certainty equivalents used in this article rests, in part, on a theory of generalized concatenation structures with singular points worked out in Luce (1992a). Those aspects relevant to the CE theory used in the article are discussed in this appendix.

Let $\gtrsim$ be a binary ordering of a set X, e.g., the ordering of consequences by preference. $\langle X, \gtrsim \rangle$ forms a continuum if and only if it is order isomorphic to $\langle \mathrm{Re}, \geq \rangle$, where Re denotes the real numbers. Because we are assuming money lies in the real numbers, this is a plausible assumption.

Let $F: X^n \to X$ be a function on n arguments from X into X. The main assumptions are

1. $\gtrsim$ is a total order, i.e., it is transitive, connected, and antisymmetric.
2. The structure is nontrivial in the sense that there exist x, y in X such that $x > y$.
3. The function F is monotonic in the usual sense that for each position i and each $x_1, \ldots, x_i, x_i', \ldots, x_n$,

$$x_i \gtrsim x_i' \text{ if and only if } F(x_1, \ldots, x_i \ldots, x_n) \gtrsim F(x_1, \ldots, x_i', \ldots, x_n). \tag{B1}$$

In addition, we shall suppose that 0 is a singular point in the intuitive sense of dividing the universe into gains and losses and in the technical sense of being invariant under all automorphisms (symmetries) of the structure.

As is true in most theories of utility, we shall suppose that the structure $\mathscr{X} = \langle X, \gtrsim, F, 0\rangle$ is highly symmetric in the sense that it is isomorphic under a mapping U onto a real relation structure $\mathscr{R} = \langle \mathrm{Re}, \geq, G, 0\rangle$ whose automorphisms are all multiplications by positive constants (ratio scale), i.e., for all real u_i and $r > 0$,

$$G_\pi(ru_1, \ldots, ru_n) = rG_\pi(u_1, \ldots, u_n), \tag{B2}$$

and

$$U(0) = 0. \tag{B3}$$

The technical conditions, formulated as properties of the automorphisms of $\mathscr{X}$, that underlie such a representation are provided in Luce (1992a).

Moreover, it can be shown that 0 has the property of being a generalized zero in the sense that there are constants $W_{s,i} > 0$, $s = +, -$, such that for u_i in Re,

$$G(0, \ldots, 0, u_i, 0, \ldots, 0) = W_{s,i}u_i, \qquad s = \operatorname{sgn} u_i. \tag{B4}$$

This fact is key to the particular representation of equations (3) and (4).

In our interpretation, we will have different functions for each n and each partition of an event into n subevents; the arguments of each F are sums of money, and F is the CE of the corresponding gamble. We assume, as is true in almost all theories of utility, that there is a common isomorphism U that takes all the different CEs into ratio scale representations.

Our next assumption concerns gambles with both gains and losses. It says that the CE can be computed as follows: Determine the CE of the gains, conditional on a gain occurring, and call it CE^+; determine the CE of the losses, conditional on a loss occurring, and call it CE^-; and then determine the CE of the binary gamble of CE^+ pitted against CE^-. The assumption is that this number is identical to the CE of the gamble. Formally, suppose $x_i > e$ for $i = 1, \ldots, k$, $x_i \lesssim e$ for $i = k + 1, \ldots, n$. Set $A = \pi_1 \cup \ldots \cup \pi_k$ and $B = \pi_{k+1} \cup \ldots \cup \pi_n$. Then,

$$F_\pi(x_1, \ldots, x_n) = F_{\{A,B\}}\left[F_{\pi|A}(x_1, \ldots, x_k), F_{\pi|B}(x_{k+1}, \ldots, x_n)\right], \tag{B5}$$

where $\pi|A$ denotes the restriction of the partition π to the event A.

Suppose we add to equation (B5) the assumption that two events that have the same consequence can be collapsed in the partition, in particular that

$$F_{\{A,B,C\}}(x,e,e) = F_{\{A,B\cup C\}}(x,e). \tag{B6}$$

Then the weights can be shown to have the form

$$W_{s,2}(A,B) = W_s(A)/W_s(A \cup B), \tag{B7}$$

where $W_s(A) = W_{s,2}(A,A^c)$ and A^c is the complement of A relative to some universal event Ω. In general, $A \cup B$ is a proper subset of Ω.

It follows from equation (B2) that for fixed $\{A,B\}$, the function $g_{\{A,B\}}(w) = -G_{\{A,B\}}(-w,-1)$ of one variable is strictly increasing, $g_{\{A,B\}}(w)/w$ is strictly decreasing, and for $u > 0 > v$:

$$G_{\{A,B\}}(u,v) = vg_{\{A,B\}}(u/v). \tag{B8}$$

The assumptions to this point do not determine the form for $g_{\{A,B\}}$. All the sign-dependent theories that have been formulated so far (Kahneman and Tversky, 1979; Luce, 1992b; Luce and Fishburn, 1991; Tversky and Kahneman, 1990; Wakker, 1989) have, one way or another, forced the linear for $g_{\{A,B\}}(w) = aw + b$, where b need not equal $1 - a$. However, as each variable approaches 0, monotonicity and equation (B4) imply

$$a = W_{+,2}(A,B) \text{ and } b = W_{-,2}(B,A),$$

yielding

$$G_{\{A,B\}}(u,v) = uW_{+,2}(A,B) + vW_{-,2}(B,A), \tag{B9}$$

which with equation (B7) is equation (3) of the text. Luce (1992b) describes two properties that lead to such a weighted average, but it needs to be studied further. There is the real possibility that the theory should not be linear across gains and losses.

Next, consider a gamble of just gains, and suppose that the events have been numbered so that $u_1 \geq \ldots \geq u_n > 0$. The first general idea is that the gamble can, using a term of Kahneman and Tversky (1979), be edited by subtracting away the value of the consequnce that is nearest the status quo. In particular, suppose that each $u_i > 0$ and that $u_n = \min\{u_i\}$; then

$$G_{\pi}(u_1, \ldots, u_n) = G_{\pi}(u_1 - u_n, \ldots, u_{n-1} - u_n, 0) + u_n. \tag{B10}$$

A similar assertion, using the maximum value, holds when all arguments are negative.

This editing principle, which is a restricted version of Pfanzagl's (1959) consistency axiom, is then used recursively along with equations (B5), (B7), and (B9) to arrive at the form given in equation (4).

Notes

1. Becker, DeGroot, and Marschak (1964) proposed a payoff procedure that for a person maximizing subjective expected utility (SEU) (Fishburn, 1982; Savage, 1954) leads a person to reveal his true certainty equivalent. Its failure outside the framework of SEU is discussed by Chew, Karni, and Safra (1981), Karni and Safra (1987), Keller, Segal, and Wang (1991), and Safra, Segal, and Spivak (1990a,b).
2. Sometimes these have been defined to be judged prices, selling or buying, sometimes as attractiveness ratings, and sometimes as that sum of money that is judged indifferent to the gamble in a choice experiment.

3. The mapping is, of course, actually into money, but if one assumes a fixed monetary unit, which may be done without loss of generality, then the mapping can be modeled as if it is into the real numbers.
4. For a finite set X, denote the number of its elements by $|X|$.
5. This development rests upon as analysis of a general class of ordered structures with an n-ary, monotonic operation defined on it. Following the pioneering work of Narens (1981a,b) and Alper (1987) (for a summary of these and related developments, see chapter 20 of Luce, Krantz, Suppes, and Tversky, 1990), Luce (1992a) studied those structures that are homogeneous except at isolated singularities, of which the status quo is thought to be an example. This is described more fully in appendix B.
6. A major development in utility theory during the 1980s was the flowering of various versions of rank-dependent representations. Some of the relevant papers are Gilboa (1987); Luce (1988, 1991); Luce and Fishburn (1991); Luce and Narens (1985); Quiggin (1982); Segal (1987a, 1987b, 1989); Wakker (1989, 1990, in press); Yaari (1987).
7. An alternative dominates another if both its outcome and probability of occurring are larger.

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