Independence Properties Vis-à-vis Several Utility Representations

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Abstract

A detailed theoretical analysis is presented of what five utility representations – subjective expected utility (SEU), rank-dependent (cumulative or Choquet) utility (RDU), gains decomposition utility (GDU), rank weighted utility (RWU), and a configural-weight model (TAX) that we show to be equivalent to RWU – say about a series of independence properties, many of which were suggested by M. H. Birnbaum and his coauthors. The goal is to clarify what implications to draw about the descriptive aspects of the representations from data concerning these properties. The upshot is a sharp rejection of SEU and RDU and no clear choice between GDU and TAX, but a list of 8 properties is given that should receive more attention to discriminate between the latter two models.

Key Words: independence properties, rank-dependent utility, rank weighted utility, TAX utility, weighted utility

Economics Classification: D46, D81

This article focuses on discovering exactly what five classes of utility representations predict about a series of independence properties found in the literature, many of which were suggested by M. H. Birnbaum and his coauthors. We begin by defining the general mathematical forms of these representations. This is followed by a systematic list of independence properties, each satisfied by subjective expected utility (SEU). Next we derive what the various models predict about such properties. We compile these results in three Tables (Sections 3.4, 4.3, and 5.5) and in two derivative tables where the prediction of at least one model is unambiguously positive or negative and report how the models fare vis-à-vis the existing data. These tables may be viewed as a refinement and expansion of Table 1 of Birnbaum, Patton, and Lott (1999, p. 53). The main discovery is that two of the models have similar properties, and seem to account for much of the available data. Nonetheless, their predictions differ on 8 properties. Not a great deal of data on these distinguishing properties yet exists, and we urge experimenters to focus on them.

1 Form of Models

Suppose that X is a set of "pure" consequences and $\mathcal{E}$ is an algebra of chance events underlying a particular gamble. In general, we have several different algebras for gambles based on different chance experiments. A typical *first-order gamble of size* n has the form

$$g_{\overrightarrow{\mathbf{C}}_n} = (x_1, C_1; ...; x_i, C_i; ...; x_n, C_n),$$

where the $x_i \in X$, and $(C_1, ..., C_i, ..., C_n)$ is an ordered[1] partition, denoted $\overrightarrow{\mathbf{C}}_n$, of the non-null event $C(n) = \bigcup_{i=1}^{n} C_i$, which is "universal" for the gamble. The size of the gamble is determined by the number of consequence-event pairs (x_i, C_i), each of which is called a *branch* of $g_{\overrightarrow{\mathbf{C}}_n}$. Some properties are stated below only for the case where there is a finitely additive probability measure Pr such that $\Pr(C_i) = p_i$, $C_i \in \mathcal{E}$, and $\Pr(C(n)) = 1$. In that case, the gamble is written

$$g_{\overrightarrow{\mathbf{p}}_n} = (x_1, p_1; ...; x_i, p_i; ...; x_n, p_n) \qquad \left(\sum_{i=1}^{n} p_i = 1\right).$$

If one or more of the x_i is replaced by a first-order gamble, the result is a *second-order (compound) gamble*. For simplicity, we often refer to the set of pure consequences and first- and second-order gambles simply as 'gambles.'

We assume that the preferences between gambles satisfy a weak order $\succsim$, i.e., $\succsim$ is transitive and connected, and that $\succ$ and $\sim$ are defined in the usual fashion.

The set X is endowed with a unique element, called "no change from the status quo," denoted e. Any $x \succsim e$ or any gamble g with all of its consequences $x_i \succsim e$ is called a *gain*, and any $x \precsim e$ or g with all of its consequences $x_i \precsim e$ is called a *loss.* Cases of gambles with mixed gains and losses are very important, but in this article we restrict our attention to gambles of all gains. The theory for losses is completely parallel to that for gains.

A gamble for which $x_1 \succsim x_2 \succsim ... \succsim x_n$ is said to be in *ranked* form. In this paper, the ranked form holds unless we state otherwise. Written in this way, the cumulative subevents $C(j) = \bigcup_{i=1}^{j} C_i$, $1 \le j \le n$, sometimes play a role. We assume that people are indifferent to permutations of the indices.

Suppose g is a gamble. We assume that there exists $CE(g) \in X$ such that

$$CE(g) \sim g, \tag{1}$$

which is called a *certainty equivalent* of g.

Co-monotonic consequence monotonicity is satisfied iff for every $i \in \{1, 2, .., n\}$, when $x_i, x'_i \in X$ have the same rank position among the other consequences, then

$$x'_i \succsim x_i \;\; \text{iff} \;\; (x_1, C_1, ..., x'_i, C_i, ..., x_n, C_n) \succsim (x_1, C_1, ..., x_i, C_i, ..., x_n, C_n) \tag{2}$$

We assume co-monotonic consequence monotonicity holds.

Idempotence is satisfied iff for every $y \in X$ and every ordered partition $\overrightarrow{\mathbf{C}}_n := (C_1, \cdots, C_i, \cdots, C_n)$,

$$(y, C_1; \cdots; y, C_i; \cdots; y, C_n) \sim y. \tag{3}$$

Although we do not assume idempotence in general, which is a bit unusual, we will mostly examine models for which it does hold. More attention needs to be paid to the non-idempotent cases which allow the chance experiment underlying the gamble to have value, per se, positive or negative. This is a way to approach the issue of the utility of gambling (see Luce and Marley, 2000; Meginniss, 1976).

A function U from the domain of pure consequences and gambles of gains to the non-negative real numbers is called a *utility function* if it is order preserving and maps the status quo into 0, i.e.,

$$g \succsim h \quad \text{iff} \quad U(g) \ge U(h), \tag{4}$$

$$U(e) \quad = \quad 0. \tag{5}$$

The existence of certainty equivalents, (1), plus the assumption of consequence co-monotonicity, justifies using the same notation U over both gambles and pure consequences. Given that $\succsim$ has the numerical representation of (4), then $\succsim$ must be a weak order. Whenever a function U occurs in the remainder of the paper it is to be interpreted as such a utility function.

Our focus will be, first, on several well known forms of utility representations. All but one is well understood from an axiomatic perspective (Luce & Marley, 2004). Mostly, we will work with the representation rather than with the underlying axioms although several proofs are simpler if we use a qualitative defining property and so we cite these properties. Next, we explore what these representations predict about several independence properties that have been discussed in the literature, with our focus on differential predictions. Finally, we compare these differential predictions with existing data.

1.1 Rank weighted utility

The most general representation we shall use is the following:

Definition 1 *Let S_i, $i \in \{1, 2, .., n\}$, be mappings from ordered event partitions $\overrightarrow{\mathbf{C}}_n := (C_1, ..., C_i, ..., C_n)$ to the non-negative real numbers. The* ***rank weighted utility (RWU)***[2] *of a ranked gamble is of the form*

$$U(\cdots; x_i, C_i; \cdots) = \sum_{i=1}^{n} U(x_i) S_i(\overrightarrow{\mathbf{C}}_n) \quad (S_i(\overrightarrow{\mathbf{C}}_n) \geq 0,\ i \in \{1, 2, \cdots, n\}). \quad (6)$$

Note that assuming that the weights $S_i(\overrightarrow{\mathbf{C}}_n)$ are non-negative is equivalent to assuming co-monotonic consequence monotonicity.

The other representations we examine are special cases of this form which we describe in terms of specializations of the weights $S_i(\overrightarrow{\mathbf{C}}_n)$.

Idempotence holds in RWU iff $\sum_{i=1}^{n} S_i(\overrightarrow{\mathbf{C}}_n) = 1$.

A RWU representation is said to be *simple* iff there exist functions $S_{C(n)} : \mathcal{E} \to [0, 1]$ such that, for each i, $S_i(\overrightarrow{\mathbf{C}}_n) = S_{C(n)}(C_i)$. This means that the weight $S_i(\overrightarrow{\mathbf{C}}_n)$ is independent of all other events C_j, $j \neq i$, of the partition. A simple and idempotent utility representation with $S_{C(n)}$ finitely additive is called *subjective expected utility* (SEU).

An example of a simple representation is

$$S_{C(n)}(C_i) = \frac{W(C_i)}{W(C(n))}.$$

When W is finitely additive, this representation is idempotent and so is an example of an SEU representation as defined above. Note that these weights satisfy the choice property of Luce (1959); see (16) below.

Observe that by its definition, simple utility (and therefore SEU) does not depend at all on the ordered partition, only on C_i, and so not on the ranking of consequences. The theory is thus simpler.

1.2 Rank-dependent utility

Definition 2 *A RWU representation for gains, (6), is a* ***rank-dependent utility (RDU)*** *representation iff there is a function $W_{C(n)}$ from events into $[0, 1]$ with $W_{C(n)}(\varnothing) = 0,\ W_{C(n)}(C(n)) = 1$, such that the weights are of the form*

$$S_i(\overrightarrow{\mathbf{C}}_n) = W_{C(n)}(C(i)) - W_{C(n)}(C(i-1)) \quad (i \in \{1, ..., n\}), \quad (7)$$

where $C(0) = 0$.

Note that the above representation satisfies idempotence.

Substituting (7) into the weighted utility expression, (6), we see that RDU can be written in the equivalent form:

$$U(\cdots;x_i,C_i;\cdots)=\sum_{i=1}^{n}U(x_i)\left[W_{C(n)}(C(i))-W_{C(n)}(C(i-1))\right]. \qquad (8)$$

Because the structure is idempotent, it can also be written in the form:

$$U(\cdots;x_i,C_i;\cdots)=\sum_{i=1}^{n-1}[U(x_i)-U(x_{i+1})]W_{C(n)}(C(i))+U(x_n). \qquad (9)$$

The class of RDU models for gains alone and for losses alone agrees with the general form of cumulative prospect theory (CPT, Tversky & Kahneman, 1992), and includes such special cases as subjective expected utility (SEU), where $W_{C(n)}$ is finitely additive and so

$$\begin{aligned} &W_{C(n)}(C(i))-W_{C(n)}(C(i-1)) \\ &\quad = W_{C(n)}(C_i\cup C(i-1))-W_{C(n)}(C(i-1)) \\ &\quad = W_{C(n)}(C_i), \end{aligned}$$

and expected utility (EU), where the events are replaced by probabilities and so $W_{C(n)}(p)=p$.

For mixed gains and losses, which we do not deal with here, rank- and sign-dependent utility, RSDU, and CPT may differ materially (Luce, 2000, Chs. 6 and 7).

A major, defining, necessary property of RDU is *coalescing* which says that if two events of a gamble have the same consequence, then that gamble is indifferent to the one in which the union of the two events is treated as a single event with the same consequence (see Luce & Marley, 2004). Formally, for all ordered partitions and ordered consequences $x_1 \succsim \cdots \succsim x_n \succsim e$, $n>2$, with $x_{k+1}=x_k=x$, $k<n$:

$$\begin{aligned} &(x_1,C_1;\cdots;x,C_k;x,C_{k+1}\cdots;x_n,C_n) \\ &\quad \sim (x_1,C_1;\cdots;x,C_k\cup C_{k+1};\cdots;x_n,C_n) \quad (k=1,\cdots,n-1). \end{aligned} \qquad (10)$$

In terms of the "bottom line" the two sides are identical. Of course, that does not automatically mean that they are perceived as indifferent.

It is easy to see that a simple and idempotent utility representation is an SEU representation iff it satisfies coalescing (Luce & Marley, 2004).

1.3 Gains decomposition utility

Definition 3 *Within the domain of second-order (compound) gambles of gains, a* ***(lower) gains-decomposition utility*** *(GDU) representation holds iff there is a family of binary weights* $W_{C(i)}, i=1,...,n$, *with* $C(n)$ *the universal event, such that RWU, (6), holds, with the RWU weights*

$$S_i(\overrightarrow{\mathbf{C}}_n)=W_i(\overrightarrow{\mathbf{C}}_n)-W_{i-1}(\overrightarrow{\mathbf{C}}_n), \qquad (11)$$

where

$$W_i(\overrightarrow{\mathbf{C}}_n) = \begin{cases} 0, & i = 0 \\ \prod_{j=i}^{n-1} W_{C(j+1)}\left(C(j)\right), & 1 \le i \le n-1 \\ 1, & i = n \end{cases} . \tag{12}$$

By (11) and (12),

$$S_i(\overrightarrow{\mathbf{C}}_n) = \begin{cases} \prod_{j=1}^{n-1} W_{C(j+1)}\left(C(j)\right), & i = 1 \\ \left(1 - W_{C(i)}\left(C(i-1)\right)\right) \prod_{j=i}^{n-1} W_{C(j+1)}\left(C(j)\right), & 2 \le i \le n-1 \\ 1 - W_{C(n)}\left(C(n-1)\right) & i = n \end{cases} . \tag{13}$$

In particular, in the binary case,

$$S_1(\overrightarrow{\mathbf{C}}_2) = W_{C(2)}(C_1),\ S_2(\overrightarrow{\mathbf{C}}_2) = 1 - W_{C(2)}(C_1),$$

which agrees with binary RDU. Although RDU and GDU agree for binary gambles, they do not in general for $n > 2$.

For $n > 2$, these forms may not look terribly natural, but they correspond to a surprisingly simple behavioral property which we describe because we use it in a few of the proofs. For a gamble $g_{\overrightarrow{\mathbf{C}}_n}$, $n > 2$, with $x_1 \succsim \cdots \succsim x_n \succsim e$, consider the following sub-gamble

$$g_{\overrightarrow{\mathbf{C}}_{n-1}} := (x_1, C_1; \cdots; x_{n-1}, C_{n-1}). \tag{14}$$

Note that $g_{\overrightarrow{\mathbf{C}}_{n-1}}$ is based on the sub-experiment with the universal event $C(n-1)$ but run independently of the $\overrightarrow{\mathbf{C}}_n$ experiment. Within the domain of second-order (compound) gambles of gains, *lower gains decomposition* states that

$$g_{\overrightarrow{\mathbf{C}}_n} \sim \left(g_{\overrightarrow{\mathbf{C}}_{n-1}}, C(n-1); x_n, C_n\right), \tag{15}$$

where $\left(g_{\overrightarrow{\mathbf{C}}_{n-1}}, C(n-1); x_n, C_n\right)$ is a second-order (compound) binary gamble. Note that in terms of the bottom line, the two sides of (15) are the same. Gains decomposition is a special case of a property that is frequently invoked as "rational" in the form of "reducing compound lotteries to the corresponding first-order one."

Although Meginniss (1976) implicitly used the lower gains decomposition axiom in his interesting approach to non-idempotent, un-ordered gambles, the concept was first explicitly introduced by Liu (1995) in an attempt to axiomatize RDU in the case of known probabilities. Luce (2000, p. 187) generalized it (without the adjective "lower") to events and used that together with (16) below to arrive at RDU. However, he did not work out fully the implications found in Luce and Marley (2004), which improved on Marley and Luce (2001), that any two of the following properties implies the third: (i) RDU, (ii) GDU, and (iii) for events with $C \subseteq D \subseteq E$, the *choice property* (Luce, 1959) holds:

$$W_{\mathbf{E}}(C) = W_{\mathbf{D}}(C)W_{\mathbf{E}}(D). \tag{16}$$

The property of lower gains decomposition, (15), suggests also looking at *upper gains decomposition* defined by the decomposition

$$g_{\overrightarrow{\mathbf{C}}_n} \sim (x_1, C_1; g_{\overrightarrow{\mathbf{C}}_n\backslash C_1}, C(n)\backslash C_1),$$

where

$$g_{\overrightarrow{\mathbf{C}}_n\backslash C_1} := (x_2, C_2; ...; x_n, C_n).$$

We use sometimes the unmodified term 'gains decomposition' to refer to the lower case. As needed, we use the modifier *upper* to refer to the upper case, that is, *upper gains decomposition* and *upper GDU.*

Indeed, for any i, we may define $g_{\overrightarrow{\mathbf{C}}_n\backslash C_i}$ in the obvious way and define gains decomposition relative to branch (x_i, C_i) as

$$g_{\overrightarrow{\mathbf{C}}_n} \sim \begin{cases} (x_i, C_i; g_{\overrightarrow{\mathbf{C}}_n}\backslash_{C_i}, C(n)\backslash C_i, & x_i \succsim g_{\overrightarrow{\mathbf{C}}_n}\backslash_{C_i} \\ (g_{\overrightarrow{\mathbf{C}}_n}\backslash_{C_i}, C(n)\backslash C_i; x_i, C_i), & x_i \prec g_{\overrightarrow{\mathbf{C}}_n}\backslash_{C_i} \end{cases}.$$

Luce and Marley (2004) show that if idempotent RWU is satisfied, and gains decomposition holds for all three branches of a gamble of size 3, then the choice property, (16), is satisfied, and the weights are finitely additive.

RDU ($\neq$ SEU) violates gains decomposition; GDU predicts it. We are not aware of any direct experiments that have tested gains decomposition in isolation.

1.4 Configural weighted utility

M. H. Birnbaum, with various collaborators, in a series of papers, some of which are cited explicitly later, has explored a class of representations called *configural weighted utility*[3]. In contrast to the above utility models that have been axiomatized in terms of behavioral properties (see Luce & Marley, 2004), the configural weighted representations are stated only at the representational level. No defining properties are known; however, see Proposition 6 below. Rather, Birnbaum and his collaborators have shown that certain special cases do or do not exhibit certain behavioral properties, and they have reported experiments comparing how various models fare relative to them. The majority of this work has focussed on two particular classes of configural weighted representations, called RAM and TAX - the reasons for these names are given later. Both RAM and TAX can explain various choice and judgment data, but, as discussed below, TAX is overall the better model.

1.4.1 RAM utility

According to Birnbaum (2003), the RAM model has the following form:

Definition 4 *A RWU representation for gains, (6), is a* ***RAM*** *representation iff there is a function W from events into $[0,1]$ with $W(\varnothing) = 0$ and positive constants $a_n(i)$, $i = 1, ..., n$, such that the weights are of the form*

$$S_i(\overrightarrow{\mathbf{C}}_n) = \frac{a_n(i)W(C_i)}{\sum_{j=1}^{n} a_n(j)W(C_j)}. \tag{17}$$

The name RAM arises as the acronym for *rank affected multiplicative* (Birnbaum, Coffey, Mellers, and Weiss, 1992, Birnbaum and MacIntosh, 1996). The models of Karmarkar (1979), Viscusi (1989), and Lattimore, Baker, and Witte (1992) are of this form. For choice studies, Birnbaum focuses mostly on the case where the weights $a_n(i) = i$. Note that the RAM representation is idempotent, and that for nonnull events C_i,C_j, $i \neq j$,

$$\frac{S_i(\overrightarrow{\mathbf{C}}_n)}{S_j(\overrightarrow{\mathbf{C}}_n)} = \frac{a_n(i)W(C_i)}{a_n(j)W(C_j)}.$$

Since this constraint does not hold for a general RWU representation, there are RWU representations that are not RAM representations.

1.4.2 TAX utility

As stated at the beginning of this section, the configural weighted representation that has best survived empirical test is called TAX (for reasons stated below) and we will focus on it here. According to Birnbaum and Navarrete (1998), it has the following form[4]:

Definition 5 *Let U be a utility function over ranked gambles and pure consequences, T a function from events into the non-negative real numbers, and $\omega_{i,j}(\overrightarrow{\mathbf{C}}_n)$ mappings from ordered event partitions to real numbers. Then, TAX is the following representation over gambles in ranked order:*

$$U(g_{\overrightarrow{\mathbf{C}}_n}) = \frac{\sum_{i=1}^{n} U(x_i)T(C_i) + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} [U(x_i) - U(x_j)]\,\omega_{i,j}(\overrightarrow{\mathbf{C}}_n),}{T(\overrightarrow{\mathbf{C}}_n)}, \tag{18}$$

where $T(\overrightarrow{\mathbf{C}}_n) := \sum_{i=1}^{n} T(C_i)$.

The name TAX arises because Birnbaum describes the term on the right as imposing a tax from (resp., to) lower ranked consequences to (resp., from) higher ranked ones depending on whether the relevant weight is positive (resp., negative). Birnbaum usually imposes a particular form on the $\omega_{i,j}$, which we discuss later.

Because all of the $U(x_i)$ terms appear linearly, it is obvious that this is a RWU representation.[5] It is less obvious how the $S_i(\overrightarrow{\mathbf{C}}_n)$ of (6) relate to the T and $\omega_{i,j}$ of (18). This is formulated as:

Proposition 6

(i) *Any TAX representation, (18), is an idempotent RWU representation, (6), with*

$$S_i(\overrightarrow{\mathbf{C}}_n) = \frac{T(C_i) + \sum_{j=i+1}^{n+1} \omega_{i,j}(\overrightarrow{\mathbf{C}}_n) - \sum_{j=0}^{i-1} \omega_{j,i}(\overrightarrow{\mathbf{C}}_n)}{T(\overrightarrow{\mathbf{C}}_n)}, \tag{19}$$

where

$$\omega_{0,i}(\overrightarrow{\mathbf{C}}_n) \quad : \quad = 0, \tag{20}$$

$$\omega_{i,n+1}(\overrightarrow{\mathbf{C}}_n) \quad : \quad = 0, \tag{21}$$

(ii) *Any idempotent RWU representation, (6), can be put (in many ways) in the form of a TAX representation, (18). One such has*

$$T(C_i) > 0 \tag{22}$$

and

$$\omega_{i,j}(\overrightarrow{\mathbf{C}}_n) = \begin{cases} T(\overrightarrow{\mathbf{C}}_n) \sum_{k=1}^{i} S_k(\overrightarrow{\mathbf{C}}_n) - \sum_{k=1}^{i} T(C_k), & i = 1, ..., n-1, \\ & j = i+1 \\ 0, & i = 0 \text{ or } j = n+1, \\ & \text{or } j \neq i+1 \end{cases} . \tag{23}$$

All proofs are in the appendix.

Observe that to satisfy co-monotonic consequence monotonicity, we must have $S_i(\overrightarrow{\mathbf{C}}_n) \geq \mathbf{0}$, and that places some (unknown) constraints on the $\omega_{i,j}$ in the general form (19).

Because of this result, we will state how properties fare for RWU and then, in some cases, provide necessary and/or sufficient conditions for the TAX formulation to predict the property.

The second part of this proposition means that RAM is a special case of TAX.

2 Issues of Design and Analysis

Before we turn to the analysis of, and data on, several specific independence properties, it is appropriate to discuss some of the problems that have to be confronted.

2.1 Choices or certainty equivalents in testing

The most obvious way to test properties that involve comparing two gambles that have a common branch, which is true in many of the independence properties that we shall explore, is to present the two gambles to the respondent and request a choice. Although obvious, in fact it is fraught with difficulties. Respondents seem to engage in various forms of "editing" when comparing the gambles directly. One cannot be sure what it is that they do, but as we shall see, they seem to be doing something different from the kind of combining that the several RWU models describe.

Perhaps the situation least subject to editing is to ask each respondent to give a certainty equivalent for each gamble, and for the experimenter to construct the preference order $\succsim$ from these certainty equivalents. Doing so is trivial if just one certainty equivalent is established for each gamble; however, across repetitions, the estimates vary. In that case, various options are available that have been widely explored; we do not go into all of them here. Earlier, we defined a certainty equivalent to be the certain consequence indifferent to the gamble, (1). There are two standard ways to obtain such a CE: by asking the respondent to report his or her judged CE or by choice procedures. These methods do not yield the same estimates (Luce, 2000, p. 44).

Among the choice procedures used, two important ones are PEST and Quick Indifference (QI). They are both up-down methods that differ primarily in the size of the step used in adjusting the money alternative that is used in the immediately subsequent presentation of this gamble among the several others simultaneously under study. In PEST the changes are carried out in steps that are independent of how far the procedure is from the true CE. In QI, the respondents also indicate their strength of preference for the choice made and an algorithm is used to determine the size of the step to use in generating the next presentation of a money amount. The idea is to get into the neighborhood of the CE as rapidly as possible.

In a study of consequence co-monotonicity, von Winterfeldt, Chung, Luce, & Cho (1997) compared CEs determined by PEST and QI. Using medians, they concluded that QI was inferior to PEST in confirming consequence co-monotonicity. However, Ho, Regenwetter, Niederée, and Heyer[6] reanalyzed the von Winterfeldt et al. (1997) data using quantile methods and reached exactly the opposite conclusion: Under QI, consequence monotonicity was strongly sustained and under PEST it was not. This may relate to the conjecture of Cho, Luce, and Truong (2002) that PEST, as implemented in these experiments, is subject to premature terminations which sometimes yield very poor estimates of the certainty equivalent.[7]

The following danger in using CEs, especially judged ones, has not been deeply explored. Suppose that we are dealing with money gambles of gains. If the choices are made by comparing the money amounts with the gamble and choosing, there is no problem. But suppose that the respondent simply subtracts the amount of money, y, either presented or judged, from all of the consequences, i.e., considers the gamble

$$(x_1 - y, C_1; ...; x_i - y, C_i; ...x_n - y, C_n),$$

and compares it with 0. If, as is usually the case, $x_1 > y > x_n$, this has moved us from the domain of all gains to that of mixed gains and losses. The form of the RDU models is not usually the same in the mixed domain (Luce, 2000, Chs. 6 and 7; Tversky & Kahneman, 1992). Thus, were this to happen, the data would be quite misleading in evaluating models designed just for gains. This concern is particularly pertinent to judged buying and selling prices.

Most of the studies discussed below used either direct choices or the respondents reported buying and/or selling judged prices of gambles. Such judged forms of CE are by no means the same as choice ones (Luce, 2000, pp. 39-44) and so it is desirable to have data on the latter.

2.2 Data analyses

In most experimental studies of choice between gambles, there are usually some respondents whose data satisfy a property under study and others whose data do not. Thus, the data of some individuals may be compatible with a relatively constraining model, such as SEU, whereas those of others may be compatible with a less constraining model, such as RDU or GDU, and still others may not be compatible with any RWU model. Much of the available data has been reported at the aggregate, or group, level, and has been interpreted as showing that a particular condition does not hold if a 'significant' portion of the aggregated data does not satisfy that condition, where 'significant' is defined in different ways in different papers. The condition is said to hold if it does not fail in the above sense. Sometimes authors follow up such an analysis with an inspection of the pattern of results for individual subjects, as is the case with Birnbaum and McIntosh (1996) and Birnbaum and Navarette (1998). When such analysis of individual participants is performed, a condition that does not hold at the group level is, perhaps not surprisingly, usually found not to hold for some, or even a majority, of individuals. Nonetheless, often there are still numerous individuals who satisfy the condition under study. In this article, we report conclusions regarding particular conditions as they are stated by the original authors, which usually amounts to stating that a condition does not hold if it does not hold for a (significant) proportion of the respondents, even though it does hold for the remainder.

For various reasons, such studies may be either understating or overstating the case for a particular condition holding. For instance, a study may underestimate the case for a particular condition holding as a result of aggregating "noisy" individual data. And a study may overestimate the case for a particular condition holding as a result of a failure to select gambles in an appropriate region of the gamble space for the selected participants. For instance, testing branch indepedence (defined later) requires the study of a set of common consequences z, z' that covers a broad enough range of values that a failure can be detected. Figures 1, 2 of Birnbaum and Beeghley (1997) show such failures of branch independence that could have been missed in a study with different gambles, or, possibly, with different participants.

2.3 Goals of the article

The balance of this article concerns somewhat complex independence properties, not used in standard axiomatizations of the representations, that do and do not follow from each of the representations presented above. Our approach, insofar as we know how, is to state necessary and sufficient conditions for the property to hold under RWU. Then in a corollary we establish, in terms of the forms for the weights for each of the special cases of RWU, to what this necessary and sufficient condition reduces. This permits us to compare the models to each other and to data. Although in existing articles Birnbaum has stated all of the properties for the cases where the probabilities of events are known, whenever possible we replace the probabilities by general events. Also he has usually worked with special cases of TAX and sometimes even with special forms for U over money and weights over probabilities. In contrast, we examine both the general TAX = idempotent RWU model and special cases of it that force the property in question. To some extent this strategy breaks down with some of the conditions in Section 5 because we have only sufficient conditions for the property to hold.

3 Branch Cancellation and Independence

The first of these properties, branch independence, holds that if two gambles have a common consequence for a particular event, i.e., they have a common branch, then the value of that consequence should have no effect on the preference order induced by the branches (Cohen and Jaffray, 1988, Birnbaum and McIntosh, 1996). We will see that this is a very strong condition. For example, we show below that, given RWU, a simple representation holds on gambles of size $n = 3$ if and only if a property called branch cancellation holds for all branches – branch cancellation is equivalent to branch independence provided the comparatively weak (34), below, is satisfied.

3.1 Branch types and locations

We now define various terms that help in stating conditions and results compactly. We say that a branch (z, E) of a 3-component gamble is in *position* i, $i = 1, 2, 3$, if z is in rank position i. A branch (z, E) that occurs in each of two gambles based on the same "universal" event is of *type* (i, j) if z is in position i in the first gamble and in position j in the second gamble. For convenience, we refer occasionally to the given pair of gambles as of type (i, j). The definitions of branch cancellation and independence given below considers pairs of such gambles with the restriction that $x' \succ x \succ y \succ y' \succ e$, in which case there are

5 distinct possibilities for the type of z, namely $(1,1)$, $(1,2)$, $(2,2)$, $(3,2)$, $(3,3)$. The three symmetric cases, $(1,1)$, $(2,2)$, and $(3,3)$, are called *co-ranked* with, respectively, upper (U), intermediate (I), and lower (L) positions.

The gamble pair is called *restricted* when $C' = C, D' = D$; otherwise, it is called *unrestricted.*

Finally, for two pairs of pairs of such gambles, say

$$(x, C; y, D; z, E), (x', C'; y', D'; z, E),$$

and

$$(x, C; y, D; z', E), (x', C'; y', D'; z', E),$$

we say that the branches (z, E) and (z', E) have a *common location* (i, j) iff each is of type (i, j). In the theoretical development below, we restrict attention mainly to the 5 common locations, which we denote by $(1,1)^2$, $(1,2)^2$, $(2,2)^2$, $(3,2)^2$, $(3,3)^2$. Consistent with our prior use of the term, the cases $(1,1)^2$, $(2,2)^2$, $(3,3)^2$ are called *co-ranked.* This limitation to common locations is adequate for our purposes because these 5 cases, with RWU, give the simple utility representation (see Proposition 8 below), which, as its name suggests, is the simplest RWU representation, plus they are sufficient to distinguish between SEU, RDU, GDU, and TAX, which distinctions are the focus of the present article.

Nonetheless, note that the above 5 cases are a small subset of the possible combinations of types for the z, z' pair. Because there are 5 distinct possibilities for the type of each of z, z', then, noting the symmetry of the roles of z, z', there are 15 different combinations of types for z, z' in the above 4 gambles (see Table 1). These possibilities can be identified by symbols such as $[(i, j), (i', j')]$ which, as proposed above, may in the case of common locations be written $(i, j)^2$. And for all possible cases of branch independence to hold, the same preference pattern must hold in the above two pairs of gambles for all 15 possibilities. Numerous of these non-common locations appear in several experimental articles including Birnbaum (2003), Birnbaum and McIntosh (1996), Birnbaum and Navarrete (1998); in particular, see Tables 2 and 3 of Birnbaum and McIntosh (1996). The one non-common location that we do consider is $[(1,1), (3,3)]$.

Insert Table. 1 about here

3.2 Branch cancellation

In working with branch cancellation, it is essential that the notation keep track of the ordering imposed on the partitions $\{C, D, E\}$ and $\{C', D', E\}$ where $C \cup D \cup E = C' \cup D' \cup E$. To this end, define:

$$\overrightarrow{\mathbf{C}}_3^{(i)} = \left\{ \begin{array}{c} (E, C, D) \\ (C, E, D) \\ (C, D, E) \end{array} \right\} \quad \text{for} \quad \left\{ \begin{array}{c} i = 1 \\ i = 2 \\ i = 3 \end{array} \right\},$$

and similarly $\overrightarrow{\mathbf{C}}_3'^{(i)}$ for the partition $\{C', D', E\}$.

Definition 7 *Suppose that a RWU representation, (6), holds. For all events C, C', D, D', E where E is nonnull, $\{C, D, E\}$ and $\{C', D', E\}$ are both partitions of the same event, and a pair of gambles $(x, C; y, D; z, E)$, $(x', C'; y', D'; z, E)$ with $x' \succ x \succ y \succ y' \succ e$, $z \succ e$ of type (i, j) but otherwise arbitrary,* ***branch cancellation (BC) of type*** (i, j) *holds iff*

$$S_i(\overrightarrow{\mathbf{C}}_3^{(i)}) = S_j(\overrightarrow{\mathbf{C}}_3'^{(j)}). \tag{24}$$

Restricted BC of type (i, j) *holds iff the above definition holds with the restriction that $C' = C, D' = D$.*
Co-ranked BC of type (i, i) *holds iff BC of type (i, i) holds.*

Note that branch cancellation is automatically satisfied in any co-ranked, restricted case. Also, it is trivially satisfied if E is null. We nonetheless include the condition E nonnull to keep the definition of BC and the later definition of branch indepedence, Def. 10, of the same form.

To be completely precise, the above should be called 3 branch cancellation (3-BC), with a natural generalization of the definition to gambles of size $n > 3$. We use the briefer terminology here, and in the later presentation of branch independence, since we only study these conditions for gambles of size 3.

Proposition 8 *Suppose that a RWU representation holds for gambles of size 3. It is a simple utility representation iff BC holds for all five common locations.*

Proposition 9 *Suppose that a RWU representation holds. Then:*

(i) *SEU predicts all forms of BC.*

(ii) *RDU*[8] *predicts that:*

- **(a)** *In the unrestricted case: Upper BC and lower BC are satisfied, but BC of types $(1, 2)$, $(2, 2)$ (intermediate), and $(3, 2)$ are not satisfied.*
- **(b)** *In the restricted case: co-ranked BC is satisfied, but BC of types $(1, 2)$ and $(3, 2)$ are not satisfied.*

(iii) *GDU predicts that:*

- **(a)** *In the unrestricted case: Upper BC is not satisfied. Intermediate BC is satisfied iff*

$$W_{C\cup E}(C)W_{C\cup D\cup E}(C \cup E) = W_{C\cup D\cup E}(C \cup E) - W_{C\cup D\cup E}(E). \tag{25}$$

Lower BC is satisfied. For the non-co-ranked types, $(1, 2)$ BC is not satisfied, and $(3, 2)$ BC is satisfied iff (25) holds.

- **(b)** *In the restricted case: co-ranked BC is satisfied. For the non-co-ranked cases, $(1, 2)$ BC is satisfied iff*

$$W_{C\cup E}(C) + W_{C\cup E}(E) = 1; \tag{26}$$

and $(3, 2)$ BC is satisfied iff

$$1 - W_{C\cup D\cup E}(C \cup D) = [1 - W_{C\cup D\cup E}(C)]W_{C\cup D\cup E}(C \cup E). \tag{27}$$

(iv) *TAX predicts:*

(a) *Restricted co-ranked BC.*

(b) *Non-co-ranked BC of type* (i, j) *iff*

$$\frac{T(E) + \sum_{k=i+1}^{4} \omega_{i,k}(\overrightarrow{\mathbf{C}}_3) - \sum_{k=0}^{i-1} \omega_{k,i}(\overrightarrow{\mathbf{C}}_3)}{T(C) + T(D) + T(E)} = \frac{T(E) + \sum_{k=j+1}^{4} \omega_{j,k}(\overrightarrow{\mathbf{C}}'_3) - \sum_{k=0}^{j-1} \omega_{k,j}(\overrightarrow{\mathbf{C}}'_3)}{T(C') + T(D') + T(E)}. \qquad (28)$$

Several observations:

1. These conditions entail restrictions only on the weights, not on the utilities.

2. Neither (24) nor (28) is a single condition but is dependent on whether or not the pairs of gambles are restricted or not and on the values of $i, j = 1, 2, 3$. These conditions are not necessarily consistent with each other, thus making it very difficult, indeed, to say whether either the RWU or TAX model does or does not predict the property.

3. Under RAM, i.e., RWU = TAX with (17), co-ranked upper, intermediate, and lower BC hold, but BC of types (1,2) and (3,2) do not.

4. Consider the following form for the weights in the left hand side of (28) but stated for general size n, not just $n = 3$. For all $r < s$ (the only terms we need to specify),

$$\omega_{r,s}(\overrightarrow{\mathbf{C}}_n) = \begin{cases} \lambda_{r,s}^{(n)} T(C_i) & \text{if} \quad r = i \text{ or } s = i \\ \text{arbitrary} & \text{otherwise} \end{cases}, \qquad (29)$$

with a parallel form for the weights in the right hand side of (28). It is important to note that, in general, (29) cannot hold for two distinct values i, j, with, say, $i < j$, for then we have, by (29),

$$\lambda_{i,j}^{(n)} T(C_i) = \omega_{i,j}(\overrightarrow{\mathbf{C}}_n) = \lambda_{i,j}^{(n)} T(C_j),$$

and so either $\lambda_{i,j}^{(n)} = 0$ or $T(C_i) = T(C_j)$.

Now consider the case of (28) corresponding to co-ranked upper BC, for which $i = j = 1$. Then using (29) we have

$$\omega_{r,s}(\overrightarrow{\mathbf{C}}_n) = \begin{cases} \lambda_{r,s}^{(n)} T(E) & \text{if} \quad r = 1 \text{ or } s = 1 \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

on both sides of (28).

Substituting the above weight values in (28) gives the condition

$$\frac{\lambda_{1,2}^{(n)} + \lambda_{1,3}^{(n)}}{T(C) + T(D) + T(E)} = \frac{\lambda_{1,2}^{(n)} + \lambda_{1,3}^{(n)}}{T(C') + T(D') + T(E)}$$

iff

$$T(C) + T(D) + T(E) = T(C') + T(D') + T(E)$$

which is true in the restricted case or when $T(C) + T(D) = T(C') + T(D')$, which holds if T is finitely additive because $C \cup D = C' \cup D'$. The other two cases, intermediate and lower BC, are similar.

Birnbaum has considered two special cases for the TAX weights: For all $r < s$ (the only terms we need to specify), there are constants $\lambda_{r,s}^{(n)}$ such that:

Case 1.

$$\omega_{r,s}(\overrightarrow{\mathbf{C}}_n) = \lambda_{r,s}^{(n)} T(C_r). \tag{30}$$

Case 2.

$$\omega_{r,s}(\overrightarrow{\mathbf{C}}_n) = \lambda_{r,s}^{(n)} T(C_s). \tag{31}$$

Note that, when $r = i = 1$, (30) is a special case of (29), and so by the earlier argument upper BC holds for (30) in the restricted case or when T is finitely additive. Similarly, when $s = i = 3$, (31) is a special case of (29), and so by the earlier argument lower BC holds for (31) in the restricted case or when T is finitely additive.

There are other restrictions that might suggest themselves as of potential interest. For instance, one might consider the assumption that the weights $\omega_{i,j}(\overrightarrow{\mathbf{C}}_n)$ are restricted to the form $\lambda_{i,j}^{(n)} \omega(C_i, C_j)$ where the $\lambda_{i,j}^{(n)}$ only depend on the indices, not the events; (29), (30), and (31) are special cases of this form. Later general results are easily restated under such restrictions, and it is clear that they are of relatively limited interest for the various independence conditions studied here. Turning to the $\omega_{i,i+1}$ of (23), one might consider the assumption that they are all positive, or that they are all negative; routine calculations show that such is always possible for a given partition $\overrightarrow{\mathbf{C}}_n$, but not necessarily simultaneously for more than one such partition. In summary, our overall conclusion will be that special cases of TAX have limited relevance to whether or not TAX fits available data.

3.3 Branch independence

Branch cancellation is not directly testable, but it does imply the following property that is indeed testable.

Definition 10 ***Branch independence (BI)*** *of type* $(i, j)^2$ *is defined by: Given consequences* x, x', y, y', z, z' *with* $x' \succ x \succ y \succ y' \succ e$*,* z*,* z' *with common location* (i, j) *in the gambles below but otherwise arbitrary, and all events* C*,* C'*,* D*,* D'*, and non-null* E *where* $\{C, D, E\}$ *and* $\{C', D', E\}$ *are both partitions of the same event,*

$$(x, C; y, D; z, E) \succsim (x', C'; y', D'; z, E) \tag{32}$$

iff

$$(x, C; y, D; z', E) \succsim (x', C'; y', D'; z', E). \qquad (33)$$

Restricted BI of type $(i, j)^2$ *holds iff the above definition holds with the restriction* [9] *that* $C' = C$, $D' = D$.

Co-ranked BI of type $(i, i)^2$ *holds iff BI of type* $(i, i)^2$ *holds. These cases are called upper, intermediate, and lower BI, respectively, iff* $i = 1, 2, 3$.[10] *If this condition is satisfied for all* 3 *types, then we simply say that* ***co-ranked BI*** *holds.*

The term *co-ranked* is introduced to link our concepts to those in the literature – see, for instance, Birnbaum (1997, 1999, p. 31), Birnbaum and Chavez (1997), and Birnbaum and Navarrete (1998).

This definition of branch independence for gambles of size $n = 3$ can be immediately generalized to gambles of size $n > 3$, in which case there is more than one intermediate case.

Proposition 11 *Suppose that a RWU representation holds, that gambles are idempotent, that* $\overrightarrow{\mathbf{C}}_3$ *and* $\overrightarrow{\mathbf{C}}'_3$ *are ordered event partitions of the same event underlying gambles* g_3 *and* g'_3*, and that these gambles are of type* $(i, j)^2$*, with* (i, j) *one of* $(1, 1)$, $(1, 2)$, $(2, 2)$, $(3, 2)$, *or* $(3, 3)$. *Then, the following statements are equivalent:*

(i) *Branch cancellation of type* $(i, j)^2$ *holds.*

(ii) *Branch independence of type* $(i, j)^2$ *holds, and there exist* $x' \succ x \succ y \succ y' \succ e$ *and* z *in position* (i, j) *such that*

$$(x, C; y, D; z, E) \sim (x', C'; y', D'; z, E). \qquad (34)$$

This proposition shows us that the properties formulated in Proposition 9 under the hypothesis of branch cancellation hold equally for branch independence provided only that the comparatively weak (34) is satisfied.

3.4 Data on BI and summary

Wakker, Erev, and Weber (1994), Weber and Kirsner (1997), Birnbaum and McIntosh (1996), Birnbaum and Chavez (1997), and Birnbaum and Navarrete (1998) each ran empirical studies of the BI properties. Wakker et al. (1994), using choices where the cancellations were fairly obvious, concluded that all cases of co-ranked, restricted BI were sustained. Weber and Kirsner (1997) argued that choices invite direct cancellation which may not appear when forms of certainty equivalents are used such as buying or selling prices. Their choice data satisfy co-ranked, restricted BI, and their price data exhibit somewhat more violations, although to our eyes not impressively more. Birnbaum and McIntosh (1996) studied choices between gambles with three equally likely consequences.

The data satisfy co-ranked restricted BI, but, by and large, reject the non-co-ranked restricted cases of BI.

Note that the available studies of BI (Tables 1 and 2) reject all versions of BI except co-ranked restricted BI, with no studies having been carried out of co-ranked unrestricted BI. Both the Weber and Kirsner (1997) and Birnbaum and Macintosh (1996) studies used presentation formats for the gambles that displayed the outcomes in order of magnitude - in descending order from top to bottom in Weber and Kirsner's graphic displays, and in ascending order from left to right in Birnbaum and Macintosh's text displays. Thus, in the case of choices, the common branches in the co-ranked cases may have been more "transparent" than in the non-co-ranked cases, leading to BI being satisfied in the former, but not in the latter (as found by Birnbaum and Macintosh). This suggestion is compatible with Weber and Kirsner's finding that co-ranked restricted BI was somewhat less strongly supported in their judged price condition. Thus we recommend that these conditions be re-run using a certainty equivalent method such as Quick Indifference.

On the assumption that (34) is satisfied, which is not a problematic assumption, then BC and BI are equivalent (Proposition 11). So, although the theoretical results are stated for BC and the data for BI, we proceed as if both concern BI. The pattern of predictions and the experimental results for BI are summarized in Table 2.

Insert Table 2 about here

4 Distribution Independence

4.1 3-Distribution independence

Definition 12 ***3-distribution independence (3-DI)*** *is defined by: For* $x' \succ x \succ y \succ y' \succ e$, $z \succ e$, $z' \succ e$, $p, p' \in (0, 1/2]$,

$$(x, p; y, p; z, 1-2p) \succ (x', p; y', p; z, 1-2p),$$

iff

$$(x, p'; y, p'; z', 1-2p') \succ (x', p'; y', p'; z', 1-2p').$$

Lower 3-DI (L3-DI) *holds if* $y' \succ z = z'$.
Upper 3-DI (U3-DI) *holds if* $z = z' \succ x'$.
Lower/Upper 3-DI (L/U3-DI) *holds if* $z' \succ x', y' \succ z$.

Birnbaum (2003), who introduced the concept of the Lower/Upper case, viewed it as a case of non-co-ranked restricted branch independence, Def. 10. This is clearly consistent with the definitions. We place it here because the branches $(z, 1-2p)$ and $(z', 1-2p')$ do not have a common location since $z' \succ x' \succ y' \succ z$, and we have elected to develop the BI properties of the several theories only for the co-ranked BI cases.

Proposition 13 *Suppose that a RWU representation holds. Then Lower 3-DI holds iff, for some constant $K_L > 0$,*

$$\frac{S_1(p,p,1-2p)}{S_2(p,p,1-2p)} = K_L \quad (0 < p \leq 1/2). \tag{35}$$

Upper 3-DI holds iff, for some constant $K_U > 0$,

$$\frac{S_2(1-2p,p,p)}{S_3(1-2p,p,p)} = K_U \quad (0 < p \leq 1/2). \tag{36}$$

Lower/Upper 3-DI holds iff

$$\frac{S_1(p,p,1-2p)}{S_2(p,p,1-2p)} = \frac{S_2(1-2p,p,p)}{S_3(1-2p,p,p)} \quad (0 < p \leq 1/2). \tag{37}$$

Corollary.

(i) *Under EU, Upper, Lower, and Lower/Upper 3-DI are satisfied.*

(ii) *Under RDU, Lower 3-DI is satisfied iff, for γ* defined by

$$\gamma = -\frac{\ln\left(\frac{K_L}{1+K_L}\right)}{\ln 2} \quad (K_L > 0),$$

$$W(p) = p^{\gamma} P\left(\frac{\ln p}{-\ln 2}\right) \quad (0 < p \leq 1), \tag{38}$$

where P is periodic with period 1 and W is strictly increasing. This simplifies to

$$W(p) = p^{\gamma} \quad (0 < p \leq 1) \tag{39}$$

iff the right derivative of W exists at 0.

Upper 3-DI is satisfied iff, for γ defined by

$$\gamma = -\frac{\ln\left(\frac{1}{1+K_U}\right)}{\ln 2} \quad (K_U > 0),$$

$$W(p) = 1-(1-p)^{\gamma} P\left(\frac{\ln(1-p)}{-\ln 2}\right) \quad (0 \leq p < 1), \tag{40}$$

where P is periodic with period 1 and *W is strictly increasing. This simplifies to*

$$W(p) = 1-(1-p)^{\gamma} \quad (0 \leq p < 1) \tag{41}$$

iff the left derivative of W exists at 1.

Lower/Upper 3-DI is satisfied iff W satisfies the following functional equation:

$$\frac{W(p/2)}{W(p)-W(p/2)} = \frac{W(1-p/2)-W(1-p)}{1-W(1-p/2)} \quad (0 < p \leq 1)\,. \tag{42}$$

(iii) *Under GDU,*

Lower 3-DI is satisfied iff, for some constant $K_L > 0$,

$$W_p(p/2) = \frac{K_L}{1 - K_L} \quad (0 < p \leq 1). \tag{43}$$

Upper 3-DI is satisfied iff, for some constant $K_U > 0$,

$$W_{1-p/2}(1-p) = 1 - K_U \left(\frac{1}{W_1(1-p/2)} - 1 \right) \quad (0 < p \leq 1)\,. \tag{44}$$

Lower/Upper 3-DI is satisfied iff

$$\frac{W_p(p/2)}{1 - W_p(p/2)} = \frac{W_1(1-p/2)[1 - W_{1-p/2}(1-p)]}{1 - W_1(1-p/2)} \quad (0 < p \leq 1)\,. \tag{45}$$

(iv) *Under TAX Lower 3-DI is satisfied iff, for some constant* $K_L > 0$,

$$\frac{T(p) + \omega_{1,2}(p,p,1-2p) + \omega_{1,3}(p,p,1-2p)}{T(p) - \omega_{1,2}(p,p,1-2p) + \omega_{2,3}(p,p,1-2p)} = K_L. \tag{46}$$

Upper 3-DI is satisfied iff, for some constant $K_U > 0$,

$$\frac{T(p) - \omega_{1,2}(1-2p,p,p) + \omega_{2,3}(1-2p,p,p)}{T(p) - \omega_{1,3}(1-2p,p,p) - \omega_{2,3}(1-2p,p,p)} = K_U. \tag{47}$$

Lower/Upper 3-DI is satisfied iff

$$\frac{T(p) + \omega_{1,2}(p,p,1-2p) + \omega_{1,3}(p,p,1-2p)}{T(p) - \omega_{1,2}(p,p,1-2p) + \omega_{2,3}(p,p,1-2p)} = \frac{T(p) - \omega_{1,2}(1-2p,p,p) + \omega_{2,3}(1-2p,p,p)}{T(p) - \omega_{1,3}(1-2p,p,p) - \omega_{2,3}(1-2p,p,p)}. \tag{48}$$

There are several comments about these results.

1. The conditions are entirely in terms of the weights; the utility of the consequences do not appear.

2. Under RDU, the form of W is known for lower and upper 3-DI, and if one is willing to assume that W has a right derivative at 0 and a left one at 1, respectively, the results are quite simple. It is easy to see, depending on $\gamma \gtrless 1$, that these functions are either everywhere concave or convex. Thus, they are inconsistent with much data on estimates of W (Luce, 2000, pp. 90-100). For the Lower/Upper 3-DI case, we have a functional equation, (42), for W which has not yet been solved. If $W(p) = p$, i.e., EU, were the only family of solutions, it would be a discriminating property, but we conjecture that there are other solutions.

3. Under GDU, Lower 3-DI is satisfied iff, for all $x \succsim y \succsim e$ and all $p \in]0, \frac{1}{2}]$, $(x, p; y, p) \sim (x, \frac{1}{2}; y, \frac{1}{2})$. This follows from the fact that (43) is equivalent to $W_p(p/2) = W_1(\frac{1}{2})$.

4. Under TAX, (46) holds when (29) holds for $i = 1$ or $i = 2$, and in particular for Birnbaum's (30). Also, (47) holds when (29) holds for $i = 2$ or $i = 3$, and in particular for Birnbaum's (31). Finally, (48) holds for (29) only if the latter holds for both $i = 1$ and $i = 3$, which we have shown is very restrictive, and (48) does not hold for either (30) or (31).

5. Under RAM, the special case of RWU = TAX with (17), Lower and Upper 3-DI hold and Lower/Upper 3-DI holds iff $[a_3(2)]^2 = a_3(1)a_3(3)$.

6. As we shall see in Table 3 below, the data do not support the Lower/Upper 3-DI case, and it fails in a particular way, namely, with $z' \succ x' \succ x \succ y \succ y' \succ z \succ e$,

$$(x, p; y, p; z, 1 - 2p) \succ (x', p; y', p; z, 1 - 2p), \tag{49}$$

and

$$(z', 1 - 2p; x, p; y, p) \prec (z', 1 - 2p; x', p; y', p). \tag{50}$$

If we assume that RDU is valid, then this violation implies

$$\frac{W(2p) - W(p)}{W(p)} > \frac{U(x') - U(x)}{U(y) - U(y')} > \frac{1 - W(1 - p)}{W(1 - p) - W(1 - 2p)}. \tag{51}$$

Birnbaum (2003) then goes on to argue, correctly, that this inequality is incompatible with the empirical findings of studies using binary gambles which show that, assuming binary RDU, W has an inverse-S form (Tversky & Kahneman, 1992; Wu & Gonzalez, 1996). Thus, either RDU is false or the inverse-S form is incorrect. Because we will present much other data (see Table 5 below) that rejects RDU, we really cannot reach any conclusion about the form of the weights from this study. The point of this observation is to warn the reader that the data from a single study can easily be misleadingly interpreted.

4.2 4-Distribution independence

Definition 14 ***4-distribution independence (4-DI)*** *is defined by: For* $z' \succ x' \succ x \succ y \succ y' \succ z \succ e$ *and* $p, r, r', r - 2p, r' - 2p \in [0, 1]$, *then*

$$(z', 1 - r - 2p; x, p; y, p; z, r) \succ (z', 1 - r - 2p; x', p; y', p; z, r), \tag{52}$$

iff

$$(z', 1 - r' - 2p; x, p; y, p; z, r') \succ (z', 1 - r' - 2p; x', p; y', p; z, r'). \tag{53}$$

Note that when $r = r'$, 4-DI can be considered to be a special case of restricted 4-BI, as was done by Birnbaum and Veira (1998)—see the earlier parallel discussion of the interpretation of Lower/Upper 3-DI. Also note that this condition, paralleling the form for L/U3-DI, can be renamed Lower/Upper 4-DI, and can be extended to (at least) Lower and Upper cases.

Proposition 15 *Suppose that the RWU representation holds. Then, 4-DI is satisfied iff, for all $p, r, r+2p \in [0,1]$ and some constant $K > 0$,*

$$\frac{S_2(1-r-2p,p,p,r)}{S_3(1-r-2p,p,p,r)} = K. \tag{54}$$

Corollary to Proposition 15. *For all $p, r, r+2p \in [0,1]$,*

(i) *Under EU, 4-DI is satisfied.*

(ii) *Under RDU, 4-DI is not satisfied.*

(iii) *Under GDU, 4-DI is satisfied iff, for some constant $K > 0$,*

$$W_{1-r-p}(1-r-2p) = 1-K\left(\frac{1}{W_{1-r}(1-r-p)} - 1\right) \quad \left(0 \le p < \frac{1}{2}(1-r)\right). \tag{55}$$

(iv) *Under TAX, 4-DI is satisfied iff, for some constant $K > 0$,*

$$\frac{T(p) - \omega_{1,2}(p,r) + \omega_{2,3}(p,r) + \omega_{2,4}(p,r)}{T(p) - \omega_{1,3}(p,r) - \omega_{2,3}(p,r) + \omega_{3,4}(p,r)} = K \quad \left(0 \le p < \frac{1}{2}(1-r)\right). \tag{56}$$

where $\omega_{i,j}(p,r) := \omega_{i,j}(1-r-2p,p,p,r)$. Under RAM, i.e., RWU with (17), 4-DI holds.

Note that (56) is satisfied if the assumption (29) is made either for $i = 2$ or for $i = 3$, but not both without major restrictions arising. And (56) is not satisfied for either assumption (30) or assumption (31). An important unsolved problem is to discover the form of any nontrivial solutions to (56).

4.3 Data on DI and summary

Birnbaum and Chavez (1997) tested 3-DI and 4-DI using choices with 100 respondents. At the aggregate level, they found approximately 30% and 40% violations of 3-DI, 4-DI, resp.; the data of individual respondents showed a similar pattern of violations. Birnbaum and Veira (1998) present violations in judged price (buying, selling) data. Birnbaum's (2003) choice data give strong evidence for Lower 3-DI holding and Upper 3-DI failing. In addition, Birnbaum (2003) presents various choice tests of Lower/Upper 3-DI. He constructed his gambles in such a way that he could test whether or not Lower/Upper 3-DI always holds—it does not—plus he could compare various special cases of RDU and TAX. Overall, the special cases of TAX fit the violations of Lower/Upper 3-DI much better than do the special cases of RDU. These interpretations of the data are highly dependent on the selected parameter values, which Birnbaum has kept fixed successfully in fitting several data sets. Other violations of Lower/Upper 3-DI in choice are presented in Birnbaum and Navarette (1998), Birnbaum and McIntosh (1996), Birnbaum and Chavez (1997) and Birnbaum, Patton and Lott (1999); and violations in price (buying, selling) appear in Birnbaum and Beeghley (1997).

Table 3 summarizes the predictions and the data. Note that no general model unqualifiedly predicts the success or failure of any distribution property except for RDU for 4-DI (failure) and GDU for L3-DI (success). However, RAM, i.e., RWU with (17), satisfies lower and upper 3-DI and 4-DI, counter to the data, and the special case, (30), of TAX satisfies lower 3-DI and fails upper 3-DI and 4-DI, in agreement with the data. These are the crucial differences that lead Birnbaum (2003) to favor the special case, (30), of TAX over RAM, given that they otherwise provide very similar fits to the available data[11] (e.g., Birnbaum, Patton, and Lott, 1999).

Insert Table 3 about here

5 Other Forms of Independence

5.1 Common consequence and common ratio independence

Birnbaum (1999) defines the following two additional types of independence in the context of comparing model predictions regarding the occurrence or otherwise of the Allais paradox and of choices satisfying or not stochastic dominance.

Definition 16 ***Common consequence independence*** *(CCI) is satisfied if, for all* $p, q, r, p+r, q+r \in [0,1]$,

$$(x, p; 0, 1-p) \succsim (y, q; 0, 1-q) \tag{57}$$

is equivalent to

$$(x, p; z, r; 0, 1-p-r) \succsim (y, q; z, r; 0, 1-q-r).$$

Proposition 17 *Suppose that a RWU representation holds. Then CCI is satisfied iff, for all* $p, q, r, p+r, q+r \in [0,1]$,

$$\frac{S_1(q, 1-q)}{S_1(p, 1-p)} = \frac{S_1(q, r, 1-q-r)}{S_1(p, r, 1-p-r)} \tag{58}$$

Corollary to Proposition 17. *For all* $p, q, r, p+r, q+r \in [0,1]$,

(i) *Under EU, CCI is satisfied.*

(ii) *Under RDU, CCI is satisfied.*

(iii) *Under GDU, CCI is satisfied iff*

$$W_1(p) = W_{p+r}(p) W_1(p+r). \tag{59}$$

(iv) *Under TAX, CCI is satisfied iff*

$$\frac{T(q)+\omega_{1,2}(q,1-q)}{T(p)+\omega_{1,2}(p,1-p)} \times \frac{T(p)+T(1-p)}{T(q)+T(1-q)}$$
$$= \frac{T(q)+\omega_{1,2}(q,r,1-q-r)+\omega_{1,2}(q,r,1-q-r)}{T(p)+\omega_{1,2}(p,r,1-p-r)+\omega_{1,2}(p,r,1-p-r)}$$
$$\times \frac{T(p)+T(r)+T(1-p-r)}{T(q)+T(r)+T(1-q-r)}. \quad (60)$$

Under RAM, (Def. 4), CCI holds iff the constants $a_n(i)$ of (17) are independent of n.

Note the following:

- (59) is a special case of the choice property, (16).
- With the additional assumption (29) for $i = 1$, and in particular for Birnbaum's (30), then (60) reduces to

$$\frac{T(p)+T(1-p)}{T(q)+T(1-q)} = \frac{T(p)+T(r)+T(1-p-r)}{T(q)+T(r)+T(1-q-r)} \quad (61)$$

  holding, which it clearly does if the sum of T over each partition (of size three) adds to a constant.

- The CCI property can be recast in event form, namely, for events C, C', C'', D, D', D'', E such that the relevant cases below are partitions, $C \cup C' = D \cup D'$ and $C \cup E \cup C'' = D \cup E \cup D''$, then

$$(x, C; e, C') \succsim (y, D; e, D')$$

  iff

$$(x, C; z, E; e, C'') \succsim (y, D; z, E; D'').$$

There are, to our knowledge, no relevant data about CCI aside from Birnbaum's (1999) specialization of this property to the Allais paradox.

Definition 18 ***Common ratio independence*** *is satisfied if, for all $p, q \in [0, 1]$, (57) is equivalent to*

$$(x, ap; e, 1-ap) \succsim (y, aq; e, 1-aq) \qquad (a \le 1/\max(p, q)).$$

This property does not seem to generalize in any simple way to events.

Proposition 19 *Any separable model, i.e., for all $x \in X, p \in [0, 1]$,*

$$U(x, p; e, 1-p) = U(x)W(p),$$

satisfies common ratio independence iff W is a power function of p.

Note that all RWU representations are separable. Estimates of the binary weights based on binary data for the separable case such as Tversky and Kahneman (1992) or Wu and Gonzalez (1996), which find both $W(p) > p$ for small p and $W(p) < p$ for large p, reject that W is a power function.

5.2 Cumulative independence

The following concepts are defined by Birnbaum (1997) and further elaborated by Birnbaum and Navarrete (1998) and by Birnbaum et al (1999).

Definition 20 *The following forms of* ***cumulative independence (CI)*** *are defined where* $z' \succ x' \succ x \succ y \succ y' \succ z \succ e$:

Lower CI

$$(x, C; y, D; z, E) \succ (x', C; y', D; z, E) \tag{62}$$

implies

$$(x, C \cup D; y', E) \succ (x', C; y', D \cup E). \tag{63}$$

Upper CI

$$(z', E; x, C; y, D) \prec (z', E; x', C; y', D) \tag{64}$$

implies

$$(x', E; y, C \cup D) \prec (x', C \cup E; y', D). \tag{65}$$

Proposition 21 *Suppose that a RWU representation is satisfied. Then lower CI is satisfied iff*

$$\frac{S_1(C \cup D, E)}{S_1(C, D \cup E)} \geq \frac{S_2(C, D, E)}{S_1(C, D, E)}. \tag{66}$$

And upper CI is satisfied iff

$$\frac{S_2(C \cup E, D)}{S_2(E, C \cup D) - S_2(C \cup E, D)} \leq \frac{S_3(E, C, D)}{S_2(E, C, D)} \tag{67}$$

Corollary to Proposition 21.

(i) *Under SEU, both lower and upper CI are satisfied.*

(ii) *Under RDU, both lower and upper CI are satisfied.*

(iii) *Under GDU, lower CI is equivalent to*

$$W_{C \cup D \cup E}(C \cup D) W_{C \cup D}(C) \geq W_{C \cup D \cup E}(C)[1 - W_{C \cup D}(C)], \tag{68}$$

and upper CI is equivalent to

$$W_{C \cup D \cup E}(C \cup E) W_{C \cup E}(E) \geq W_{C \cup D \cup E}(E). \tag{69}$$

(iv) *Under TAX, lower CI is equivalent to*

$$\frac{T(C \cup D) + \omega_{12}(C \cup D, E)}{T(C) + \omega_{12}(C, D \cup E)} \times \frac{T(C) + T(D \cup E)}{T(C \cup D, E)} \geq \frac{T(D) - \omega_{12}(C, D, E) + \omega_{23}(C, D, E)}{T(C) + \omega_{12}(C, D, E) + \omega_{13}(C, D, E)}, \tag{70}$$

and upper CI is equivalent to

$$\frac{T(D)-\omega_{12}(E\cup C,D)}{[T(C\cup D)-\omega_{12}(E,C\cup D)]\frac{T(C\cup E)+T(D)}{T(C\cup D)+T(E)}-T(D)+\omega_{12}(E\cup C,D)} \leq \frac{T(D)-\omega_{13}(E,C,D)-\omega_{23}(E,C,D)}{T(C)-\omega_{12}(E,C,D)+\omega_{23}(E,C,D)}. \qquad (71)$$

Under RAM (Def. 4) CI does not hold.

Birnbaum et al. (1999) discuss experimental violations of both lower and upper CI.

5.3 Interval independence

Birnbaum et al (1999) formulated the following concept[12]:

Definition 22 *Let*

$$\begin{aligned} A_k &= (x_1, C_1; x_2, C_2; \cdots; x, C_k;; ...; x_n, C_n), \\ B_k &= (x_1, C_1; x_2, C_2; \cdots; y, C_k; ...; x_n, C_n), \\ A'_k &= (y_1, D_1; y_2, D_2; \cdots; x, D_k; ...; x_n, D_n), \\ B'_k &= (y_1, D_1; y_2, D_2; \cdots; y, D_k; ...; x_n, D_n), \end{aligned}$$

where $\overrightarrow{\mathbf{C}}_n, \overrightarrow{\mathbf{D}}_n$ *are ordered partitions with*

$$\bigcup_{j=1}^{n} C_j = \bigcup_{j=1}^{n} D_j, \text{ and } D_k = C_k.$$

Then ***interval independence at position*** k ***(kII)*** *is satisfied provided that*

$$U(A_k) - U(B_k) = U(A'_k) - U(B'_k).$$

Lower II *holds when* $k = n$ *and* ***upper II*** *holds when* $k = 1$.

Proposition 23 *Suppose that a RWU representation is satisfied. Then, for any* k,

$$\frac{U(A_k) - U(B_k)}{U(A'_k) - U(B'_k)} = \frac{S_k(\overrightarrow{\mathbf{C}}_n)}{S_k(\overrightarrow{\mathbf{D}}_n)}, \qquad (72)$$

and kII is satisfied iff $S_k(\overrightarrow{\mathbf{C}}_n) = S_k(\overrightarrow{\mathbf{D}}_n)$.

Corollary to Proposition 23.

(i) *Under SEU, kII is satisfied for every* k.

(ii) *Under RDU, lower and upper II are satisfied, and, for $2 \leq k \leq n-1$, kII is not satisfied. However,*

$$\frac{U(A_k) - U(B_k)}{U(A'_k) - U(B'_k)} = \frac{W\left(C_k \cup C(k-1)\right) - W\left(C(k-1)\right)}{W\left(C_k \cup D(k-1)\right) - W\left(D(k-1)\right)}. \tag{73}$$

(iii) *Under GDU,*

$$\frac{U(A_k) - U(B_k)}{U(A'_k) - U(B'_k)} = \frac{\prod_{j=k}^{n-1} W_{C(j+1)}(C(j))[1 - W_{C(k)}(C(k-1))]}{\prod_{j=k}^{n-1} W_{D(j+1)}(D(j))[1 - W_{D(k)}(D(k-1))]}. \tag{74}$$

Thus, lower II is satisfied, but kII is not in general satisfied for $k < n$.

(iv) *Under TAX, a sufficient condition for kII is that*

$$T(\overrightarrow{\mathbf{C}}_n) = T(\overrightarrow{\mathbf{D}}_n),\ \omega_{i,j}(\overrightarrow{\mathbf{C}}_n) = \omega_{i,j}(\overrightarrow{\mathbf{D}}_n)\ (i < j). \tag{75}$$

In particular, (29) satisfies kII and (31) satisfies lower II provided that either T is finitely additive or if $T(C_n)$ is a constant for all partitions of a fixed event; and (30) satisfies upper II.

Birnbaum et al. (1999, Fig. 4, p. 69) provide evidence that both lower and upper II are violated.

Note that we do not know necessary and sufficient conditions for GDU or TAX to imply kII.

5.4 Tail independence (= Ordinal Independence)

Ordinal independence was first formulated by Green and Jullien (1988; see their important erratum, 1989). They used it to axiomatize RDU. In Wu's (1994) empirical study of it, he suggested that it might be better called *tail independence*, which term we adopt. To formulate this, consider a gamble $A = (x_1, C_1; ...; x_k, C_k; x_{k+1}, C_{k+1}; ...; x_n, C_n)$, $x_1 \succsim x_2 \succsim ... \succsim x_n \succsim e$, of size n. Let k be an index with $2 \leq k \leq n-1$. The upper tail of A is the portion $A_u(k) := (x_1, C_1; ...; x_k, C_k)$ and the lower tail is the portion $A_l(k) := (x_{k+1}, C_{k+1}; ...; x_n, C_n)$.

Definition 24 *Let A, B, A', B' be gambles of size n with a common universal set C and the same ordered consequences. Let $k \in \{2, ..., n-1\}$.*

Lower tail independence (LTI) *holds if $A_l(k) = B_l(k)$, $A'_l(k) = B'_l(k)$, $A_u(k) = A'_u(k)$, and $B_u(k) = B'_u(k)$ implies that $A \succsim B$ iff $A' \succsim B'$.*

Upper tail independence (UTI) *holds if the same condition is true with the u and l interchanged.*

The following result is not satisfactory for RWU and so for TAX because we do not have a necessary and sufficient condition, but only a sufficient one which we do not believe is necessary.

Proposition 25 *Suppose a RWU representation holds. Then neither UTI nor LTI is satisfied in general. A sufficient condition for LTI to hold is:*

$$\begin{aligned} A_l(k) &= B_l(k) \ \textit{implies} \ S_i(\overrightarrow{\mathbf{C}}_A) = S_i(\overrightarrow{\mathbf{C}}_B) \quad (i = k+1, .., n), \\ A'_l(k) &= B'_l(k) \ \textit{implies} \ S_i(\overrightarrow{\mathbf{C}}_{A'}) = S_i(\overrightarrow{\mathbf{C}}_{B'}) \quad (i = k+1, .., n), \\ A_u(k) &= A'_u(k) \ \textit{implies} \ S_i(\overrightarrow{\mathbf{C}}_A) = S_i(\overrightarrow{\mathbf{C}}_{A'}) \quad (i = 1, .., k), \\ B_u(k) &= B'_u(k) \ \textit{implies} \ S_i(\overrightarrow{\mathbf{C}}_B) = S_i(\overrightarrow{\mathbf{C}}_{B'}) \quad (i = 1, .., k), \end{aligned} \tag{76}$$

where $\overrightarrow{\mathbf{C}}_A$ *denotes the ordered event partition of gamble* A*, etc. A sufficient condition for UTI is*

$$\begin{aligned} A_u(k) &= B_u(k) \ \textit{implies} \ S_i(\overrightarrow{\mathbf{C}}_A) = S_i(\overrightarrow{\mathbf{C}}_B) \quad (i = 1, .., k), \\ A'_u(k) &= B'_u(k) \ \textit{implies} \ S_i(\overrightarrow{\mathbf{C}}_{A'}) = S_i(\overrightarrow{\mathbf{C}}_{B'}) \quad (i = 1, .., k), \\ A_l(k) &= A'_l(k) \ \textit{implies} \ S_i(\overrightarrow{\mathbf{C}}_A) = S_i(\overrightarrow{\mathbf{C}}_{A'}) \quad (i = k+1, .., n), \\ B_l(k) &= B'_l(k) \ \textit{implies} \ S_i(\overrightarrow{\mathbf{C}}_B) = S_i(\overrightarrow{\mathbf{C}}_{B'}) \quad (i = k+1, .., n). \end{aligned} \tag{77}$$

Corollary to Proposition 25.

(i) *SEU implies both LTI and UTI.*

(ii) *RDU implies both LTI and UTI.*

(iii) *GDU implies LTI but not UTI.*

(iv) *A sufficient condition on TAX for LTI or for UTI to hold is obtained by replacing the* S_i *terms of the sufficient conditions of Part (i) by the form given in (19).*

A weaker sufficient condition for TAX, (18), to hold is that the $\omega_{i,j}$ are independent of the event partition $\overrightarrow{\mathbf{C}}_n$ and that T is finitely additive. The latter implies $T(\overrightarrow{\mathbf{C}}_n)$ is independent of $\overrightarrow{\mathbf{C}}_n$ because $\bigcup_{i=1}^{k} C_i = \bigcup_{i=1}^{k} C'_i$ and $\bigcup_{i=k+1}^{n} C_i = \bigcup_{i=k+1}^{n} C'_i$.

Note that the above results generalize to the case where, for example in LTI, the $x_1, ..., x_k$ in the gambles B and B' are replaced by $y_1, ..., y_k$ where the y_j , $j \leq k$, may differ from the $x_j, j \leq k$. A similar remark is true for UTI.

The data reported by Wu (1994) and somewhat replicated by Birnbaum (2001) showing violations of UTI are, from our perspective, limited in two respects[13]. First, in constructing the experimental gambles, they assumed coalescing even though it is not a part of the definition of TI.[14] Second, the smallest consequence was 0. So, we do not believe that UTI has been adequately tested experimentally.

LTI has not been studied empirically.

5.5 Summary of other forms of independence

Table 4 summarizes what we have shown about these several forms of independence and the upshot of the relevant data. One does not learn much from these conditions other than that RDU has been tested using choices in 6 of 7 cases where the property is predicted and it failed in all 6.

Insert Table 4 about here

6 Conclusions and Open Problems

In our opinion, the only important tests of a model are of two types: (1) When the model, or a well-motivated special case of it, predicts that a property always holds, in which case negative data weigh in against the model. (2) When the model, or a well-motivated special case of it, predicts that a property never holds, in which case positive data are evidence against the model. Table 5 summarizes the tests of type (1) and Table 6, of type (2).

Insert Tables 5 and 6 about here

We now make several observations about these Tables, which must be read in the context of our comments in Section 2 regarding the way these data were analyzed, and interpreted, by the researchers who collected them.

- Table 5 shows that the only cases where RWU (and therefore all of SEU, RDU, GDU and TAX) predicts properties to hold are lower, intermediate, and upper BI in the restricted case. The 3 choice and one price studies on this agree with the prediction. Notice that various of the properties do not appear at all in Table 5. Most notable are the distribution independence ones except for L3-DI.

- Twelve of the 12 entries of Table 5 for RDU are positive. For the 5 positive predictions with respect to BI, both the choice and price data are positive in 3 cases, and not available in 2. For the remaining 7 positive predictions of Table 5 for RDU, the choice data are negative in 6 cases, and not available in 1; there are no price data for any of these cases. Various of these failures have been confirmed also at the level of individual participants (see, for instance, Birnbaum and McIntosh, 1996; Birnbaum and Navarette, 1998). These failures of RDU should have major implications for the field of utility under risk and uncertainty which, after all, has for the most part been restricted to RDU for gains in one variant or another—EU, SEU, and CPT. Birnbaum has made this point in many recent papers, but so far the implications of these data seem not to have had much impact on theorists.

- Six of the 12 entries of Table 5 for GDU are positive. For the 4 positive predictions with respect to BI, both the choice and price data are positive in 3 cases and not available in 1. GDU also predicts lower interval independence (LII) for which the choice data are negative, and lower tail independence (LTI) for which there are no choice data; there are no price data for these three cases. Of course, upper GDU gives the opposite pattern of predictions. We urge additional experimental focus on the 3 cases in Table 5 where lower GDU predicts a property to hold and TAX does not except for special cases. These properties are unrestricted co-ranked LBI, LII, and LTI.

- For the 7 entries of Table 6 for which RDU predicts that a property does not hold, the choice data are negative for 3 and not available for 4; the price data are negative for 1 and not available for 6. For the 5 entries of Table 6 for which GDU predicts that the property does not hold, the choice data are positive for 1, negative for 3, and not available for 1; the price data are negative for 1 and not available for 4. Thus, we have a rather mixed picture for both RDU and GDU. Because we fear that direct choices tend to elicit direct cancellation independent of the model we are testing, we recommend that a number of these crucial conditions be re-run using a certainty equivalence method such as Quick Indifference.

- Much more complex and perplexing are the various necessary and/or sufficient conditions for a property to hold. Given a currently empirically acceptable model, and new data that either confirm or reject a particular property of that model, is it reasonable to develop new variants of the model that handle both the old and new data, and then test new predictions of the revised model? Birnbaum has taken this approach, leading him to develop and test numerous independence properties of numerous configural weighted models. He argues currently for the special case, (30), of TAX which does make several important predictions, including that lower 3-DI holds and upper 3-DI fails, in agreement with the available data. Nonetheless, there are two limitations to this type of approach. First, TAX is as flexible as RWU. Second, different conditions are required for different positive predictions (see the various TAX equations in Tables 2-4), and we do not know whether or not one can invoke a single set of consistent special conditions that predict all the available data. The task is formidable – witness all of the equation numbers in the tables, and, additionally, take into account that the apparently common (28) is really different conditions for $i = 1, 2, 3$.

- A major open theoretical problem is to axiomatize special cases of the class of configural weighted models, such as the currently most successful special case, (30), of TAX.

Appendix: Proofs

Proposition 6

Proof. (i) Expanding (18) we have,

$$
\begin{aligned}
&T(\overrightarrow{\mathbf{C}}_n)U(g_{\overrightarrow{\mathbf{C}}_n}) \\
&= \sum_{i=1}^{n} U(x_i)T(C_i) + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} U(x_i)\omega_{i,j}(\overrightarrow{\mathbf{C}}_n) - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} U(x_j)\omega_{i,j}(\overrightarrow{\mathbf{C}}_n) \\
&= \sum_{i=1}^{n} U(x_i)T(C_i) + \sum_{i=1}^{n-1} U(x_i) \sum_{j=i+1}^{n} \omega_{i,j}(\overrightarrow{\mathbf{C}}_n) - \sum_{j=2}^{n} U(x_j) \sum_{i=1}^{j-1} \omega_{i,j}(\overrightarrow{\mathbf{C}}_n) \\
&= \sum_{i=1}^{n} U(x_i)T(C_i) + \sum_{i=1}^{n-1} U(x_i) \sum_{j=i+1}^{n} \omega_{i,j}(\overrightarrow{\mathbf{C}}_n) - \sum_{i=2}^{n} U(x_i) \sum_{j=1}^{i-1} \omega_{j,i}(\overrightarrow{\mathbf{C}}_n) \\
&= \sum_{i=1}^{n} U(x_i)T(C_i) + \sum_{i=2}^{n-1} U(x_i) \left[\sum_{j=i+1}^{n} \omega_{i,j}(\overrightarrow{\mathbf{C}}_n) - \sum_{j=1}^{i-1} \omega_{j,i}(\overrightarrow{\mathbf{C}}_n)\right] \\
&\quad + U(x_1)\left(\sum_{j=2}^{n} \omega_{1,j}(\overrightarrow{\mathbf{C}}_n)\right) - U(x_n)\left(\sum_{j=1}^{n-1} \omega_{j,n}(\overrightarrow{\mathbf{C}}_n)\right) \\
&= \sum_{i=1}^{n} U(x_i)T(C_i) + \sum_{i=2}^{n-1} U(x_i) \left[\sum_{j=i+1}^{n} \omega_{i,j}(\overrightarrow{\mathbf{C}}_n) - \sum_{j=1}^{i-1} \omega_{j,i}(\overrightarrow{\mathbf{C}}_n)\right] \\
&\quad + U(x_1)\left(\sum_{j=2}^{n} \omega_{1,j}(\overrightarrow{\mathbf{C}}_n) - \omega_{0,1}(\overrightarrow{\mathbf{C}}_n)\right) \\
&+ U(x_n)\left(\omega_{n,n+1}(\overrightarrow{\mathbf{C}}_n) - \sum_{j=1}^{n-1} \omega_{j,n}(\overrightarrow{\mathbf{C}}_n)\right) \qquad \text{[by (20) and (21)]} \\
&= \sum_{i=1}^{n} U(x_i)\left(T(C_i) + \left[\sum_{j=i+1}^{n+1} \omega_{i,j}(\overrightarrow{\mathbf{C}}_n) - \sum_{j=0}^{i-1} \omega_{j,i}(\overrightarrow{\mathbf{C}}_n)\right]\right) \\
&\qquad \text{[by (20) and (21)]} \\
&= T(\overrightarrow{\mathbf{C}}_n)\sum_{i=1}^{n} U(x_i)S_i(\overrightarrow{\mathbf{C}}_n),
\end{aligned}
$$

where $S_i(\overrightarrow{\mathbf{C}}_n), i = 1, ..., n$, are given by (19). Thus $U(g_{\overrightarrow{\mathbf{C}}_n})$ satisfies RWU, (6).

(ii) Assume that the idempotent version of (6) holds, and define $\omega_{i,j}(\overrightarrow{\mathbf{C}}_n)$

by (22) and (23). Then using these $\omega_{i,j}(\overrightarrow{\mathbf{C}}_n)$,

$$\begin{aligned}
&T(C_i) + \omega_{i,i+1}(\overrightarrow{\mathbf{C}}_n) - \omega_{i-1,i}(\overrightarrow{\mathbf{C}}_n) \\
&\quad = T(C_i) + T(\overrightarrow{\mathbf{C}}_n)\sum_{k=1}^{i} S_k(\overrightarrow{\mathbf{C}}_n) - \sum_{k=1}^{i} T(C_k) \\
&\qquad -T(\overrightarrow{\mathbf{C}}_n)\sum_{k=1}^{i-1} S_k(\overrightarrow{\mathbf{C}}_n) + \sum_{k=1}^{i-1} T(C_k) \\
&\quad = T(\overrightarrow{\mathbf{C}}_n) S_i(\overrightarrow{\mathbf{C}}_n).
\end{aligned}$$

Thus,

$$\begin{aligned}
S_i(\overrightarrow{\mathbf{C}}_n) &= \frac{T(C_i) + \omega_{i,i+1}(\overrightarrow{\mathbf{C}}_n) - \omega_{i-1,i}(\overrightarrow{\mathbf{C}}_n)}{T(\overrightarrow{\mathbf{C}}_n)} \\
&= \frac{T(C_i) + \displaystyle\sum_{j=i+1}^{n} \omega_{i,j}(\overrightarrow{\mathbf{C}}_n) - \sum_{j=0}^{i-1} \omega_{j,i}(\overrightarrow{\mathbf{C}}_n)}{T(\overrightarrow{\mathbf{C}}_n)},
\end{aligned}$$

i.e., the TAX representation, (18), with the above $\omega_{i,j}(\overrightarrow{\mathbf{C}}_n)$, gives the desired idempotent RWU representation, (6), with its $S_i(\overrightarrow{\mathbf{C}}_n)$ and the chosen $\omega_{i,j}(\overrightarrow{\mathbf{C}}_n)$, related by (19).

Note that this representation is highly non-unique because there are no particular restrictions on $T(C_i)$ and there may well be other choices for the $\omega_{i,j}$.

■

Proposition 8

Proof. Clearly if a simple utility representation holds, then BC holds for all 5 common locations.

So now suppose that the RWU representation holds with BC holding for the five types. Let C_i, D_i, $i = 1, ..., 6$, be such that each $\{C_i, D_i, E\}$ is an arbitrary partition of the same event $C(3)$. Using the definition of branch cancellation, Def. 10, for each of the five types of 3-BC, and selecting various of the possible pairings of two partitions of $C(3)$, we obtain:

$$\begin{aligned}
(1,1) \text{ gives } S_1(E, C_1, D_1) &= S_1(E, C_2, D_2), \\
(1,2) \text{ gives } S_1(E, C_1, D_1) &= S_2(C_3, E, D_3), \\
(2,2) \text{ gives } S_2(C_4, E, D_4) &= S_2(C_3, E, D_3), \\
(3,2) \text{ gives } S_3(C_5, D_5, E) &= S_2(C_4, E, D_4), \\
(3,3) \text{ gives } S_3(C_5, D_5, E) &= S_3(C_6, D_6, E),
\end{aligned}$$

which gives

$$\begin{aligned} & S_1(E,C_1,D_1) = S_1(E,C_2,D_2) \\ = \; & S_2(C_3,E,D_3) = S_2(C_4,E,D_4) \\ = \; & S_3(C_5,D_5,E) = S_3(C_6,D_6,E), \end{aligned}$$

and so we may define: for $i = 1,2,3$,

$$W_{C(3)}(C_i) = S_i(\overrightarrow{\mathbf{C}}_3),$$

and thus we have a simple utility representation. ■

Proposition 9

Proof. (i) Suppose that SEU holds. In any version of BC, the (z, E) branch has weight $S(E)$ and so (24) holds.

(ii) Suppose that RDU holds and write W for $W_{C\cup D\cup E} = W_{C'\cup D'\cup E}$.

(a) For $i = j = 1$, the weight on both sides of (24) is $W(E)$ and so UBC holds. For $i = j = 3$, using that $C\cup D = C'\cup D'$, the weight on both sides of (24) is $1 - W(C \cup D)$ and so LBC holds. For the unrestricted case with $i = j = 2$, the weights on the two sides of (24) are not equal as C appears on the left, and C' occurs on the right. The two non-co-ranked cases are $i = 1, j = 2$ and $i = 3, j = 2$. In the former, BC holds iff

$$W(E) = S_1(\overrightarrow{\mathbf{C}}_3^{(1)}) = S_2(\overrightarrow{\mathbf{C}}_3'^{(2)}) = W(C' \cup E) - W(C'),$$

with E and C' arbitrary, which gives finite additivity, which is SEU, which is excluded. For the $i = 3, j = 2$ case,

$$1 - W(C \cup D) = S_3(\overrightarrow{\mathbf{C}}_3^{(3)}) = S_2(\overrightarrow{\mathbf{C}}_3'^{(2)}) = W(C' \cup E) - W(C').$$

Setting $C' = \varnothing$, and so $D' = C \cup D$, gives $1 - W(C \cup D) = W(E)$, and substituting this back into the general equation gives

$$W(E) = W(C' \cup E) - W(C'),$$

for all C', which gives finite additivity, which is SEU, which is excluded.

(b) In the restricted case, it is clear that BC holds iff $i = j$.

(iii) Suppose that GDU holds. Using (13) and writing $F = C \cup D \cup E = C' \cup D' \cup E$,

$$\begin{aligned} S_1(\overrightarrow{\mathbf{C}}_3^{(1)}) &= S_1(E,C,D) = W_{C\cup E}(E) W_F(C \cup E), && (78) \\ S_2(\overrightarrow{\mathbf{C}}_3^{(2)}) &= S_2(C,E,D) = [1 - W_{C\cup E}(C)] W_F(C \cup E), && (79) \\ S_3(\overrightarrow{\mathbf{C}}_3^{(3)}) &= S_2(C,D,E) = 1 - W_F(C \cup D). && (80) \end{aligned}$$

(a) For the case $i = j = 1$, (78) holds iff

$$W_{C\cup E}(E) W_F(C \cup E) = W_{C'\cup E}(E) W_F(C' \cup E).$$

Setting $C' = \varnothing$, and so $D' = C \cup D$, gives

$$W_{C\cup E}(E)W_F(C \cup E) = W_{D'\cup E}(E) = W_F(E),$$

which is the choice property. That with GDU implies RDU, which we have previously shown satisfies BC only in the special case of SEU, which is excluded.

For the case $i = j = 2$, (79) yields

$$[1 - W_{C\cup E}(C)]W_F(C \cup E) = [1 - W_{C'\cup E}(C')]W_F(C' \cup E). \tag{81}$$

Setting $C' = \varnothing$, and so $D' = C \cup D = F$, gives

$$W_{C\cup E}(C)W_F(C \cup E) = W_F(C \cup E) - W_F(E),$$

which is (25). Note that if this has the common value $W_F(C)$, then both the choice property and finite additivity hold, reducing us to SEU. But otherwise, this is not the case.

For the case $i = j = 3$, (80), and the fact that $C \cup D = C' \cup D'$ imply that

$$S_3(C, D, E) = 1 - W_F(C \cup D) = 1 - W_F(C' \cup D') = S_3(C', D', E),$$

and so cancellation occurs.

In the non-co-ranked case $i = 1, j = 2$,

$$\begin{aligned} S_1(\overrightarrow{\mathbf{C}}_3^{(1)}) &= S_2(\overrightarrow{\mathbf{C}}_3^{\prime(2)}) \text{ iff} \\ W_{C\cup E}(E)W_F(C \cup E) &= [1 - W_{C'\cup E}(C')]W_F(C' \cup E). \end{aligned} \tag{82}$$

Setting $C' = \varnothing$, and so $D' = C \cup D$, gives

$$W_{C\cup E}(E)W_F(C \cup E) = W_F(E),$$

which is the choice property. That with GDU implies RDU which is excluded.

The other non-co-ranked case $i = 3, j = 2$ yields

$$1 - W_F(C \cup D) = [1 - W_{C'\cup E}(C')]W_F(C' \cup E).$$

Setting $C' = \varnothing$, and so $D' = C \cup D$, we obtain that $1 - W_F(C \cup D) = W_F(E)$, which substituted back in the equation gives

$$W_{C'\cup E}(C')W_F(C' \cup E) = W_F(C' \cup E) - W_F(E),$$

which is the same expression as for IBC, (25).

(b) For the restricted case with $i = j$, (11) implies that the weights are identical, and so co-monotonic BC holds.

In the non-co-ranked case, $i = 1, j = 2$, we have from (78) and (79) that

$$\begin{aligned} S_1(\overrightarrow{\mathbf{C}}_3^{(1)}) &= S_2(\overrightarrow{\mathbf{C}}_3^{\prime(2)}) \text{ iff} \\ W_{C\cup E}(E)W_F(C \cup E) &= [1 - W_{C'\cup E}(C)]W_F(C \cup E), \end{aligned}$$

i.e., iff

$$W_{C\cup E}(C) + W_{C\cup E}(E) = 1,$$

which is (26). In the other non-co-ranked case, $i = 3, j = 2$, we have from (80) and (79) that

$$\begin{aligned} S_1(\overrightarrow{\mathbf{C}}_3^{(3)}) &= S_2(\overrightarrow{\mathbf{C}}_3^{\prime(2)}) \text{ iff} \\ 1 - W_F(C \cup D) &= [1 - W_{C\cup E}(C)]W_F(C \cup E), \end{aligned} \tag{83}$$

which is (27).

(iv) Suppose that TAX holds. By Def. 7 coupled with (24), the necessary and sufficient condition is (28) which clearly holds in the co-ranked cases, i.e., when $i = j$. ■

Proposition 11

Proof. (i) implies (ii). By the RWU representation we have

$$\begin{aligned} &U(g_3) - U(g_3') \\ &= \sum_{k=1,k\neq i}^{3} U(x_k)S_k(\overrightarrow{\mathbf{C}}_3^{(i)}) - \sum_{k=1,k\neq j}^{3} U(x_k')S_k(\overrightarrow{\mathbf{C}}_3^{\prime(j)}) \\ &\quad +U(z)\left[S_i(\overrightarrow{\mathbf{C}}_3^{(i)}) - S_j(\overrightarrow{\mathbf{C}}_3^{\prime(j)})\right]. \end{aligned} \tag{84}$$

By branch cancellation, the term on the last line is 0, and so z may be replaced by any z', establishing branch independence.

To show the existence of consequences such that (34) holds, consider non-null events C, C', D, D', E, and the initial choice $x_0' = x_0 = y_0 = y_0' = z_0 \succ e$, which because gambles are idempotent means that, because of BC, for any choice of (i, j),

$$\begin{aligned} &U(g_3) - U(g_3') \\ &= \sum_{k=1,k\neq i}^{3} U(x_k)S_k(\overrightarrow{\mathbf{C}}_3^{(i)}) - \sum_{k=1,k\neq j}^{3} U(x_k')S_k(\overrightarrow{\mathbf{C}}_3^{\prime(j)}) \\ &\quad +U(z)\left[S_i(\overrightarrow{\mathbf{C}}_3^{(i)}) - S_j(\overrightarrow{\mathbf{C}}_3^{\prime(j)})\right]. \end{aligned} \tag{85}$$

Now, because by BC, $S_i(\overrightarrow{\mathbf{C}}_3^{(i)}) = S_j(\overrightarrow{\mathbf{C}}_3^{\prime(j)})$, we may ignore z_0 as it plays no role in this difference, and because the representation is onto the positive real numbers we can increase x_0' to x' and decrease y_0' to y' by a compensating amount so that the utility difference remains 0 and $x' \succ x_0 = y_0 \succ y'$. In like manner, increase x_0 to $x \prec x'$ and decrease y_0 to $y \succ y'$ by a compensating amount continuing to maintain the 0 difference and now with $x' \succ x \succ y \succ y'$. Once done, select z to yield type (i, j) which does not affect the difference because of BC. This (highly non-unique) gamble pair satisfies (34).

(ii) implies (i). Observe that BI applied to (34) implies, for any z' also of type (i, j), that

$$(x, C; y, D; z', E) \sim (x', C'; y', D'; z', E), \tag{86}$$

and so by RWU, (85) holds for these two gambles for all z. Now suppose, contrary to what is asserted, that $S_i(\overrightarrow{\mathbf{C}}_3^{(i)}) \neq S_j(\overrightarrow{\mathbf{C}}_3^{\prime(j)})$, then the last line of (85) is not 0. So when z is replaced by z', still yielding a gamble pair of type (i, j), the equality is destroyed. Thus, (86) is violated. Because this is impossible, branch cancellation must hold. ■

Proposition 13

Proof. Applying RWU to the left side of the hypothesis of lower 3-DI, we obtain:

$$\begin{aligned} U(x, p; y, p; z, 1-2p) &= U(x)S_1(p, p, 1-2p) + U(y)S_2(p, p, 1-2p) \\ &\quad + U(z)S_3(p, p, 1-2p). \end{aligned}$$

The right side is the same with y, x replaced by y', x', respectively. So the hypothesis is equivalent to

$$\frac{U(y) - U(y')}{U(x') - U(x)} > \frac{S_1(p, p, 1-2p)}{S_2(p, p, 1-2p)}.$$

The conclusion is of the same form, but with p replaced by p'. If the right ratio differs for p and p', we may select values of x, x', y, y', subject to the ordering restraint, so that the left ratio lies between them, thus violating the condition. Therefore the ratio has to be independent of p, i.e., a constant. The argument for Lower/Upper and upper 3-DI is parallel. ■

Corollary to Proposition 13

Proof. (i) SEU yields $S_1(p, p, 1-2p) = S_2(p, p, 1-2p) = S_3(1-2p, p, p) = W(p)$ and so the conditions are met.

(ii) For RDU and lower 3-DI, we have

$$\begin{aligned} S_1(p, p, 1-2p) &= W(p), \\ S_2(p, p, 1-2p) &= W(2p) - W(p). \end{aligned}$$

Therefore, using the Proposition,

$$\begin{aligned} K_L &= \frac{S_1(p, p, 1-2p)}{S_2(p, p, 1-2p)} = \frac{W(p)}{W(2p) - W(p)} \\ &\Leftrightarrow W(2p) = (1 + 1/K_L)\, W(p) \quad (0 < p \leq 1/2) \\ &\Leftrightarrow W(p/2) = AW(p) \quad \left(0 < p \leq 1, A = \frac{K_L}{1 + K_L}\right) \\ &\Leftrightarrow W(p/2) = (1/2)^\gamma W(p) \quad \left(\gamma = \frac{\log \frac{K_L}{1+K_L}}{\log 2}\right). \end{aligned} \tag{87}$$

This is Eq. (11) of Aczél and Kuczma (1991), and the solutions (38) and (39) are from their Theorems 3 and 9, respectively.

For upper 3-DI, we have

$$\begin{aligned} S_2(1-2p, p, p) &= W(1-p) - W(1-2p) \\ S_3(1-2p, p, p) &= 1 - W(1-p), \end{aligned}$$

which immediately implies

$$W(1-p) = AW(1-2p) + 1 - A \quad \left(0 < p < \frac{1}{2}, A = \frac{1}{1+K_U} < 1\right).$$

The following solution to this functional equation is due to János Aczél[15]. Define $F(p) = 1 - W(1-p)$. Because $W(0) = 0, W(1) = 1$, it follows that $F(0) = 0, F(1) = 1$, and F is strictly increasing. By substitution (40) is equivalent to

$$F(p/2) = AF(p) = (1/2)^\gamma F(p) \quad (0 < p < 1)\,, \tag{88}$$

which is exactly the same as (87). So the solution follows immediately which substituted into $W(p) = 1 - F(1-p)$ yields (40). Note that F has a right derivative at 0 iff W has a left derivative at 1, and so (41).

For Lower/Upper 3-DI, substituting the above expressions for $S_i, i = 1, 2, 3$ in (37) yields (42)

(iii) For GDU, by (13),

$$\begin{aligned} S_1(p,p,1-2p) &= W_{2p}(p)W_1(2p) \\ S_2(p,p,1-2p) &= W_1(2p)[1 - W_{2p}(p)], \end{aligned}$$

and

$$\begin{aligned} S_2(1-2p,p,p) &= W_1(1-p)[1 - W_{1-p}(1-2p)] \\ S_3(1-2p,p,p) &= 1 - W_1(1-p) \end{aligned}$$

Substituting these into the necessary and sufficient condition of the Proposition immediately yields that lower 3-DI holds iff

$$\frac{W_{2p}(p)}{1 - W_{2p}(p)} = K_L \Leftrightarrow W_{2p}(p) = \frac{K_L}{1 - K_K}.$$

Upper 3-DI holds iff (44), i.e.,

$$\frac{W_1(1-p)[1 - W_{1-p}(1-2p)]}{1 - W_1(1-p)} = K_U \quad (0 < p \le \frac{1}{2}),$$

which can be written in the form

$$W_{1-p/2}(1-p) = 1 - K_U\left(\frac{1}{W_1(1-p/2)} - 1\right) \quad (0 < p \le 1)\,.$$

i.e., (44).

The above expressions for $S_i, i = 1, 2, 3$, imply that Lower/Upper 3-DI holds iff (45).

(iv) For TAX we have from (19) for the terms arising in Part (i) that

$$\begin{aligned} S_1(p,p,1-2p) &= T(p) + \omega_{1,2}(p,p,1-2p) + \omega_{1,3}(p,p,1-2p), \\ S_2(p,p,1-2p) &= T(p) - \omega_{1,2}(p,p,1-2p) + \omega_{2,3}(p,p,1-2p), \end{aligned}$$

and

$$\begin{aligned} S_2(1-2p,p,p) &= T(p) - \omega_{1,2}(1-2p,p,p) + \omega_{2,3}(1-2p,p,p), \\ S_3(1-2p,p,p) &= T(p) - \omega_{1,3}(1-2p,p,p) - \omega_{2,3}(1-2p,p,p), \end{aligned}$$

where the denominator $2T(p)+T(1-2p)$ is omitted because in both ratios it cancels. Substituting these expressions in the relevant equations, i.e., (35), (36), (37), we see that (46), (47), (48) are the appropriate conditions. ■

Proposition 15

Proof. Assuming RWU simple substitution and rearrangement yields (54). ■

Corollary to Proposition 15

Proof. (i) Under EU, $S_2(1-r-2p,p,p,r) = S_3(1-r-2p,p,p,r) = p$, and so the condition of (54) is satisfied.

(ii) Under RDU it trivial to verify that the expressions for S_2 and S_3 are

$$\begin{aligned} S_2(1-r-2p,p,p,r) &= W(1-r-p) - W(1-r-2p), \\ S_3(1-r-2p,p,p,r) &= W(1-r) - W(1-r-p), \end{aligned}$$

and thus from (54), 4-DI holds iff, for some constant $K>0$,

$$\frac{W(1-r-p)-W(1-r-2p)}{W(1-r)-W(1-r-p)} = K \quad \left(0 \le p < \frac{1}{2}(1-r)\right) \tag{89}$$

The following proof that the only solution to this functional equation is $W(q) = q$, i.e., EU, is due to János Aczél[16]. Observe that if we let $q = 1-r-p, u = q-p = 1-r-2p, v = q+p = 1-r$, then $q = \frac{1}{2}(u+v)$ and (89) is equivalent to

$$W\left(\frac{u+v}{2}\right) = AW(u) + (1-A)W(v) \quad (0 \le u \le v \le 1, A = 1/(1+K)).$$

By defining

$$f(t) = W\left(\frac{t}{2}\right), g(u) = AW(u), h(v) = (1-A)W(v),$$

this becomes the Pexider equation $f(u+v) = g(u)+h(v)$. It is well known that one can extend a Pexider equation from a convex region of the non-negative quadrant to the entire domain of non-negative real numbers and that the general solution is

$$f(t) = at+b+c, g(t) = at+b, h(t) = at+c.$$

Taking into account that $W(0)=0, W(1)=1$, we see that $b=c=0$, $a=1$, and $A = 1-A$, whence $A = \frac{1}{2}$, and, we have $W(p) = p$.

(iii) Suppose GDU, we see that

$$\begin{aligned} S_2(1-r-2p,p,p,r) &= W_1(1-r)W_{1-r}(1-r-p)[1-W_{1-r-p}(1-r-2p)], \\ S_3(1-r-2p,p,p,r) &= W_1(1-r)[1-W_{1-r}(1-r-p)], \end{aligned}$$

and (55) follows from (54).

(iv) For TAX, we have

$$\frac{S_2}{S_3} = \frac{T(p) - \omega_{1,2}(p,r) + \omega_{2,3}(p,r) + \omega_{2,4}(p,r)}{T(p) - \omega_{1,3}(p,r) - \omega_{2,3}(p,r) + \omega_{3,4}(p,r)},$$

and so (56) follows from (54). ■

Proposition 17

Proof. Assuming RWU, we see that the condition holds iff

$$\frac{U(x)}{U(y)} \geq \frac{S_1(q,1-q)}{S_1(p,1-p)} \text{ and } \geq \frac{S_1(q,r,1-q-r)}{S_1(p,r,1-p-r)}.$$

Since we may choose x and y independently, we have (58) ■

Corollary to Proposition 17

Proof. For each special model, simply calculate the weights of (58) to get the four assertions. Note that GDU gives

$$\begin{aligned} \frac{W_1(q)}{W_1(p)} &= \frac{W_{q+r}(q)W_1(q+r)}{W_{p+r}(p)W_1(p+r)} \\ \Leftrightarrow \frac{W_{p+r}(p)W_1(p+r)}{W_1(p)} &= \frac{W_{q+r}(q)W_1(q+r)}{W_1(q)} = K. \end{aligned}$$

By taking the limit as $r \to 0$, we see that $K = 1$ and so this is equivalent to (59). ■

Proposition 19

Proof. Applying separability to the conditions shows that

$$\frac{U(x)}{U(y)} \geq \frac{W(q)}{W(p)} \text{ iff } \frac{U(x)}{U(y)} \geq \frac{W(aq)}{W(ap)}.$$

By the usual argument,

$$\frac{W(q)}{W(p)} = \frac{W(aq)}{W(ap)}.$$

Suppose $p \geq q$, then for $a = 1/p$ we have

$$\frac{W(q)}{W(p)} = \frac{W(q/p)}{W(1)}.$$

Setting $t := q/p$ and using $W(1) = 1$ yields the well known Cauchy equation

$$W(tp) = W(t)W(p),$$

whose strictly monotonic solutions are power functions. ■

Proposition 21

Proof. Suppose that RWU is satisfied. Then substituting the RWU form in the hypothesis and conclusion of lower CI and rearranging gives that lower CI is satisfied iff

$$\frac{U(x') - U(x)}{U(y) - U(y')} < \frac{S_2(C, D, E)}{S_1(C, D, E)} \tag{90}$$

implies

$$\frac{U(x') - U(y')}{U(x) - U(y')} < \frac{S_1(C \cup D, E)}{S_1(C, D \cup E)}. \tag{91}$$

Similarly, upper CI is satisfied iff

$$\frac{U(x') - U(x)}{U(y) - U(y')} > \frac{S_3(E, C, D)}{S_2(E, C, D)} \tag{92}$$

implies

$$\frac{U(x') - U(y')}{U(x') - U(y)} < \frac{S_2(E, C \cup D)}{S_2(E \cup C, D)}. \tag{93}$$

In the latter calculations we use the fact that in the binary case, idempotence yields $S_1 + S_2 = 1$.

Now, consider the conditions for a failure of lower CI to hold. Using those conditions plus the facts that $U(x) > U(y) > U(y')$ we obtain

$$\frac{S_1(C \cup D, E)}{S_1(C, D \cup E)} \leq \frac{U(x') - U(y')}{U(x) - U(y')} < \frac{U(x') - U(x)}{U(y) - U(y')} < \frac{S_2(C, D, E)}{S_1(C, D, E)}, \tag{94}$$

whence (66) is a necessary condition for lower CI to hold. Suppose it is violated, then since U is onto a real interval and CEs exist, we may choose $x' \succsim x \succsim y \succsim y' \succ e$ such that they are as in (94), establishing the sufficiency.

Next consider upper CI. Note that the conclusion (93) is equivalent to,

$$\begin{aligned}
\frac{U(y) - U(y')}{U(x') - U(y)} &= \frac{U(x') - U(y')}{U(x') - U(y)} - 1 \\
&< \frac{S_2(E, C \cup D)}{S_2(E \cup C, D)} - 1 \\
&= \frac{S_2(E, C \cup D) - S_2(C \cup E, D)}{S_2(E \cup C, D)} \\
\Leftrightarrow \frac{U(x') - U(y)}{U(y) - U(y')} &> \frac{S_2(E \cup C, D)}{S_2(E, C \cup D) - S_2(C \cup E, D)}.
\end{aligned}$$

Because $U(x') > U(y)$,

$$\frac{U(x') - U(y)}{U(y) - U(y')} > \frac{U(x') - U(x)}{U(y) - U(y')},$$

so a sufficient condition for the conclusion is that

$$\frac{S_3(E, C, D)}{S_2(E, C, D)} \geq \frac{S_2(E \cup C, D)}{S_2(E, C \cup D) - S_2(C \cup E, D)}.$$

It is also necessary because, otherwise, we may select consequences so that

$$\begin{aligned}
\frac{S_2(E, C \cup D)}{S_2(E, C \cup D) - S_2(C \cup E, D)} &\geq \frac{U(x') - U(y)}{U(y) - U(y')} \\
&> \frac{U(x') - U(x)}{U(y) - U(y')} \\
&> \frac{S_3(E, C, D)}{S_2(E, C, D)},
\end{aligned}$$

which means UCI fails. So, the necessary and sufficient condition for UCI to hold is (67). ■

Corollary to Proposition 21

Proof. (i) Suppose SEU holds. Then LCI and UCI follow by the RDU result of Part (ii).

(ii) Suppose RDU holds. Then by the Proposition, and using the abbreviation $W_{C \cup D \cup E} := W$, (66) is equivalent to

$$\frac{W(C \cup D)}{W(C)} = \frac{S_1(C \cup D, E)}{S_1(C, D \cup E)} \geq \frac{S_2(C, D, E)}{S_1(C, D, E)} = \frac{W(C \cup D) - W(C)}{W(C)},$$

which obviously holds.

The upper CI condition (67) is equivalent to

$$\begin{aligned}
\frac{1 - W(C \cup E)}{W(C \cup E) - W(E)} &= \frac{S_2(E \cup C, D)}{S_2(E, C \cup D) - S_2(C \cup E, D)} \\
&\leq \frac{S_3(E, C, D)}{S_2(E, C, D)} \\
&= \frac{1 - W(C \cup E)}{W(C \cup E) - W(E)},
\end{aligned}$$

which is is trivially true.

(iii) Suppose that GDU holds. By the Proposition, lower CI is equivalent to

$$\begin{aligned}
\frac{W_{C \cup D \cup E}(C \cup D)}{W_{C \cup D \cup E}(C)} &= \frac{S_1(C \cup D, E)}{S_1(C, D \cup E)} \\
&\geq \frac{S_2(C, D, E)}{S_1(C, D, E)} \\
&= \frac{W_{C \cup D \cup E}(C \cup D)[1 - W_{C \cup D}(C)]}{W_{C \cup D}(C) W_{C \cup D \cup E}(C \cup D)} \\
\Leftrightarrow W_{C \cup D}(C) W_{C \cup D \cup E}(C \cup D) &\geq W_{C \cup D \cup E}(C)[1 - W_{C \cup D}(C)]
\end{aligned}$$

which is (68).

Upper CI is equivalent to

$$\begin{aligned}
\frac{1 - W_{C\cup D\cup E}(C\cup E)}{W_{C\cup D\cup E}(C\cup E) - W_{C\cup D\cup E}(E)} &= \frac{S_2(C\cup E, D)}{S_2(E, C\cup D) - S_2(C\cup E, D)} \\
&\leq \frac{S_3(E,C,D)}{S_2(E,C,D)} \\
&= \frac{1 - W_{C\cup D\cup E}(C\cup E)}{W_{C\cup D\cup E}(C\cup E)[1 - W_{C\cup E}(E)]} \\
\Leftrightarrow W_{C\cup D\cup E}(C\cup E)[1 - W_{C\cup E}(E)] &\leq W_{C\cup D\cup E}(C\cup E) - W_{C\cup D\cup E}(E) \\
\Leftrightarrow W_{C\cup D\cup E}(C\cup E)W_{C\cup E}(E)] &\geq W_{C\cup D\cup E}(E).
\end{aligned}$$

which is (69)

(iv). For TAX and lower CI, the necessary and sufficient condition is

$$\begin{aligned}
&\frac{T(C\cup D) + \omega_{12}(C\cup D, E)}{T(C) + \omega_{12}(C, D\cup E)} \times \frac{T(C) + T(D\cup E)}{T(C\cup D, E)} = \frac{S_1(C\cup D, E)}{S_1(C, D\cup E)} \\
&\quad \geq \frac{S_2(C,D,E)}{S_1(C,D,E)} \\
&\quad = \frac{T(D) - \omega_{12}(C,D,E) + \omega_{23}(C,D,E)}{T(C) + \omega_{12}(C,D,E) + \omega_{13}(C,D,E)}.
\end{aligned}$$

And for upper CI, the necessary and sufficient condition is

$$\begin{aligned}
&\frac{T(D) - \omega_{12}(E\cup C, D)}{[T(C\cup D) - \omega_{12}(E, C\cup D)]\frac{T(C\cup E)+T(D)}{T(C\cup D)+T(E)} - T(D) + \omega_{12}(E\cup C, D)} \\
&\quad = \frac{S_2(C\cup E, D)}{S_2(E, C\cup D) - S_2(C\cup E, D)} \\
&\quad \leq \frac{S_3(E,C,D)}{S_2(E,C,D)} \\
&\quad = \frac{T(D) - \omega_{13}(E,C,D) - \omega_{23}(E,C,D)}{T(C) - \omega_{12}(E,C,D) + \omega_{23}(E,C,D)}.
\end{aligned}$$

■

Proposition 23

Proof. Suppose that RWU holds, then by (6) we see that

$$\begin{aligned}
U(A_k) - U(B_k) &= [U(x) - U(y)]S_k(\overrightarrow{\mathbf{C}}_n), \\
U(A'_k) - U(B'_k) &= [U(x) - U(y)]S_k(\overrightarrow{\mathbf{D}}_n),
\end{aligned}$$

whence (72). So kII holds iff $S_k(\overrightarrow{\mathbf{C}}_n) = S_k(\overrightarrow{\mathbf{D}}_n)$. ■

Corollary to Proposition 23

Proof. (i) Suppose that SEU holds. Then,

$$S_k(\overrightarrow{\mathbf{C}}_n) = W(C_k) = W(D_k) = S_k(\overrightarrow{\mathbf{D}}_n),$$

and so, by the Proposition, kII holds.

(ii) Suppose that RDU holds.

$$\begin{aligned} U(A_k) - U(B_k) &= [U(x) - U(y)] \left(W[C_k \cup C(k-1)] - W[C(k-1)] \right), \\ U(A'_k) - U(B'_k) &= [U(x) - U(y)] \left(W[C_k \cup D(k-1)] - W[D(k-1)] \right). \end{aligned}$$

The ratio, which is (73), is independent of the consequences. For lower II, $C_n = D_n$ and so $C(n-1) = D(n-1)$ and they are clearly equal. For upper II, the weights reduce to $W(C_1) = W(D_1)$ and so they are equal. For other values of k, the ratio is 1 means that

$$(W[C_k \cup C(k-1)] - W[C(k-1)]) = f(C_k).$$

The special case where $C(k-1) = \varnothing$ gives $f(C_k) = W(C_k)$ and so

$$W\left(C_k \cup C(k-1)\right) = W(C_k) + W\left(C(k-1)\right),$$

which establishes that W is finitely additive, which case is excluded. So kII does not hold.

(iii) Suppose that GDU holds. By the Proposition and the form of the weights (13), then (74) follows. Let $k = n$ in that and because $C(n) = D(n)$ and $C_n = D_n$, we have $C(n-1) = D(n-1)$, so lower II follows immediately.

(iv) For TAX, by (19) of Proposition 6, a sufficient condition for II to hold is clearly (75). ■

Proposition 25

Proof.

Consider LTI and RWU. Introduce the following notation:

$$U(A_u(k)) = \sum_{i=1}^{k} U(x_i) S_i(\overrightarrow{\mathbf{C}}_A).$$

Note that while the sum is over $i = 1, ..., k$, the terms $S_i(\overrightarrow{\mathbf{C}}_A)$ depend upon the entire ordered partition for gamble A. The other expression are defined similarly. Note that the sufficient condition (76):

$$\begin{aligned} A_l(k) &= B_l(k) \text{ implies } S_i(\overrightarrow{\mathbf{C}}_A) = S_i(\overrightarrow{\mathbf{C}}_B) \quad (i = k+1, .., n), \\ A'_l(k) &= B'_l(k) \text{ implies } S_i(\overrightarrow{\mathbf{C}}_{A'}) = S_i(\overrightarrow{\mathbf{C}}_{B'}) \quad (i = k+1, .., n), \\ A_u(k) &= A'_u(k) \text{ implies } S_i(\overrightarrow{\mathbf{C}}_A) = S_i(\overrightarrow{\mathbf{C}}_{A'}) \quad (i = 1, .., k), \\ B_u(k) &= B'_u(k) \text{ implies } S_i(\overrightarrow{\mathbf{C}}_B) = S_i(\overrightarrow{\mathbf{C}}_{B'}) \quad (i = 1, .., k), \end{aligned}$$

implies that

$$\begin{aligned} U(A_l(k)) &= U(B_l(k)), && (95) \\ U(A'_l(k)) &= U(B'_l(k)), && (96) \\ U(A_u(k)) &= U(A'_u(k)), && (97) \\ U(B_u(k)) &= U(B'_u(k)). && (98) \end{aligned}$$

Thus,

$$\begin{array}{rcll} A & \succsim & B & \\ \Leftrightarrow U(A) & \geq & U(B) & \\ \Leftrightarrow U(A_u(k)) + U(A_l(k)) & \geq & U(B_u(k)) + U(B_l(k)) & \\ \Rightarrow U(A_u(k)) + U(B_l(k)) & \geq & U(B_u(k)) + U(B_l(k)) & (95) \\ \Rightarrow U(A'_u(k)) + U(B_l(k)) & \geq & U(B'_u(k)) + U(B_l(k)) & (97),(98) \\ \Leftrightarrow U(A'_u(k)) & \geq & U(B'_u(k)) & \\ \Leftrightarrow U(A'_u(k)) + U(A'_l(k)) & \geq & U(B'_u(k)) + U(A'_l(k)) & \\ \Rightarrow U(A'_u(k)) + U(A'_l(k)) & \geq & U(B'_u(k)) + U(B'_l(k)) & (96) \\ \Leftrightarrow U(A') & \geq & U(B') & \\ \Leftrightarrow A' & \succsim & B', & \end{array}$$

establishing that this condition is sufficient for LTI. The argument for the sufficiency of (77) for UTI is similar. ■

Corollary to Proposition 25

Proof. (i) The SEU case follows from the RDU one.

(ii) Assume RDU. Key to the proof of LTI is the fact that all gambles are based on the same universal event C. Let the consequences be x_i and the events in the common upper tails of A, A' (resp., B, B') be C_i (resp, E_i) and in the common lower tails of A, B (resp., A', B') be C_i (resp, D_i). So

$$C(k) = \bigcup_{j=1}^{k} C_j = \bigcup_{j=1}^{k} E_j = E(k),$$

and

$$\bigcup_{j=k+1}^{n} C_j = \bigcup_{j=k+1}^{n} D_j.$$

Define $D_{C(k)}(j) := C(k) \cup D_{k+1} \cup .... \cup D_j$. Note that because $C(k) = E(k)$, the terms of the lower tail does not depend on whether the upper tail is C or

E based. Thus,

$$\begin{aligned}
& U(A) \geq U(B) \\
\Leftrightarrow \quad & \sum_{j=1}^{k} U(x_j) \left(W[C(j)] - W[C(j-1])\right. \\
& \qquad + \sum_{j=k+1}^{n} U(x_j) \left(W[C(j)] - W[C(j-1)]\right) \\
& \quad \geq \sum_{j=1}^{k} U(x_j) \left(W[E(j)] - W[E(j-1)]\right) \\
& \qquad + \sum_{j=k+1}^{n} U(x_j) \left(W[C(j)] - W[C(j-1)]\right) \\
\Leftrightarrow \quad & \sum_{j=1}^{k} U(x_j) \left[W(C(j)) - W(C(j-1)\right] \\
& \quad \geq \sum_{j=1}^{k} U(x_j) \left(W[E(j)] - W[E(j-1)]\right)
\end{aligned}$$

Using this,

$$\begin{aligned}
U(A') &= \sum_{j=1}^{k} U(x_j) \left(W[C(j)] - W[C(j-1)]\right) \\
&\quad + \sum_{j=k+1}^{n} U(x_j) \left(W[D_{C(k)}(j)] - W[D_{C(k)}(j-1)]\right) \\
&\geq \sum_{j=1}^{k} U(x_j) \left(W[E(j)] - W[E(j-1)]\right) \\
&\quad + \sum_{j=k+1}^{n} U(x_j) \left(W[D_{C(k)}(j)] - W[D_{C(k)}(j-1)]\right) \\
&= U(B').
\end{aligned}$$

The proof for UTI is parallel.

(iii) For GDU, consider the conditions for LTI. Apply gains decomposition to A and B recursively. By consequence monotonicity, the order of each subgamble is maintained because the terms (x_i, C_i), $i = k+1, ..., n$ are the same. So $A \succsim B \Leftrightarrow A_u \succsim B_u$. The argument is identical for the primed gambles. Because $A_u = A'_u$ and $B_u = B'_u$, the result is immediate.

For UTI no such argument based on gains decomposition works. Indeed, the following is a counter example. Consider the simplest case of UTI, namely

$i = 2, n = 4$ with objective probabilities a and b as shown

	x_1	x_2	x_3	x_4
$A:$	a	0	b	$1-a-b$
$B:$	a	0	$1-a-b$	b
A'	0	a	b	$1-a-b$
B'	0	a	$1-a-b$	b

By repeated applications of GDU we see that

$$A \sim \left(\left((x_1, 1; x_2, 0), \frac{a}{a+b}; x_3, \frac{b}{a+b}\right) a+b; x_4, 1-a-b\right).$$

The other three cases are similar. Thus by repeated applications of the binary RDU representation

$$\begin{aligned}
U(A) - U(B) = U(x_1) &\left[W\left(\frac{a}{a+b}\right) W(a+b) - W\left(\frac{a}{1-b}\right) W(1-b)\right] \\
&+ U(x_3)\left[\left(1 - W\left(\frac{a}{a+b}\right)\right) W(a+b) - \left(1 - W\left(\frac{a}{1-b}\right)\right) W(1-b)\right] \\
&+ U(x_4)[W(1-b) - W(a+b)]; \\
U(A') - U(B') = U(x_2) &\left[W\left(\frac{a}{a+b}\right) W(a+b) - W\left(\frac{a}{1-b}\right) W(1-b)\right] \\
&+ U(x_3)\left[\left(1 - W\left(\frac{a}{a+b}\right)\right) W(a+b) - \left(1 - W\left(\frac{a}{1-b}\right)\right) W(1-b)\right] \\
&+ U(x_4)[W(1-b) - W(a+b)].
\end{aligned}$$

Although the dependence on x_3 and x_4 is identical in the two equalities, the first depends on x_1 and not x_2 whereas the second depends on x_2 and not x_1. Therefore, from the fact that $U(A) - U(B) \geq 0$, we cannot conclude anything about the sign of $U(A') - U(B')$.

(iv) Assuming TAX, the sufficient condition for lower TI follows immediately from (19) and Part (i). ■

Notes

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1. The source of the order is discussed below.

2. Karamarkar (1978) used the term "subjectively weighted utility" for a special case of our unranked usage. Chew (1983) used "weighted utility" for a different representation. We introduced our usage in Marley and Luce (2001) and will continue to use it as defined in Definition 1.

3. Birnbaum has typically called them "configural weight models" but we feel this term is a bit more accurate.

4. We have changed their notation in two ways. First, we have reversed the order of the indices, and second, we have changed (18) on the right from $\omega_{i,j}(g_n)$ to $\omega_{i,j}(\overrightarrow{\mathbf{C}}_n)$ because we do not believe that they really want these TAX weights to depend on the consequences, beyond the ranking that they impose on the indices, as well as the event partition. If they do depend on g_n, then the model imposes no constraint at all.

5. If the ω's depended on the consequences as well as the event partition, this assertion would not be true.

6. Ho, M-H, Regenwetter, M., Niederée, & Heyer, D. (2003). Observation: An alternative perspective on von Winterfeldt et al.'s (1997) Test of Consequence Monotonicity. Submitted.

7. PEST is used effectively in psychophysics because, compared to the domain of gambles, it is far easier to obtain large numbers of responses. Thus, one can afford to use a more demanding stopping rule that is less susceptible to premature termination and misestimates of CEs.

8. In what follows, when we say RDU or GDU we exclude SEU, which is stated separately.

9. BI assumes that $x \succ y \succ e$ and $x' \succ y' \succ e$. Given these constraints, if we had either $x \succsim x'$ and $y \succsim y'$ or $x' \succsim x$ and $y' \succsim y$, then in the restricted case the equivalence of (32) and (33) would follow from stochastic dominance. Thus for the conditions to add additional constraints over stochastic dominance, we require either $x' \succ x$ and $y \succ y'$, which gives $x' \succ x \succ y \succ y' \succ e$ (the condition in the Definition) or $x \succ x'$ and $y' \succ y$, which gives the notational variant $x \succ x' \succ y' \succ y \succ e$.

10. In the literature, this is typically called *co-monotonic independence.* We consider this a misnomer because monotonicity is playing no role in the definition. Also, Birnbaum and McIntosh (1996) and Birnbaum and Navarrete

(1998) use the term ‘co-monotonic’ for restricted and co-ranked BI both holding, but we do not use that term here for the same reason.

11. Birnbaum was lead to formulate 3-DI and 4-DI as properties that distinguish between the predictions of RAM and the special case of TAX of (30). Personal communication, March 30, 2004.

12. Birnbaum et al. (1999) write x_k and y_k where we have written x and y. Their notation seems a bit confusing in making these arguments. Also, they formulate the condition in terms of judged strength of preference and demand that strength of preference be identical for A, B and A', B'. We interpret this to be a statement about equality of utility diferences.

13. Birnbaum (personal communication, March 30, 2004) has suggested that we restrict UTI to the special cases investigated by Wu and use UOI for the general definition. We have chosen not to do this.

14. This trap is set whenever one thinks that money lotteries can be represented by random variables—in which case coalescing is automatically assumed.

15. Personal communication January 13, 2003.

16. Personal communication January 13, 2003.

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Table 1. The types of branch independence. The data are organized with $z' \succsim z$, therefore the only off-diagonal entries $(i, j), (i', j')$ filled in are those above the main diagonal. A + means that the data so far collected satisfy the property, − means that violations have been observed. No data are available in the unrestricted case, and the only price data that are available are for co-ranked cases, where they agree with choice data. The full details of the data summarized in this table are presented in Tables 2 and 3.

		Conclusion				
	z'	$(1,1)$	$(1,2)$	$(2,2)$	$(3,2)$	$(3,3)$
	z					
	$(1,1)$	+	$-^{(a)}$	$-^{(a)}$	$-^{(a)}$	−
	$(1,2)$		$-^{(a)}$	$-^{(a)}$	$-^{(a)}$	$-^{(a)}$
Hyp.	$(2,2)$			+	$-^{(a)}$	$-^{(a)}$
	$(3,2)$				$-^{(a)}$	$-^{(a)}$
	$(3,3)$					+

[a] Birnbaum and McIntosh (1996, Tables 2 and 3). The data about these non-co-ranked cases are not discussed explicitly individually, but Tables 2 and 3, and the general discussion of the data, suggest that they all fail.

Table 2. Branch Independence (BI) and the Several Models

The following codes are used: r = restricted, u = unrestricted. The cases $(i,j)^2$ are each described plus the symmetric ones are called, respectively, upper ($i = j = 1$), intermediate ($i = j = 2$) and lower ($i = j = 3$) BI. A + in a condition column means that the condition is met and − means that it is not. A + in a model column means that the model predicts the condition, − means that there are violations. A + in the data column means that the data so far collected satisfy the property, − means that violations have been observed, and ND means no data have been reported. Equation numbers are shown for special cases of a model for which the property holds. Note that SEU satisfies all of the properties and is not explicitly shown. Also note that the entries (24) and (28) are, in fact, different conditions from row to row.

rest.	BI Type $(i,j)^2$		RWU	RDU	GDU	TAX	Data Choices	Data Judged Prices
u	$(1,2)^2$		(24)	−	−	(28)	ND	ND
u	$(3,2)^2$		(24)	−	(25)	(28)	ND	ND
r	$(1,2)^2$		(24)	−	(26)	(28)	$-^{(a)}$	ND
r	$(3,2)^2$		(24)	−	(27)	(28)	$-^{(a)}$	ND
u	$(1,1)^2$	UBI	(24)	+	−	(28)	ND	ND
u	$(2,2)^2$	IBI	(24)	−	(25)	(28)	ND	ND
u	$(3,3)^2$	LBI	(24)	+	+	(28)	ND	ND
r	$(1,1)^2$	UBI	+	+	+	+	$+^{(a)}, +^{(b)}, +^{(c)}$	$+^{(c)}$
r	$(2,2)^2$	IBI	+	+	+	+	$+^{(a)}, +^{(b)}, +^{(c)}$	$+^{(c)}$
r	$(3,3)^2$	LBI	+	+	+	+	$+^{(a)}, +^{(b)}, +^{(c)}$	$+^{(c)}$

$^{(a)}$ Birnbaum and McIntosh (1996, Tables 2 and 3). The data about the (restricted) cases $(1,2)$ and $(3,2)$ are not explicitly discussed, but Tables 2 and 3 and the general discussion, suggest that both fail.

$^{(b)}$ Wakker, Erev, & Weber (1994); see criticism of choices in Weber and Kirsner (1997).

$^{(c)}$ Weber & Kirsner (1997). The exact criterion for acceptance or not is subject to debate. For the co-ranked cases, the number of cases of failure is larger for judged prices than for choices, but the differences in judged prices that lead to such failures are really not very large.

Table 3. Distribution Independence (DI) and the Several Models
The codes are as in Table 2. Note that SEU satisfies all of the properties and is not explicitly shown.

Type	RWU	RDU	GDU	TAX	Data Choices	Judged Prices
U3-DI	(36)	(40)	(44)	(47)	$-^{(a)}, +^{(b)}$	ND
L3-DI	(35)	(38)	(43)	(46)	$+^{(a)}, -^{(b)}$	ND
L/U3-DI	(37)	(42)	(45)	(48)	$-^{(c)}, -^{(e)}$	$-^{(d)}$
4-DI	(54)	–	(55)	(56)	$-^{(e)}$	$-^{(f)}$

(a) Birnbaum (2003), Tables 1 and 2.

(b) Birnbaum (2003), Table 3. These data are inconsistent with those of (a). The result is not statistically significant according to Birnbaum's criterion.

(c) Birnbaum and Navarette (1998), Birnbaum and McIntosh (1996), Birnbaum and Chavez (1997), Birnbaum, Patton and Lott (1999), Birnbaum (2003). In each of these papers, L/U3-DI was treated as a special case of restricted 3-BI. The condition fails as it does not always hold, but it does hold for some special gambles, in agreement with Birnbaum's (2003) predictions based on special parameter values.

(d) Birnbaum and Beeghley (1997), where M3-DI was treated as a special case of restricted 3-BI.

(e) Birnbaum and Chavez (1997).

(f) Birnbaum and Veira (1998), where 4-DI was treated as a special case of restricted 4-BI.

Table 4. Remaining Forms of Independence and the Several Models

In addition to the codes of Table change: 2, CCI = common consequence independence, CRI = common ratio independence, CI = cumulative independence, II = interval independence, TI = tail independence. Note SEU satisfies all of the properties save CRI where it only holds if W is a power function.

Type	RWU	RDU	GDU	TAX	Data Choices
CCI	(58)	+	(59)	(60)	−[b]
CRI		−[a]	−[a]	−[a]	−[b]
UCI	(67)	+	(69)	(71)	−[c],[d]
LCI	(66)	+	(68)	(70)	−[c],[d]
UII	(72)	+	−	(75)	−[d]
III	(72)	−	−	(75)	−[d]
LII	(72)	+	+	(75)	−[d]
UTI	(77)	+	−	(77) & (19)	−[e]
LTI	(76)	+	+	(76) & (19)	ND

[a] It is satisfied iff W is a power function.

[b] Birnbaum (1999) cites a mix of well known empirical results and the existence of the Allais paradox in support of rejecting this property.

[c] Birnbaum and Navarrete (1998).

[d] Birnbaum, Patton, and Lott (1999).

[e] The experiments in the literature, Wu (1994) and Birnbaum (2001) that exhibit violations of TI are flawed in presuming coalescing and having a 0 consequence.

Table 5. Properties predicted to hold by at least one general model

Condition Rest.	$(i,j)^2$	Type	RDU	GDU	TAX	Data Choices	Judged Prices
u	$(1,1)^2$	UBI	+			ND	ND
u	$(3,3)^2$	LBI	+	+		ND	ND
r	$(1,1)^2$	UBI	+	+	+	+, +, +	+
r	$(2,2)^2$	IBI	+	+	+	+, +, +	+
r	$(3,3)^2$	LBI	+	+	+	+, +, +	+
		CCI	+			−	ND
		UCI	+			−	ND
		LCI	+			−	ND
		UII	+			−	ND
		LII	+	+		−	ND
		UTI	+			−	ND
		LTI	+	+		ND	ND

Table 6. Properties predicted not to hold by at least one general model

Condition rest.	(i, j)	Type	RDU	GDU	Data Choices	Data Judged Prices
u	$(1, 2)$	BI	–	–	ND	ND
u	$(3, 2)$	BI	–		ND	ND
r	$(1, 2)$	BI	–		–	ND
r	$(3, 2)$	BI	–		ND	ND
u	$(1, 1)$	UBI		–	+	–
u	$(2, 2)$	IBI	–		ND	ND
		4-DI	–		–	–
		III	–	–	–	ND
		UII		–	–	ND
		UTI		–	–	ND