Dynamics of Network Formation Processes in the Co-Author Model

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Abstract

This article studies the dynamics in the formation processes of a mutual consent network in game theory setting: the Co-Author Model. In this article, a limited observation is applied and analytical results are derived. Then, 2 parameters are varied: the number of individuals in the network and the initial probability of the links in the network in its initial state. A simulation result shows a finding that is consistent with an analytical result for a state of equilibrium while it also shows different possible equilibria.

1. INTRODUCTION

Game theoretical network has grown into a distinctive research field (Back and Flache 2006; Bala and Goyal 2000; Belleflamme and Bloch 2004; Currarini and Morelli 2000; Dutta and Mutuswami 1997; Dutta, van den Nouweland; and Tijs 1998; Kranton and Minehart 2001; Skyrms and Pemantle 2000). To study the strategic aspect of cooperative game theoretic network, Jackson and Wolinsky (1996) introduce the Co-Author Model and analyze it analytically. The Co-Author Model has interesting simultaneous maximum and minimum optimization in its game theoretic setting, making it plausible for many applications[1]. While originally aimed to study the behaviors of research collaborations, any network structures that reward direct links while penalizing indirect links under a condition that requires a sharing of limited resources can be studied using this model. For examples, one can adjust this model to fit anything from a network of activity partners, a trade zone, to a defense pact.

Despite the usefulness of the previous analytical results, there is still room for improvements. First, the assumptions are often too strong. In the Co-Author Model, there is a strong assumption that each individual in the network is capable of global-observation, which allows her to observe other individuals' utilities and network configurations accurately. Yet, this assumption about complete information has been considered as too strong in sociology literatures (Friedkin 1983; Kumbasar; Romney, and Batchelder 1994; Wu, Huberman, Adamic, and Tyler 2004). Yaari and Bar-Hillel (1983) found that different individuals' beliefs, as a result of observational limitations, could yield different equilibria that will affect the social utility. Hence, a refinement by limiting

1 See section 2.2 for the details of the model.

individuals' observations such that there exists a profile of beliefs, in which each individual does not receive a message that contradicts to her own beliefs, makes the network models become more realistic (McBride 2006a). Second, due to the complexity of combinatorial possibilities, analytical models often prove only an existence of a certain equilibrium starting from certain initial states of the network, leaving the dynamics of the models unexplored. While one can argue that many variations can be made to the Co-Author Model, there are certain variations that are essential to explain the applicability of the network in reality.

This paper progresses as follows: first, I will present a proof of existence of an efficient network that is also stable, along with its necessary condition, to show the effects of limited observation in the Co-Author Model. Second, I will present the dynamical processes of the network formation and equilibrium in the Co-Author Model using two parameters, the number of individuals in the network and the probability of the links in the network in its initial state, under limited observation that allows a change in the individuals' beliefs.

2. GAME AND CONCEPTS

2.1. Network Definitions

N is the number of individuals i in a network. A *complete network*, denoted g^N, consists of all subsets of two distinct elements in the set $N(g) = \{1, ..., N\}$ of nodes in g^N. The set of all possible graphs on N is then $\{g \mid g \subseteq g^N\}$.

Let ij denote the 2 element subset of N in g containing i and j and referred to as the *link ij*, L_{ij}, or $ij \in g$, which means that *i* and *j* are directly connected. If *i* and *j* are not connected, then $ij \notin g$. Links are bi-directional, so $ij \in g \Leftrightarrow ji \in g$.

Let *g* + *ij* means adding a new *link ij* to the existing network *g* and *g* – *ij* means detaching *ij* from the existing network *g*.

The *effective number of links*, L_e, is the sum of all links *ij*, $\Sigma L_{ij}, ij \in g$.

A *path* in *g* is a set of distinct nodes $\{i_1, i_2, \ldots, i_n\} \subset N(g)$ connecting i_1 *and* i_n such that $\{i_1 i_2, i_2 i_3, \ldots, i_{n-1} i_n\} \subset g$. A graph *g* is *connected* if there exists a path between every pair of nodes. A property *P* on a graph *g* is *maximal* (*minimal*) if there is no graph $g' \supset g$ ($g' \subset g$) that is, larger (smaller) than *g*, for which property *P* holds.

A *component* $g' \subseteq g$ is a maximal connected subgraph of *g* such that, if for all $i \in N(g')$ and $j \in N(g')$, $i \neq j$, there exists a path in *g′* connecting *i* and *j*.

A graph $g' \subseteq g$ is a *clique* if it is a maximal complete subgraph of *g* with 2 or more nodes.[2] A clique that consists only of 2 nodes and 1 link is a *dyad*.

Average number of links A_{g^N} is a ratio the effective number of links, L_e, over the number of individuals *N*, and is formulated with

$$A_{g^N} = \frac{L_e}{N}, ij \in g.$$ [3]

A *state* S_t is a configuration of a network *g* in time *t*. The *initial probability*

2 This definition differs from the definition used in Sociology that requires at least 3 nodes. This definition also differs in that a dyad is defined as a link *ij* that is not contained in a larger clique.

3 Another similar alternative measurement is *network density*. However, A_{g^N} is used in this model since it describes the properties better. The equation for *network density* would be

$$D_{g^N} = \frac{L_e}{N(N-1)/2}, ij \in g.$$

P(i) assigns the occurrence likelihood of links in the initial state of network, S_0.

Let $v(g), v: g \Rightarrow \mathbb{R}$ be the *value function* that assigns a real number value to network g and *v* be the set of all value functions. Network *g* is *efficient* iff $v(g) \geq v(g')$ for all $g' \subset g^N$. The efficiency is in the notion of more sum of utility *v*(*g*) rather than in paretian sense.

Let $Y_i: g\, x\, v \Rightarrow \mathbb{R}$ assigns a *utility function* for each individual *i* in network *g*. *Social utility* Y_s is the sum of all individual utilities Y_i in the network g, $Y_s = \sum_{i:ij \in g} Y_i$. *Average individual utility* is the arithmetic average of social utility with respect to each individual *i* in the network *g*, $Y_a = \frac{Y_s}{N} = \sum_{j:ij \in g} \frac{Y_i}{N}$.

2.2. The Co-Author Model

In the Co-Author Model, two nodes represent individuals and a link between them represents their collaboration. The *fundamental utility function* is

$Y_i(g) = \sum_{j:ij \in g} w_i(n_i, j, n_j) - c(n_i)$. $\sum_{j:ij \in g} w_i(n_i, j, n_j)$ is the rewards derived from the direct links, n_i, and the penalty derived from one collaborators' indirect links, n_j. [4] The cost of maintaining the direct links is $c(n_i)$. Specifically, the *fundamental utility function* of each individual, *i*, in network *g*, as a function of the inverse of the path that link her and her collaborators directly or indirectly, is given by

4 Assuming that resources are limited, the opportunity cost of one's co-author sharing her resources with more people is lesser resources can be given to oneself. A good analogy will be sharing one cake. The more slices given to somebody else, the lesser slice oneself can get.

$Y_i(g) = \sum_{j:ij\in g} [\frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_i n_j}] = 1 + (1 + \frac{1}{n_i}) \sum_{j:ij\in g} \frac{1}{n_j}$ for $n_i > 0$, and $Y_i(g) = 0$ for $n_i = 0$. Since the model derives utility only from the inverse of links, a non-collaborated individual does not gain utility. The focus of this model is in the collaborations. Hence, this condition forces all individuals to be linked. Further, the model implies that (i) if $n_i = n_j$, then $ij \in g$; any individuals with the same number of links will connect. This implies that a component will consist of individuals with the same number of links, making it a clique. In addition, it also implies that (ii) if $n_k \leq Max\{n_j | ij \in g\}$, then *i* wants to link to *k;* an individual will want to link to another individual outside her clique if that outside individual has a lesser number of links than each of the individual's current collaborators' number of links, or an individual will not want to link to another individual who has more number of links than she has.

For an example of (i), consider a network of 4 individuals in the configuration of 2 dyads (figure 1). Each individual has a utility of $1 + (1 + \frac{1}{1})(\frac{1}{1}) = 3$. If 2 individuals deviate and create another link to connect their dyads, the deviating individuals now have a utility of $1 + (1 + \frac{1}{2})(1 + \frac{1}{2}) = 3\frac{1}{4}$, while the non-deviating individuals now have a utility of $1 + (1 + \frac{1}{1})(\frac{1}{2}) = 2$. For an example of (ii), consider a network of 8 individuals with 2 individuals who form a dyad and 6 individuals who form a 6-individual clique (figure 2). Similar to the previous example, the individuals in the dyad have a utility of 3.

The individuals in the clique have a utility of $1+(1+\frac{1}{5})(\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5})=2\frac{1}{5}$. Each individual in the clique will want to form a new link with an individual in the dyad since her new utility will be $1+(1+\frac{1}{6})(\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{2})=2\frac{3}{4}$. However, the individuals in the dyad will not want to form a new link to a member of the clique since her utility will be reduced to the same number of $1+(1+\frac{1}{2})(\frac{1}{1}+\frac{1}{6})=2\frac{3}{4}$.

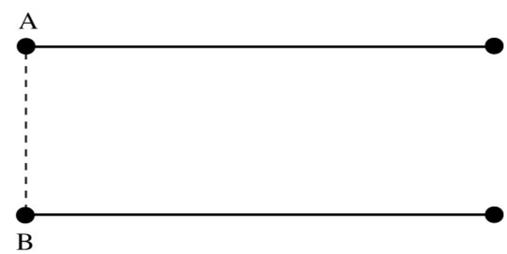

Figure 1. A network of 2 pairs with 2 individuals from each component deviate

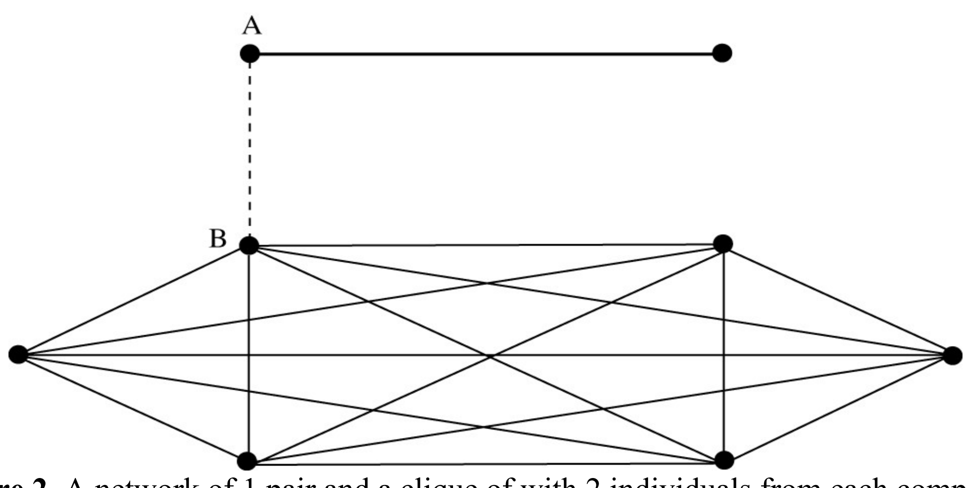

Figure 2. A network of 1 pair and a clique of with 2 individuals from each component deviate

3. ANALYTICAL MODEL AND RESULTS

3.1. Pairwise Stability

Jackson and Wolinsky (1996) also introduce the concept of *Pairwise Stability* for such a network.

Definition 1: A network g is *Pairwise Stable* with respect to v and Y_i if:

(i) for all $ij \in g$, $Y_i(g,v) \geq Y_i(g-ij,v)$ and $Y_j(g,v) \geq Y_j(g-ij,v)$ and

(ii)for all $ik \notin g$, if $Y_i(g,v) < Y_i(g+ik,v)$, then $Y_k(g,v) > Y_k(g+ik,v)$

Condition (i) states that no individual in a Pairwise Stable network is willing to detach from any of her existing links. Condition (ii) implies that a new link between 2 individuals that are not in the same clique can be formed if one individual strictly prefers the new link while the other individual is indifferent. Notice that in this concept, every individual is assumed to be fully aware of the number of links for all individuals.

In their findings, an efficient network for this model is a pairwise network; an efficient network consists of individuals in pairs as dyads. This network configuration yields the highest social utility, Y_s. [5] However, *the Pairwise Stability concept shows that an efficient network is not necessarily stable*. In Jackson & Wolinsky's finding, individuals in an efficient network can make a single bilateral deviation to make a personal gain, Y_i, while decreasing the social utility Y_s of the network. They show that a stable network must consist of cliques. The largest clique is $C_{i=1}$ and the next largest clique is $C_{i=2}$. The relation of the cliques is as described by $C_i > C_{i+1}^2, i=1,2,\ldots,c.$

5 Recall that social utility is the sum of all individual utilities in the network. See section 2.1. for the definition.

3.2. Conjectural Pairwise Stability

To weaken the strong global-observation assumption, I will use concepts developed by McBride to analyze the Co-Author Model under limited observations. In *x-link observation* (McBride 2006b), each individual observes other individuals within *x-links* in her network such that a path of length x-steps exists between them. It implies that each individual only possibly observes other individuals that are in the same component as her regardless of the steps. For *Conjectural Pairwise Stability,* McBride (2006a) combines *Conjectural Equilibrium* (Battigalli, Gilli, and Morinari 1992) with Pairwise Stability. Conjectural Equilibrium is when each individual chooses a best response based on her belief if it is not contradicted by her observation although the belief does not necessarily match the reality.

For this concept, another notation needs to be introduced. First, let Π_i be *i*'s *beliefs,* $\Pi_i \in \Delta(G \, x \, V)$, a probability distribution of possible states of network *(g,v)*. Second, Let $m_i\colon \; G \, x \, V \Rightarrow M_i$ be *i*'s *signal function* such that each state of the network gives a *signal* m_i in *message space* M_i.

Definition 2: A network is *Conjectural Pairwise Stable* with respect to *v*, Y_i, m_i, and Π_i if:

(i) for all $ij \in g$, $Y_i(g,v) \geq \Sigma_{(g',v') \in G x V} \Pi_i(g',v') \, Y_i(g'-ij,v')$ and

$Y_j(g,v) \geq \Sigma_{(g',v') \in G x V} \Pi_j(g',v') \, Y_j(g'-ij,v')$ and

(ii) for all $ik \notin g$, if $Y_i(g,v) < \Sigma_{(g',v') \in G x V} \Pi_i(g',v') \, Y_i(g'+ik,v')$ then

$Y_k(g,v) > \Sigma_{(g',v') \in G x V} \Pi_k(g',v') \, Y_k(g'+ik,v')$ and

(iii) for all i, $\Pi_i(g', v') > 0$ for any $(g', v') \in G X V$ implies

$m_i(g',v') = m_i(g,v)$.

Condition (i) states that each individual will not detach from any of the existing links in the network given her beliefs. Condition (ii) implies that, given their beliefs, a new link between 2 individuals that are not in the same component can be formed if one individual strictly prefers the new link while the other individual is indifferent. Condition (iii) states that the belief system will sustain as long as each individual's belief system is not contradictory to her observation.

Preposition 1: Suppose *x-link observation* and fix $x < \infty$. Consider *g* with cliques $C_1, C_2, \ldots, C_c$ and $C_1 \geq C_2 \geq \ldots \geq C_c$. If $C_1 < \sqrt{N - C_1}$, then there exists a profile of beliefs $\{\Pi_i\}_{i \in g}$ such that the network *g* is *Conjectural Pairwise Stable*.

Proof: I will construct a profile of belief for *i*, show that conditions (i), (ii), and (iii) of definition 2 are satisfied given those beliefs, then explain it why it will be true for all *i.*

Consider a condition for the largest clique C_1, such that $C_1 < \sqrt{N - C_1}$. There exists beliefs of $\Pi_i, i \in g$ such that each individual *i* in network *g* assigns a probability $\Pi_i = 1, i \in g$ for each individual *k* not in her clique, $\forall k \notin C_i$, to belong in the largest clique, $\forall k \in C_1$, implying that $C_{\sim i} = C_1$.

First, to show that condition (i) is satisfied, it is necessary to show that no link will be deleted, or $Y_i(g_1) \geq \Sigma_{g_1' \in G} \Pi_i(g_1')\, Y_i(g_1'\text{-}ij), ij \in g_1$. This inequality means that detaching from any existing link will give *i* a lower utility. Recall that, for the Co-Author Model, any *j* such that $ij \in g_1$ has the same number of links as *i* since they belong to the same clique, which is also a component. Hence, each *j* can be treated as an *i* so that they have the same utility function of $Y_i(g_1) = 1 + (1 + \frac{1}{n_i}) \sum_{ij \in g_1} \frac{1}{n_j}$. For any *i* to remove a link that connects her to another group member *j* in the clique, the new utility will become $Y_i(g_1 - ij) = 1 + (1 + \frac{1}{n_i - 1}) \sum_{ij \in g_1 - ij} \frac{1}{n_j}$. Hence, $Y_i(g_1) > Y_i(g_1 - ij), ij \in g_1$ since $1 + (1 + \frac{1}{n_i - 1}) \sum_{ij \in g_1} \frac{1}{n_j} > 1 + (1 + \frac{1}{n_i}) \sum_{ij \in g_1 - ij} \frac{1}{n_j}$. Therefore, in this model, no *i* is willing to remove a link from such a clique since removing a link will result in a lower utility for her.

Second, to show that condition (ii) is satisfied, it is necessary to show that no new link will be added. Let *i*, $i \in C_1$, has beliefs that assigns probability of 1 $\Pi_i = 1, i \in C_1$ to g_1, where g_1 is a network that consists only of cliques C_1 and $C_{\sim 1}$ with $C_1 < \sqrt{N - C_1}$. Hence, $C_1 < \sqrt{C_{\sim i}}$, or $C_1 < C_{\sim i}$. In g_1, let *i* belongs to C_1. Hence, each individual *k* who is not connected to *i,* which means that they are not in C_1, belongs to $C_{\sim 1}$. Individual *i,* $i \in C_1$, is not willing to make a new link to *k,* $k \in C_{\sim 1}$ in g_1 since $C_1 < C_{\sim i}$. For individual *i*, adding a new link to *k* , whom she believes to have more links than herself, will result in a worse utility than not adding the link, or $Y_i(g_1) \geq \Sigma_{g_1' \in G} \Pi_i(g_1')\, Y_i(g_1' + ik),\ ik \notin g_1$. Since $C_c \leq C_2 \leq ... \leq C_1$, it implies that $C_c \leq C_2 \leq ... \leq C_1 < C_{\sim i}$. Hence, the beliefs of each individual ~*i* that belongs in any clique $i \in C_c, C_{c-1}, ..., C_3, C_2$ in network *g* are the same with each individual *i* who is in the largest clique $i \in C_1$, such that

$\Pi_{\sim i}, \sim i \in C_c, C_{c\text{-}1}, ..., C_3, C_2 = \Pi_i, i \in C_1$. Therefore, no new links will be added.

To show that condition (iii) in definition 2 holds, it is necessary to show that each $i, i \in g_1$ does not receive a signal that contradicts her beliefs. Given that g_1 consists only of C_1 and $C_{\sim 1}$, the total number of individuals in the population *N* in network g_1 must be equal to the sum of the number of the individuals in C_1 and $C_{\sim 1}$, or $N = C_1 + C_{\sim 1}$. However, because for the largest clique must be larger than the square of the next clique and so on (see section 3.1), $C_i > C^2_{i+1}, i = 1,2, ..., c$, it must be true that $C_{\sim 1} \geq C^2_1$. Then it follows that $N > C^2_1 + C_1$. Also recall that each *i* is only capable of monitoring only her clique and is not capable of monitoring other cliques given finite x-link observation $(x < \infty)$. Hence, no observation of each $i, i \in g_1$ contradict the beliefs since $N > C^2_1 + C_1 > C^2_2 + C_2 > ... > C^2_c + C_c$ for any x-link observation with $x < \infty$, meaning that each $i, i \in g_1$ believes that each individual *k* not in her clique belongs to a larger clique than her own, $\forall\ j \in C_{\sim i},\ j \in g_1$.

Further, by similar logic, the aforementioned conditions apply for each individual *j* in the same clique with *i* since the conditions for *i* in a clique will extend to $\forall\ j \in C_1,\ ij \in g_1$. Hence, $\forall\ j \in C_1,\ ij \in g$ will have the same properties in the Co-

Author Model. Extending the logic further, these conditions apply not only to each *i* in C_1 of g_1, but also to each *i* in C_1 of any *g*. As long as $C_1 < \sqrt{N - C_1}$, there could exists beliefs of Π_i such that each *i* in network *g* assigns a probability $\Pi_i = 1, i \in g$ for other individuals *j* not in her clique C_i to belong in the largest clique C_1, or $\forall j \in C_1, j \in g$, since each individual's observation is limited to her own clique by x-link observation. Extending the similar logic, the conditions also apply for $\forall i \notin C_1$ for any *g* because C_1 is the largest clique of $C_i, i = 1,2,\ldots,c$. Recall that it is necessary that $C_1 > C_2^2$, $C_2 > C_3^2$, ..., $C_{c-1} > C_c^2$. Hence, it is also true that $C_1 \geq C_2 \geq \ldots \geq C_c$. Also recall that $N > C_1^2 + C_1$. From the conditions above, then it is also true that $N > C_i^2 + C_i, i = 2,3,\ldots,c$. Therefore, the proof also extends to $\forall i \notin C_1$ in any *g* and there exists a profile of beliefs $\{\Pi_i\}_{i \in g}$ that is not contradicted by observations. □

The proof above shows that the observational limitation causes the Co-Author network to have a different stability property than in a full observation situation. Thus, information control could affect the profile of beliefs and cause the network to have different properties. Any configuration of the Co-Author Model whose cliques have a property of $N > C_i^2 + C_i, c = 2,3,\ldots,c$ is conjectural pairwise stable.

Next, recall Jackson and Wolinksy's findings that the efficient network of the Co-Author Model, consisting of dyads consisting only 2 individuals working together (pairwise), cannot be stable. However, an efficient network can also be a stable network under limited observation if there are enough individuals in the network.

Corollary 1: Suppose *x-link observation* and fix $x<\infty$. If $N>6$, then for any efficient network *g*, there exists a profile of beliefs $\{\Pi_i\}_{i\in g}$ that sustains *g* as *Conjectural Pairwise Stable*.

Proof: I will construct a profile of belief for *i*, show that conditions (i), (ii), and (iii) of definition 2 are satisfied given those beliefs, then explain why it will be true for all *i*.

Consider $N>6$ and each *i* belongs to a dyad. There exist beliefs of $\Pi_i, i\in g$ such that each *i* in network *g* assigns $\Pi_i = 1, i\in g$ for each individual *k* not in her clique, C_i, to belong in a larger clique $\forall\, k\in C_{\sim i}$.

First, I will show that condition (i) in definition 2 is true, that no links will be deleted. The smallest possible number of individuals in a clique is in a dyad, $C_c=2$. If such clique is also the largest clique in the network, then it is true that $C_1=C_2=...=C_c=2$, making it a pairwise (dyad) network. Suppose that *i* is an individual in C_1, the largest clique. Then, *i* will not want to detach from her only existing link because her utility, Y_i, will be reduced to zero, which is strictly worse than any utility she possibly has by having any number of links to another individuals.

Second, I will show that the condition (ii) in definition 2 is true, that no new links will be added. By proposition 1, *i* will not want to add a new link if $C_1<\sqrt{N-C_1}$, or $N>C_1^2+C_1$. So, it must follow that if $N>2^2+2$, or $N>6$, then *i* is not willing to add a new link.

Third, each individual's beliefs will not be contradicted by her observations since it is limited only to her own clique by x-link observation.

By similar logic, the conditions above also apply for $\forall j \in C_1, ij \in g$ in the same clique since $i = j$ in a dyad. Moreover, the conditions also apply for $\forall i \notin C_1$ since all cliques are identical that $C_1 = C_2 = \ldots = C_c = 2$. Therefore, there exists there a profile of beliefs $\{\Pi_i\}_{i \in g}$ that is not contradicted by observations when $N > 6$. □

Similar to the conditions in proposition 1, the conditions above show that a pairwise network *g* can be Conjectural Pairwise Stable when $N > 6$. Recall that Jackson and Wolinsky (1996) describe a pairwise network as an efficient network[6]. The probability for the beliefs for all individuals in the network is the same: $\Pi_i = 1, i \in g$ in which each individual believes that every other individual not in her clique belongs to a clique where the potential collaborator has more links than herself such that adding a new link will yield to a worse utility for her. Therefore, an efficient network *g* can also be stable under Conjectural Pairwise Stability conditions, which means that the efficient network of the Co-Author Model can be stable under observational limitations.

When $N \leq 6$, the network will not be stable since each individual will have the incentive to make a new link. In this situation, the only stable configuration is when every individual is a member of the only clique in the network.

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4. SIMULATION MODEL AND RESULTS

The analytical results show that the efficient network can also be stable under

6 See the definition of network's efficiency in section 2.1.

Conjectural Pairwise Stability. However, the analytical results do not prove that such condition will be the mode of the Co-Author Model's network configuration. To analyze how such network behaves, a simulation is conducted with 2 varying parameters, the initial probability *P(i)* that represents the possible initial configurations of the network, and the number of individual *N* in the network. Our interests are to see if the properties of the Co-Author Model change as these 2 parameters vary.

4.1. Simulation Model with Variations in *N* and *P(i)*

A random number generator assigns links according to initial probability *P(i)*={0.0, 1.0}with increment of 0.1 at state S_0 to generate a complete network g^N.[7] Initial probability is *P(i)*=0.0 when no individuals are connected, whereas *P(i)*=1.0 is when network g^N consists only of one clique at S_0. The utility function *Yi* for each individual is assigned according to definition in section 2.1. Next, each individual considers a deviation. For all $ij \in g$, if

$Y_i(g, v) \geq \Sigma_{(g', v') \in G x V} \Pi_i(g', v')\ Y_i(g' - ij, v')$, then *i* will delete her link with *j*. For all $ik \notin g$, if $Y_i(g, v) < \Sigma_{(g', v') \in G x V} \Pi_i(g', v')\ Y_i(g' + ik, v')$ then *i* will add link to *k*. Note that both of the second conditions, which define mutual response, in (i) and (ii) of Conjectural Pairwise Stability do not apply in this model due to the sequential process of the interactions that maintain the non-global x-link observation. However, this model allows a change in the profile of beliefs $\{\Pi_i\}_{i \in g}$ in condition (iii) by

7 P(i)={0.0, 1.0} covers all initial probability of the state of connectivity for the model. Increment of 0.1 is chosen since it exhibits differences in the results due to the change in initial probability without showing overt details.

allowing each individual to initiate a new link when the sum of her links $\Sigma L_{ij} > \frac{N}{2}$ [8].

Yet, this initiated link will be severed by the recipient individual sequentially if it yields a lower utility for the recipient. This network formation process is sequentially iterated until an equilibrium state, S_e, is reached, that is $S_t = S_{t-1}$. In addition to a variation in *P(i)*, the number of individuals in the network *N* is also varied. Hence, there are two varying parameters: *P(i)*={0.0,1.0} and *N*={8, 20000}[9]. From these iterations, the subsequent values of $Y_s, Y_a, L_e, and\ D_{g^N}$ are derived accordingly.

To further describe and analyze the behaviors of the model, I use Newman's finding (in press) of the *tipping-point* of information diffusion, where there is a transition in network from no diffusion of information to diffusion of information. An independent probability *r* assigns the likelihood for each individual *i* in network *g* to communicate information to other individuals $\sim i, \sim i \in g$. When $r > \frac{\Sigma L_{ij}}{\Sigma L_{ij}^2 - \Sigma L_{ij}}$, the diffusion of information occurs. This analysis is important to see if the observational limitation affects the information diffusion in the Co-Author Model. In addition, Newman also defined the attraction factor for a link to attract another link. It is defined by the number of *additional links*, *m*, that each L_{ij} will attract. The number of *m* is calculated from

8 When $\Sigma L_{ij} > \frac{N}{2}$, the individual can assign probability $\Pi_i = 1, i \in g$ for her beliefs that every individual not in her clique belongs to a clique smaller than her own component. See section 3.2. about the profile of beliefs $\{\Pi_i\}_{i \in g}$ in the equilibrium of Conjectural Pairwise Stability.

9 Conjectural Pairwise Stability require even number of N>6 for its stability condition (see corollary 1 in section 3.2.). Hence, N starts from 8 in this model. FORTRAN 90 seems to have an upper limit to handle only to less than 50,000 number of individuals. However, I ran the simulation only to N = 20,000 since it has shown a stable result.

$2\text{m}\,(m+1)=\frac{(L_{ij}/N)(L_{ij}/N+1)(L_{ij}/N+2)}{(L_{ij}/N)^3}$. This analysis is important to confirm that the Co-Author Model is a homogeneous model with no preferential attachment, which means that the larger components do not have the cumulative advantage to attract more links.

4.2 Simulation Results

Figure 3 shows different layers of social utility, *Ys*, as *N* increases. It clearly shows that *Ys* is highest at all equilibrium states S_e regardless of the value of N when *P(i)* =0.0 at S_0. The corresponding equilibrium states S_e (for each value of *N*) are in form of a pairwise network, which is also the efficient state. For connected networks, that occurs when $0.0<P(i)\leq 1.0$ at S_0, the higher initial probabilities $0.5<P(i)\leq 0.8$, yield to higher *Ys* than the lower initial probability, $0.0<P(i)<0.5$. However, there is a slight decrease in the value of all *Ys* at $0.8<P(i)\leq 1.0$ Hence, the result shows that *P(i)* consistently affects S_e across the values of *N.* Further, figure 4 gives the global picture of average individual utilities, *Ya,* as a function of number of individual,*N,* while figure 5 specifically shows that the average individual utilities, *Ya*, stabilize at $N\geq 500$, except when *P(i)*=1.0. Figure 6 reveals that the decrease of average individual utilities, *Ya,* for *P(i)*=1.0 fits a power-law at $N\geq 100$. The 3-section power-law fitting has elbows at *N*=1000 and *N*=4000.

Figure 7 shows the plot of *P(i)* versus A_{g^N} for *N*={8, 20000}. A further observation in figure 8, which is a more exploded view of figure 7, shows that the values of A_{g^N} across *P(i)* converge to lesser deviations at $N\geq 200$. Specifically, figure 9, a more exploded view of figure 7 and 8, shows that the values of A_{g^N} across *P(i)* become relatively uniform at $N \geq 2000$. There is a dichotomy with $A_{g^N} \approx 0.50, 0.0 \leq P(i) < 0.5$ and $A_{g^N} \approx 0.39, 0.5 < P(i) \leq 1.0$. The deviations of A_{g^N} corresponds to the volatility of the values of $r\, and\, m$ when $N < 200$ as shown by figure10 and 11. When $N \geq 200$, m=0.37, while $r_c < 0.0001$.

Moreover, figure 12 and 13 show a different perspective on the dichotomy. The cutline of $A_{g^N} \approx 0.50, 0.0 \leq P(i) < 0.5$ and $A_{g^N} \approx 0.39, 0.5 < P(i) \leq 1.0$ is clearly divisible and stable at $N \geq 200$. Using only visual observation, it seems that the values of A_{g^N} across *P(i)* follow constant lines as *N* increases. However, for $P(i) < 0.5$, the best fits are power-law functions with $N \Rightarrow \infty$ yields to an asymptote of $A_{g^N} \approx 0.50$. On the other hand, for $P(i) > 0.5$, neither the power-law function nor the constant line do not yield to good fits although power-law functions still fit better than the constant lines. When $N \Rightarrow \infty$, the power-law functions yield to an asymptote of $A_{g^N} \approx 0.39$. In addition, it is also shown in figure 13 that a network in Conjectural Pairwise Stability equilibrium state, as represented by P*(i)* =0.0, yields to $A_{g^N} = 0.50$.

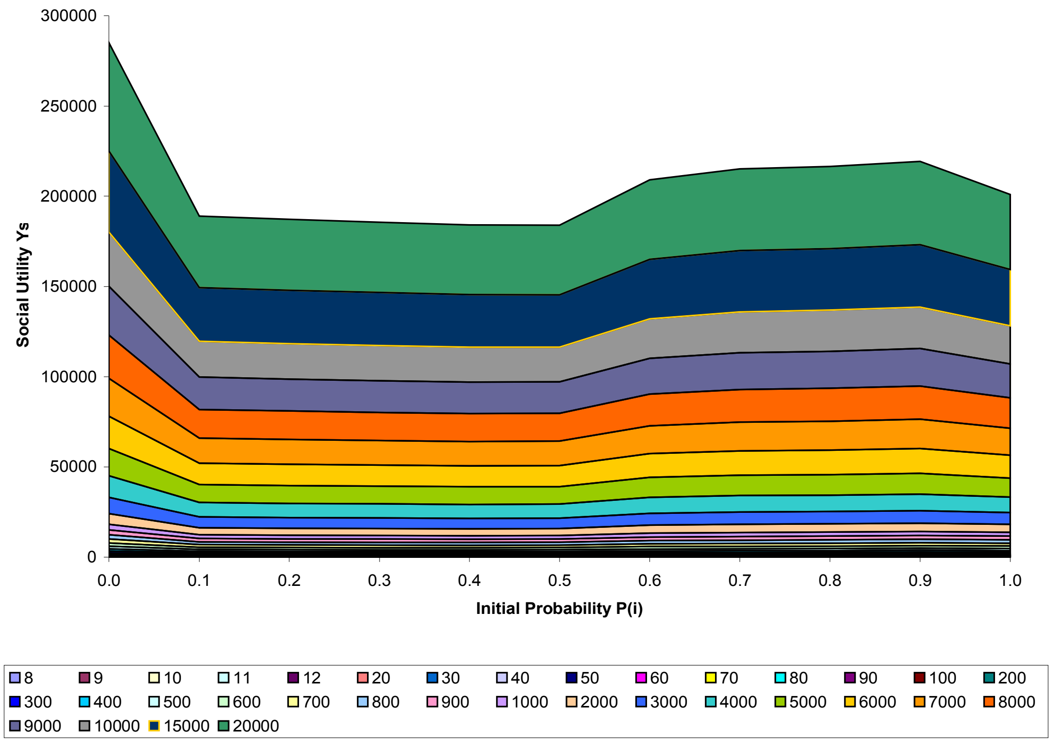

Figure 3. Social utility versus initial probability

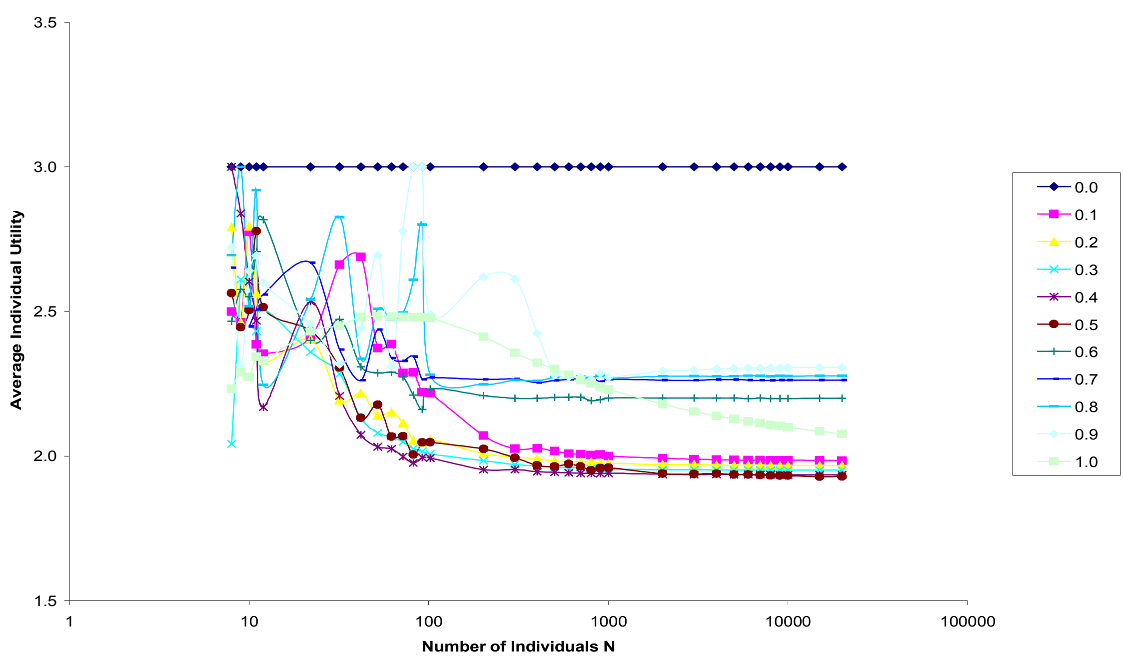

Figure 4. Average individual utility versus initial probability (global)

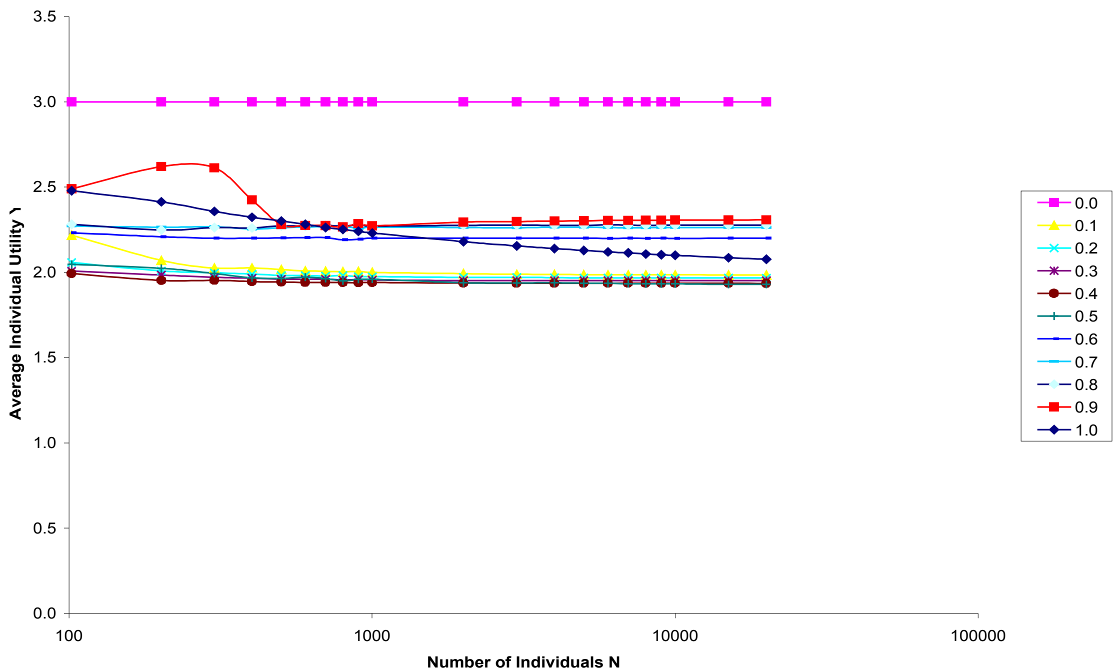

Figure 5. Average individual utility versus initial probability (N≥500)

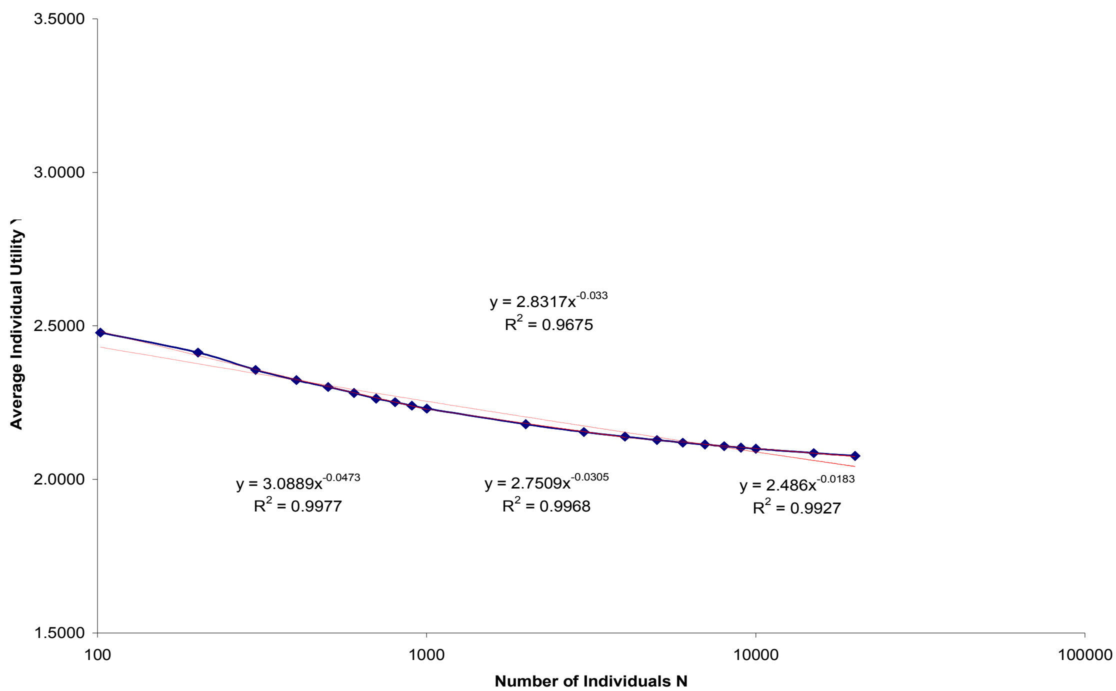

Figure 6. Average individual utility versus number of individuals

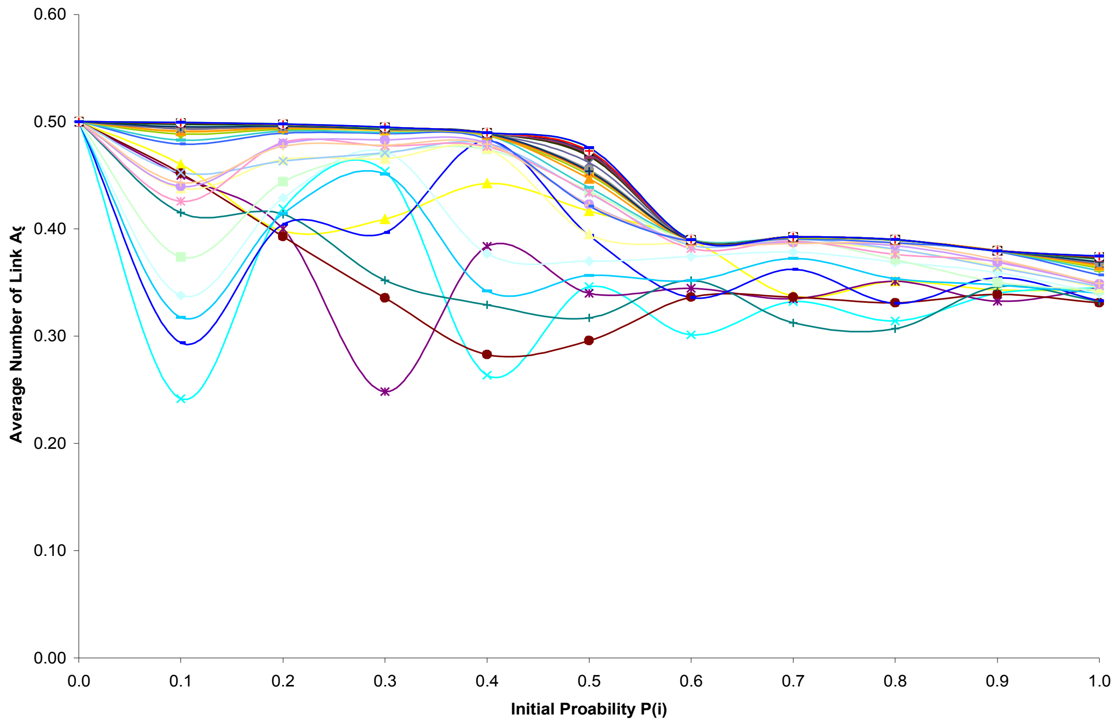

Figure 7. Average number of link versus initial probability (global)

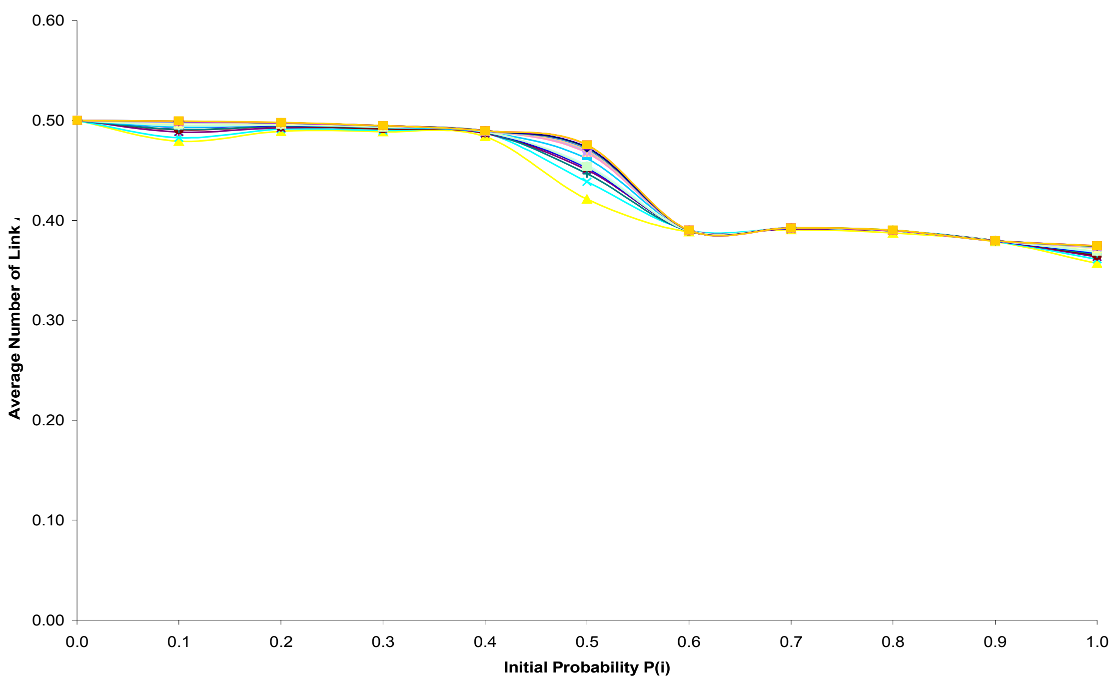

Figure 8. Average number of link versus initial probability (N≥200)

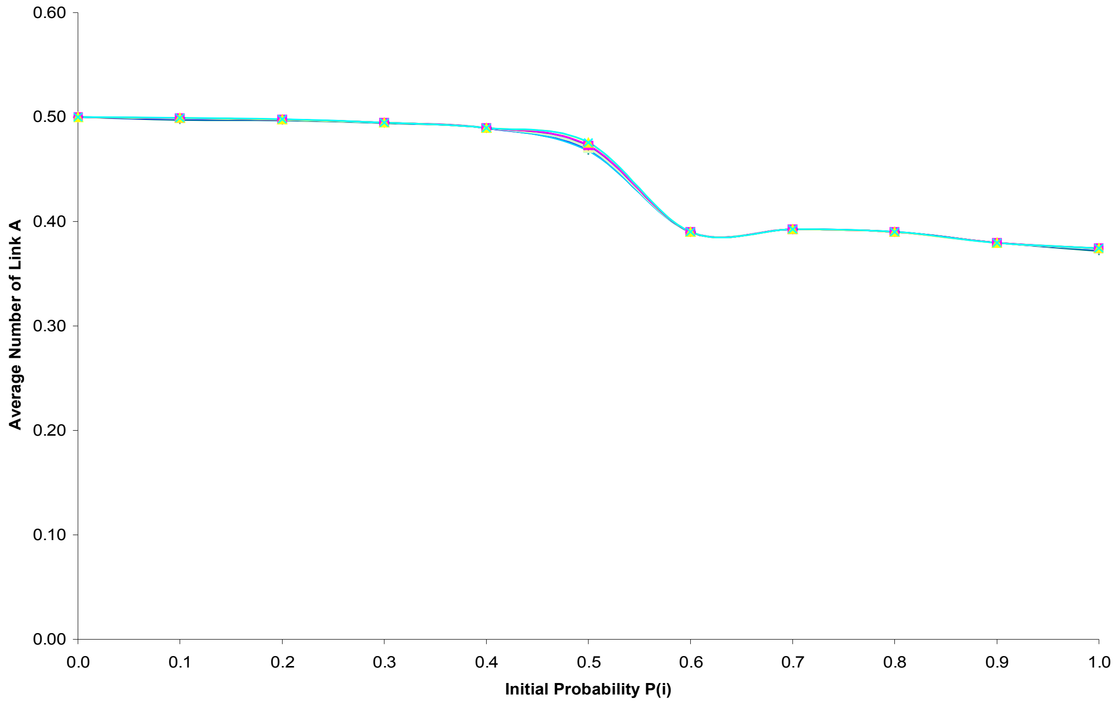

Figure 9. Average number of link versus initial probability (N≥2000)

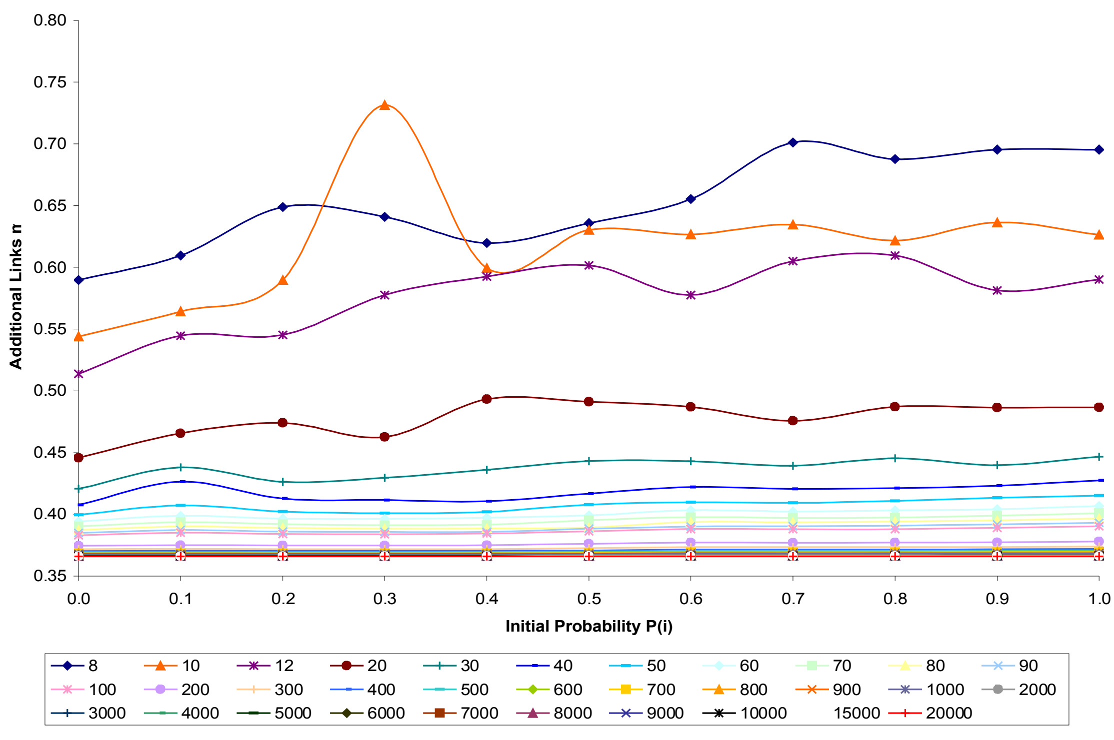

Figure 10. Additional links versus initial probability

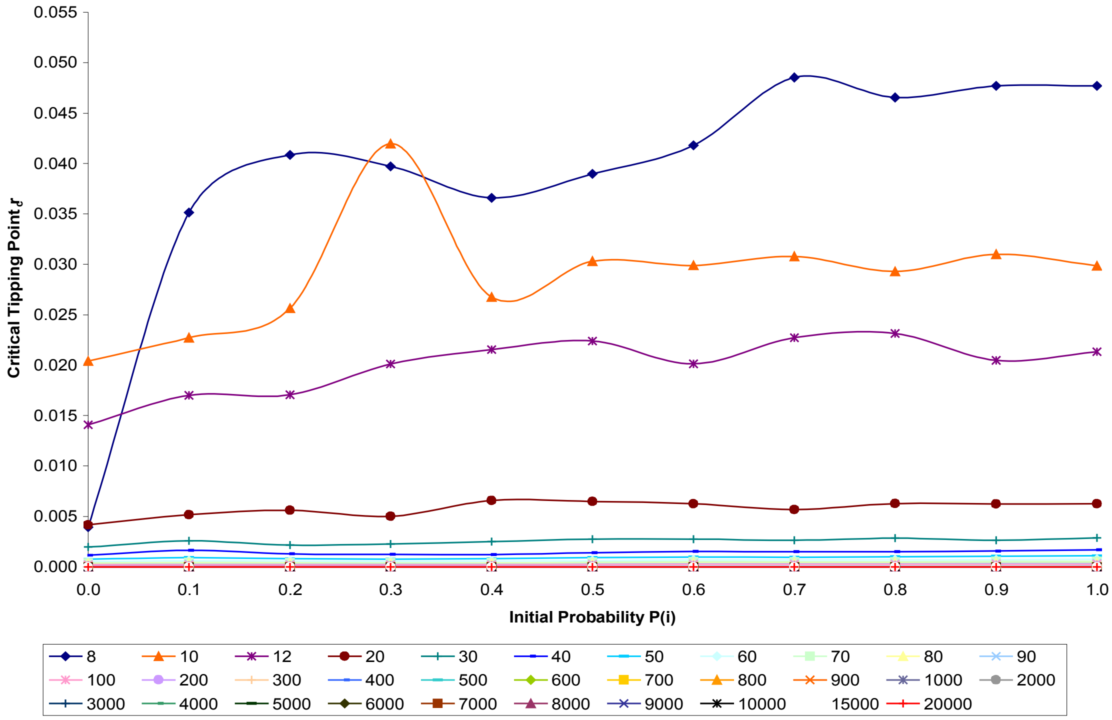

Figure 11. Critical tipping point versus initial probability

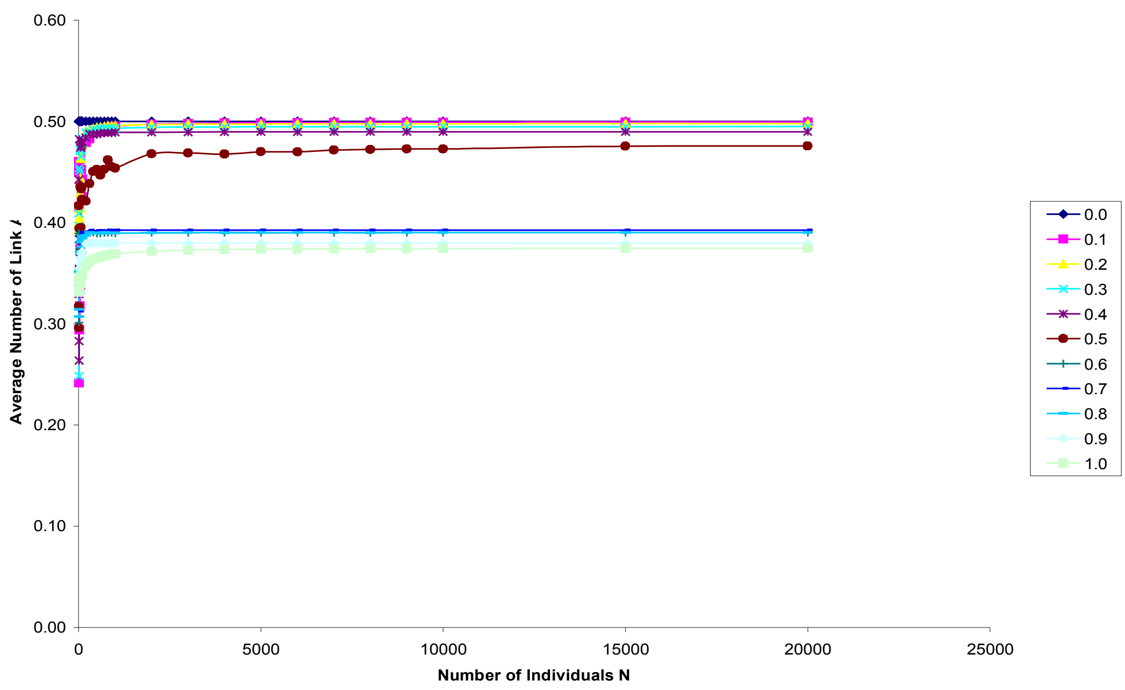

Figure 12. Average number of link versus number of individuals

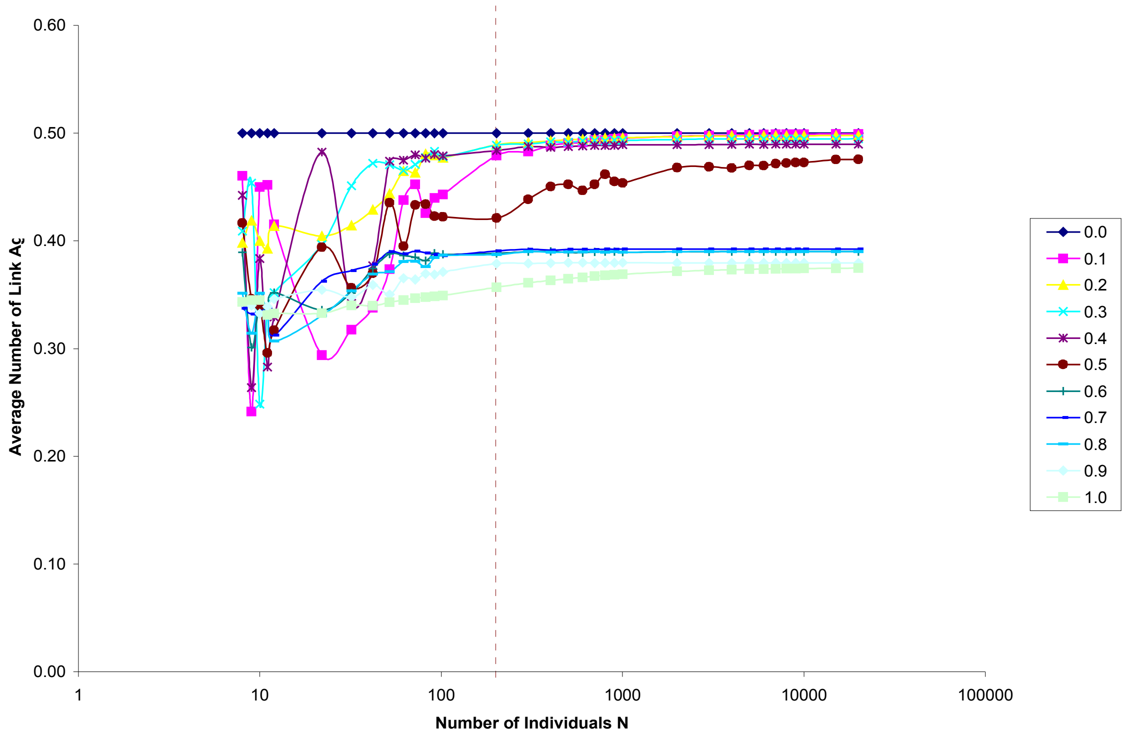

Figure 13. Average number of link versus number of individuals

5. DISCUSSIONS

While Jackson and Wolinsky found that the Co-Author Model network cannot be efficient and stable under full observation, I found the reverse under limited observation: the model can be both efficient and stable. This finding is in agreement with what McBride (2006a) found in a different network setting that some of the networks might be efficient under Conjectural Pairwise Stability conditions. Moreover, my results for the Co-Author Model show exactly under what conditions the network will be both stable and efficient. Specifically, the stable and efficient conditions could hold when the population is more than six individuals and none of the all individuals' beliefs are contradicted by their observations. In these findings, each individual in the network could have beliefs that a deviation will not bring her a personal benefit, resulting in stability of the network. However, this condition might not necessarily be the mode of equilibrium in an applied situation.

The initial probability at *P(i)*=0.0 yields to the highest social utilities for various number of individuals in the network (figure 3). This can be attributed to the certainty in the probability of each individual's beliefs when $\Pi_i = 1, i \in g$ that prevent her from initiating a new link if she believes that all other individuals not in her clique belong to a clique where the potential collaborator has more links than herself, so that adding that new link will yield to a worse utility for her. This result confirms the analytical result of Conjectural Pairwise Stability. If the processes were linear, a linear function that describes the variation in social utility across the initial probability should be expected. Yet, for $0.0 < P(i) < 0.5$, the social utilities are lower than for $0.5 < P(i) \leq 1.0$, suggesting that there is a dynamic in the network formation processes of the Co-Author Model causing the non-linearity. The initial probability of $0.5 < P(i) \leq 0.8$ is the optimum range that yields to highest social utilities for $P(i) > 0$ regardless of the number of individuals *N* in the network. It means that individuals who have initially had more links, or belong to a large group, will stay in that large group and have a better utility. This result is consistent with what Eaton, Ward, Kumar, and Reingen (1999) found in an empirical study that 84% of authors in a co-authorship network belongs to one large connected network and authors with more links tend to be more productive. Higher productivity consequently yields to a higher utility and makes individuals want to maintain their existing links.[10]

---

10 The fact that a model with homogeneous agents yield to a similar result to an empirical study with

The analysis at the average individual utility level (figure 4&5) further shows the irregularities in the dynamic of the network formation processes, especially for $N < 500$. For $N \geq 500$, except for *P(i)*=1.0, the average individual utility for each initial probability becomes relatively stable. The finding that the average individual utility decreases according to power-law at *P(i)*=1.0 in figure 6 shows the other side of the dynamic of the network formation process of the Co-Author Model. While *P(i)*=0.0 yields to constant average individual utility of 3 for all *N, P(i)*=1.0 yields to power-law decrease as the number of individuals in the network *N* increases. Hence, *P(i)*=0.0 and *P(i)*=1.0 represent 2 special cases of the network formation of the model.

Switching to average number of link analysis (figure 7, 8, and 9), there is another irregularity in the picture of the dynamic of the formation processes at $N < 200$. The average number of link of the network across all initial probabilities becomes more predictable at $N \geq 200$, specifically at $N \geq 2000$. At this level, the average number of link of the network becomes dichotomous, suggesting that a higher initial probability yields to a more connected network than a lower initial probability.

In addition to that, the small number of additional links is *m*={0.37, 0.70} throughout *N*={8, 20000} and *P(i)*={0.0, 1.0} confirm that the Co-Author Model as a decentralized homogeneous model. When $N \geq 200$, then *m*=0.37, meaning that, in average, each existing link, L_{ij}, will only attract an additional 0.37 link. Yet, figure 10 also shows that higher initial probabilities *P(i)* yield to a higher additional link *m*, which means that a more connected individuals also tend to attract more links. This is contradictory to the utility function of the Co-Author Model (see section 2.2) that

implicit heterogeneity among the agents also show the potential robustness for the Co-Author Model.

disfavors new links to individuals with more connections. The difference in the number of additional link *m* between lower and higher initial probabilities *P(i)* can be attributed more to the aversion of individuals in conditions with lower initial probabilities *P(i)* due to their beliefs that adding new links will yield to a worse-off utility than to the possibility that individuals in conditions with higher initial probabilities attract more of additional links *m*.

The finding that the tipping-point of the diffusion of information is $r < 0.05$ throughout *N*={8, 20000} and *P(i)*={0.0, 1.0}, and $r < 0.0001, N \geq 200$ shows that it does not take too much of each individual's initiative to diffuse information in the Co-Author Model (figure 11). This finding is consistent with Newman's (in press) finding that the tipping-point will be smaller as the number of the individuals in the network increases. Also recall that the observation of each individual is limited to her own component. Hence, it is very plausible that the unanimous profile of beliefs $\{\Pi_i\}_{i \in g}$ is attributed to the diffusion of information from the dynamic interactions of individuals in the sequential network formation processes. Therefore, the results show that observational limitation is not necessarily preventing information discussions as long as there are dynamic interactions of the individuals in the network.

The finding that the densities of the network at $0.0 < P(i) < 0.5$ and $N \geq 200$ are approaching the average number of link of the network in the Conjectural Pairwise Stability equilibrium state might suggest that the Co-Author Model will reach equilibrium state in pairwise format (figure 12&13). However, the average individual utility levels, which are asymptotic to the values lower than 2.5, show that such equilibriums states are unattainable since the average individual utility for a pairwise network is 3.0 regardless of the number of individuals in network. At $0.0 < P(i) < 0.5$ and $N \geq 200$, the network densities simply mimic the average number of link of Conjectural Pairwise Stability. However, the efficiency level is worse than the equilibrium states with lower densities whose initial probability are higher at $0.5 < P(i) \leq 1.0$.

6. CONCLUSIONS

These analytical results show the role of observational limitation in creating the profile of beliefs, which affects the stability and efficiency of the Co-Author Model. Although it does not show the mode for equilibrium, the proof of existence of a simultaneously efficient and stable equilibrium in the Co-Author Model diminishes the notion that an informal network cannot be efficient (Cross, Nohria, and Parker 2002). The dichotomous labeling of networks as formal or informal might not explain the behaviors as well as other variables such as information control, structure, and initial conditions.

In his book, Schelling (1978) found that micro-level individual motives can lead to a different macro-level systems behavior. In the Co-Author Model, observational limitation and low initiative at the individual level do not prevent diffusion of information in the network of the Co-Author Model, showing that the less predictable micro-level individual behaviors lead to a more predictable macro-level network behavior, which also confirms to a well-accepted notion that the increment in population diminishes the particular details, making the systems becomes more predictable. However, it also raises a question as to a generally accepted belief that a system's complexity increases as the number of its elements increases. Therefore, those who are interested in the application of the Co-Author Model need to consider the effects of the aforementioned factors on the model.

The simulation findings show that the stability and efficiency of the Co-Author Model are also affected by the initial conditions: number of individuals in the network and the probability of the links in the network in its initial state. Those two parameters affect the dynamic of the network formation processes of the model and yield to various states of equilibrium with different tipping-points of the information diffusion, additional links, and average individual utilities at the individual level and network densities and social utilities at the network level. The findings in the article show that, although the individual utility function favors pairwise format to big components, low critical tipping point for diffusion of information and higher propensity for more connected individuals to attract new links suggest that the propensity of interactions affects information transfer more than the efficiency and stability of the network. While this analysis advances the study of the Co-Author Model by including its formation process, it still does not explain the causes and the significance of the points at which the network behavior changes. There is also one interesting question left unexplored: the learning process itself, which deals with the adaptability of individual's strategy.

APPENDIX

```
!This is the code for
!"Dynamics of Network Formation Processes of the Co-Author Model"
!by Laurent Tambayong
!The code runs on FORTRAN 90

integer*1 edge(20000,20000)
```

```
integer iterations
real cedge(20000), u(20000),soc, sedge(20000),prob,sqr,sqr2,sqr3

iterations=50                          !number of steps
open(2,file=output')                   !saving all output files

do n=8,20000,2                         !change the increment of the number of individuals as desired
do prob=0.0,1,0.1                      !initial probability

!Assigning adjacency matrix
sqr=0
sqr2=0
sqr3=0
do j=1,n
 do i=1,n
  edge(i,j)=0
 enddo
 cedge(j)=0
 sedge(j)=0
enddo
do j=1,n
 do i=1,n
  if (edge(i,j).ne.1) then
   call random_number(z)
   if (z.lt.prob) edge(j,i)=1      !assigning a random relation for authors
   if (edge(j,i).eq.1.and.edge(j,i).ne.edge(i,j)) sedge(j)=sedge(j)+1
   edge(i,j)=edge(j,i)
  endif
 enddo !enddo i
 do i=1,n                              !deleting loop
  if (i.eq.j) then
   edge(j,i)=0
   edge(i,j)=0
  endif
 enddo
 do i=1,n                              !deleting assymetries
  if (edge(j,i).ne.edge(i,j)) then
  edge(j,i)=0
  edge(i,j)=0
  endif
 enddo
 do i=1,n
  if(edge(j,i).eq.1.and.sedge(i).ne.0)then
  cedge(j)=(1/sedge(i))+cedge(j)
  endif
 enddo
 if(sedge(j).eq.0) then
 u(j)=0
 else
 u(j)=1+(1+1/sedge(j))*cedge(j)   !array for initial utility function
 endif
 soc=soc+u(j)
 sqr=sqr+(sedge(j)**2)
 sqr2=sqr2+sedge(j)
```

```
enddo  !enddo j
sqr3=0.5*(sqr2**2)/sqr



do k=1,iterations
soc=0
sqr=0
sqr2=0
do j=1,n
 !reducing links?
 do i=1,n
 ut=0
  if(edge(j,i).ne.0.and.sedge(i).gt.1.and.sedge(j).gt.1) then
   cedge(j)=cedge(j)-(1/sedge(i))
   ut=1+(1+1/(sedge(j)-1))*cedge(j)
   if(ut.gt.u(j)) then
    u(j)=ut
    sedge(j)=sedge(j)-1
    else
    cedge(j)=cedge(j)+(1/sedge(i))
   endif
  endif
  ut=0
  !adding links?
  if(edge(j,i).eq.0.and.sedge(j).eq.0)then
   u(j)=3
   sedge(j)=sedge(j)+1
   edge(j,i)=1
  endif

  if(edge(j,i).eq.0.and.sedge(j).gt.n/2.and.sedge(j).lt.n-1)then
   cedge(j)=cedge(j)+(1/sedge(i))
   ut=1+(1+1/(sedge(j)+1))*cedge(j)
   if(ut.gt.u(j)) then
    u(j)=ut
    sedge(j)=sedge(j)+1
    else
    cedge(j)=cedge(j)-(1/sedge(i))
   endif
  endif
 enddo
 soc=soc+u(j)
 sqr=sqr+(sedge(j)**2)
 sqr2=sqr2+sedge(j)
enddo
enddo
sqr3=0.5*(sqr2**2)/sqr
write(2,*) n,prob,soc,soc/n,sqr3,sqr3/(n )
enddo
enddo
end
```

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