Evaluating a Model of Global Psychophysical Judgments: II. Behavioral Properties Linking Summations and Productions

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Abstract

Steingrimsson and Luce (2003a) outlined the second author's proposed psychophysical theory (Luce, 2002, 2003a,b) and tested behavioral attributes that, separately, gave rise to one psychophysical function, Ψ, that maps pairs of physical intensities onto the positive real numbers that represents subjective summation and a second one, Ψ^*,that represents a form of ratio production. This article takes up properties linking summation and production in such a way as to force $\Psi^* = \Psi$. The properties are evaluated in a series of experiments. The testing strategy is carried out in the auditory domain and concerns the subjective perception of loudness. Considerable support is provided for theoretical forms of Ψ for both summations and ratio productions.

[1] This article is based, in part, on the first author's Ph. D. dissertation (Steingrimsson, 2002).

This article, the second of three, presumes that the reader has access to the earlier one (Steingrimsson and Luce, 2003a). We do not repeat the motivating background. Although the underlying theory and experimental procedures are exactly the same as reported in (Steingrimsson and Luce, 2003a), we do give an abbreviated account of each.

Consider, for example, loudness judgments of stimuli that involve, in general, different intensities of pure tones of the same frequency and phase presented to the two ears. Let x denote the physical intensity to the left ear measured as the intensity of the signal presented less (not relative to) the participant's threshold level for that ear, and let u denote the parallel measure for the right ear. So stimuli are of the form (x, u) where, as an idealization, we assume that x and u can be any non-negative real number, $\mathbb{R}_+$.

Judgments of relative loudness are assumed to be described by a binary relation $\succsim$ such that $(x, u) \succsim (y, v)$ means (x, u) is judged to be at least as loud as (y, v). We assume that $\succsim$ agrees with the intensity ordering $\geq$ when either $u = v = 0$ or $x = y = 0$. If $\sim$ denotes a judgment of equal loudness[2], we assume that $\sim$ forms an indifference relation. This is a decided idealization. We assume that the subject can always establish matches of three types to each stimulus:

$$(x, u) \sim (z_l, 0),\ (x, u) \sim (0, z_r),\ (x, u) \sim (z_s, z_s). \tag{1}$$

The left and right matches z_l and z_r are called *asymmetric* and z_s is called a *symmetric* match.

In addition to judgments of loudness, we also assume a form of magnitude production in which the pair (x, x) and (y, y), where $x > y$, is presented to the participant who is asked to establish the intensity z such that the subjective "interval" from (y, y) to (z, z) is perceived to be p times the "interval" from (y, y) to (x, x). It is convenient to define the notation that makes explicit the dependence of z on x, y, p :

$$(x, x) \circ_p (y, y) := (z, z). \tag{2}$$

Although expression (2) is apparently more restrictive than

$$(x, u) \circ_p (y, v) := (z, w), \tag{3}$$

but, in fact, under the assumptions we have made, it is not difficult to show that this is not the case.

The theory developed by (Luce, 2002, 2003a,b) basically asks: Under reasonable assumptions about $\succsim$, such as those above, and presuming the continuous nature of physical intensity and of the psychophysical functions involved, what are necessary and sufficient behavioral properties that correspond to the following numerical representation? Using asymmetric matching, z_l or z_r in (1), one shows that there are constants $\delta \geq 0$, $\gamma > 0$ and a distortion of number

[2] As usual, $(x, u) \sim (y, v)$ is defined to mean that both $(x, u) \succsim (y, v)$ and $(y, v) \succsim (x, u)$ hold.

p to $W(p)$, where W is a strictly increasing function for the non-negative real numbers, $\mathbb{R}_+$, to itself, such that

$$\Psi(x,u) = \Psi(x,0) + \Psi(0,u) + \delta\Psi(x,0)\Psi(0,u) \quad (\delta \geq 0), \tag{4}$$
$$\Psi(x,0) = \gamma\Psi(0,x) \quad (\gamma > 0), \tag{5}$$
$$W(p) = \frac{\Psi\left[(x,x) \circ_p (y,y)\right] - \Psi(y,y)}{\Psi(x,x) - \Psi(y,y)} \quad (x \geq y \geq 0). \tag{6}$$

If, instead, symmetric matching, z_s of (1), is used one can prove that the summation property (4) holds, but it is restricted to the case $\delta = 0$, and that (6) holds. The constant bias property (5) need not hold.

Steingrimsson and Luce (2003a) explored the key properties necessary for (4) and (6) to hold separately, respectively, the Thomsen condition and (subjective) production commutativity. They seemed to be sustained. But for all we knew from those results, there are two distinct psychophysical functions, Ψ for summations and Ψ^* for productions. All we would know is that for some strictly increasing function φ, $\Psi^*(x,u) = \varphi[\Psi(x,u)]$. This article focuses on the conditions linking the two structures that force $\Psi = \Psi^*$, i.e., φ is the identity function.[3] The third article of the series (Steingrimsson and Luce, 2003b) asks what are the forms of $\Psi(x,0)$ and $\Psi(0,x)$ as functions of x and what is form of $W(p)$ as a function of p. Again, we seek testable behavioral conditions equivalent to these functional forms and then test those conditions. We also get interesting new results about ratio estimations (i.e., basically magnitude estimation with a standard).

1 Linking Summations and Productions

In Steingrimsson and Luce (2003a), the conditions corresponding to (4) and (6) separately were formulated directly in terms of the ordering $\succsim$ over $\mathbb{R}_+ \times \mathbb{R}_+$. Although we would like to be able to state conditions about how the two representation interlock in the same manner, we have not yet seen how to do that. Rather, the approach that we use entails mapping both the summation and production structures onto a single physical dimension by using one or the other of the solutions assumed to exist in (1). To that end we need to define exactly how these mappings are carried out.

1.1 Induced summation operations

First, it is useful to rename the three solutions of (1) using an operation notation that makes clear their dependence on x and u, namely,

$$x \oplus_i u := z_i \quad (i = l, r, s). \tag{7}$$

[3] This is not strictly correct. One only shows that $\Psi^* = \alpha\Psi$, where $\alpha > 0$ is a constant. But one can replace Ψ^* by Ψ^*/α and none of the expressions (4), (5), and (6) is altered.

It is not difficult to show that each of the $\oplus_i$ is, indeed, a binary operation that is defined for each pair (x, u) of intensities. It is somewhat similar to the fact that $+$ is a binary operation taking pairs of real numbers to the real numbers. This permits, among other things, the possibility of such compounding as $(x \oplus_i u) \oplus_i v$, which is not available in the joint presentation notation because $((x, u), v)$ has no direct meaning (see Appendix A). The operations $\oplus_i$ are also strictly increasing in each argument where, by Assumption 2 of Steingrimsson and Luce (2003a), we see that the ordering on a single component is just the intensity ordering $\geq$.

Each operation encodes in one intensity dimension all of the information contained in the ordering of pairs of signals, i.e., in the conjoint structure of joint presentations. Luce (2002) studied $\oplus_l, \oplus_r$ and Luce (2003a,b) improved those results and studied $\oplus_s$ as well.[4]

The definitions of (7) imply that

$$(x, 0) \sim (x \oplus_l 0, 0),\ (0, u) \sim (0, 0 \oplus_r u),\ (x, x) \sim (x \oplus_s x, x \oplus_s x).$$

Using Proposition 1 of Luce (2002), one can show that (x, u) is strictly increasing in each variable, and so 0 is a *right identity* of $\oplus_l$, i.e.,

$$x = x \oplus_l 0, \tag{8}$$

0 is a *left identity* of $\oplus_r$, i.e.,

$$u = 0 \oplus_r u, \tag{9}$$

whereas 0 is not an identity of $\oplus_s$ at all. However, the symmetric operation is *idempotent* in the sense that

$$x \oplus_s x = x. \tag{10}$$

These properties play important roles in the theory.

Note that jp-symmetry, i.e., $(x, u) \sim (u, x)$, is equivalent to $\oplus_l \equiv \oplus_r$ and $\oplus_s$ all being commutative operations. The data of Experiments 1 and 2 of Steingrimsson and Luce (2003a) pretty much ruled out such symmetry.

1.2 Induced subjective-production operations

We also define induced production operations $\circ_{p,i}$, $i = l, r, s$, by the following special cases of the general operation $\circ_p$ defined by (3): :

$$(x \circ_{p,l} y, 0) \sim (x, 0) \circ_p (y, 0), \tag{11}$$
$$(0, u \circ_{p,r} v) \sim (0, u) \circ_p (0, v), \tag{12}$$
$$(x \circ_{p,s} y, x \circ_{p,s} y) \sim (x, x) \circ_p (y, y). \tag{13}$$

[4] At the time that many of the reported experiments were carried out, the later results were not available. But, as we shall see, this does not greatly matter because we conclude that most likely $\delta = 0$ (Experiment 1).

In this notation, the form of production commutativity studied in Experiment 4 of Steingrimsson and Luce (2003a) becomes

$$(x \circ_{p,i} y) \circ_{q,i} y = (x \circ_{q,i} y) \circ_{p,i} y. \tag{14}$$

1.3 Induced representations

For these induced operations, we are interested in representations ψ_i $(i = l, r, s)$ that map intensities onto intensities as represented by the non-negative real numbers $\mathbb{R}_+$. This is done simply by relating them to the general representation Ψ by:

$$\psi_l(x) \quad : \quad = \Psi(x, 0), \tag{15}$$

$$\psi_r(u) \quad : \quad = \Psi(0, u), \tag{16}$$

$$\psi_s(x \oplus_s u) \quad : \quad = \Psi(x, u). \tag{17}$$

These are the natural projections of Ψ.

In terms of these definitions, the p-additive form, (4), the constant bias form, (5), and the subjective proportion form, (6), are equivalent, respectively, to

$$\Psi(x, u) \quad = \quad \psi_s(x \oplus_s u) = \psi_l(x) + \psi_r(u) + \delta\psi_l(x)\psi_r(u). \tag{18}$$

$$\psi_l(x) \quad = \quad \gamma\psi_r(x). \tag{19}$$

$$W(p) \quad = \quad \frac{\psi_i(x \circ_{p,i} y) - \psi_i(y)}{\psi_i(x) - \psi_i(y)} \quad (x > y \geq 0, i = l, r, s). \tag{20}$$

Because $\psi_i(0) = 0$, we also have the following separable form for ordinary ratio production

$$\psi_i(x \circ_{p,i} 0) = \psi_i(x)W(p), \tag{21}$$

which is similar to the method used by Stevens (1975) in his classic studies of magnitude methods. The relation of this theory to magnitude estimation and production is studied more thoroughly in Steingrimsson and Luce (2003b).

1.3.1 *Properties of the unbiased case*

We say the a person is *unbiased* if and only if jp-symmetry, $(x, u) \sim (u, x)$, is satisfied; otherwise they are said to be *biased.* If the constant-ratio property (5) obtains, unbiased is equivalent to $\gamma = 1$. The case $\gamma > 1$ is called *left bias* and $\gamma < 1$, *right bias.*

Assuming the unbiased case, the following two properties are important in establishing linking relations between $\oplus_i$ and $\circ_{p,i}$. Such linking is crucial in showing that, in fact, a common function Ψ exists to represent both joint presentations, (4) (with $\gamma = 1$), and subjective proportions, (6).

Joint-presentation decomposition: For every $x, u, p \in \mathbb{R}_+$, there exists $q =$

$q(x,p) \in \mathbb{R}+$, such that

$$(x \oplus_i u) \circ_{p,i} 0 = (x \circ_{p,i} 0) \oplus_i (u \circ_{q,i} 0). \tag{22}$$

Note that q does not depend on u. This property is difficult to test experimentally because we do not know the form of the function q and so it must first be estimated without any constraints and then (22) tested. Fortunately, below we are able to provide evidence that shows it suffices to study (22) only for the special case where $q = p$.

Two other consequences of the representations are:

Left segregation: For $i = l, r, s$, and all $x, u, p \in \mathbb{R}_+, p > 0$,

$$u \oplus_i (x \circ_{p,i} 0) \sim (u \oplus_i x) \circ_{p,i} (u \oplus_i 0). \tag{23}$$

Right Segregation:

$$(x \circ_{p,i} 0) \oplus_i u \sim (x \oplus_i u) \circ_{p,i} (0 \oplus_i u). \tag{24}$$

In the unbiased case, right and left segregation are equivalent, but not in the biased case discussed below. Note that (8) implies that for $i = l$ one can replace $u \oplus_l 0$ by u, and (9) implies that for $i = r$, $0 \oplus_r u$ by u.

These properties together with production commutativity—common to both the biased and the unbiased case, (14)—establish that $\Psi^* = \Psi$ in the unbiased case.

Another aspect of the unbiased case of some interest is in Appendix A.

1.3.2 *Properties of the biased case*

In the biased case, $\oplus_i$ and $\circ_{p,i}$ are also linked by segregation. If we can assume $\delta = 0$ in (4), which will be sustained in Experiment 1 (see Sections 1.3.3 and 2.2), then the joint-presentation decompsition property, (22), simplifies to $q = p$, i.e.,

Simple Joint-Presentation Decomposition:

$$(x \oplus_i u) \circ_{p,i} 0 = (x \circ_{p,i} 0) \oplus_i (u \circ_{p,i} 0) \quad (i = l, r, s). \tag{25}$$

From a testing perspective, this simplification is an enormous convenience. The role of the property is analogous to that of (22) in the unbiased case.

Assuming that Ψ is decomposable over stimuli and ratio productions in that sense that

$$\begin{aligned} \Psi(x,u) &= F[\Psi(x,0), \Psi(0,u)], \\ \Psi\left[(x,x) \circ_p (y,y)\right] &= G_p[\Psi(x,x), \Psi(u,u)], \end{aligned}$$

and that F and G_p are continuous functions, Luce (2003a) established that the key to getting the representations of (4), (5), and (6) is that left segregation holds for $i = l$ and right segregation holds for $i = r$, leading to (4) and both left and right segregation hold for $i = s$, leading to (26) given below.

1.3.3 *Properties of the biased case when* $\delta = 0$

Recall that for left and right matches, the biased case corresponds to $\gamma \neq 1$.

Assuming $\delta = 0$ in the representation (4) and using the defined functions (15), (16), and (17), one can show that

$$\psi_i(x \oplus_i u) = \mu_l(i)\psi_i(x) + \mu_r(i)\psi_i(u) \quad [i = l, r, s, \mu_j(i) \geq 0], \tag{26}$$

which is a general linear weighting with the identifications

i	$\mu_l(i)$	$\mu_r(i)$
l	1	$1/\gamma$
r	γ	1
s	$\eta = \gamma/(1+\gamma)$	$1-\eta$

.

A simple algebraic computation on the two sides of the following equation using the representation (26) establishes that the following important property holds:

Bisymmetry:

$$(x \oplus_i y) \oplus_i (u \oplus_i v) = (x \oplus_i u) \oplus_i (y \oplus_i v) \quad (i = l, r, s). \tag{27}$$

Note that the two sides of this property simply involve the interchange of y and u. Bisymmetry fails to hold, however, when $\delta \neq 0$ except for the unbiased case $\gamma = 1$. Because we have considerable evidence that $\gamma \neq 1$, then the fact of bisymmetry's holding or not tells us whether or not $\delta = 0$. When bisymmetry does hold, we are able to use symmetrical matching to test several of the properties above (see Appendices B and C).

2 Tests of Linked Summations and Productions

2.1 Experimental Methods Common to All Experiments

The several experiments reported all have in common a number of testing strategies that were outlined in detail by Steingrimsson and Luce (2003a), including details on suggested methodological improvements Steingrimsson and Luce (see Appendix A of 2003a). Here we provide only a summary of these methods.

2.1.1 Participants

A total of 17 students—graduate and undergraduate—from the University of California, Irvine, participated in the three experiments of this article. The first author was one of them.[5] Of these 17, two participants stopped for personal reasons before sufficient data had been collected for analysis; three individuals participated in piloting sessions only or in experiments whose data are not reported in full. All of P9's data involving production judgments are excluded because a strategy was followed that was inconsistent with instructions, but his data for Experiment 2 are used. (For details see Luce's web page: aris.ss.uci.edu/cogsci/personnel/luce/P9.pdf.). This leaves 12 individuals, two male and 10 female, whose data are reported in full.

All participants had normal hearing as assessed through self-reporting and by an audiometric test (Micro Audimetrics EarScan ES-AM).

All participants, except the first author, received compensation of $10 per session. Each person provided written consent and was treated in accordance with the "Ethical Principles of Psychologists and Code of Conduct" (American Psychological Association, 1992). Consent forms and procedures were designed according to the rules of and received the approval of the UC Irvine's Institutional Review Board.

Although it would have been desirable, no individual participated in every experiment.)For details on individual participation, see Luce's web page: aris.ss.uci.edu/cogsci/personnel/luce/SLParticpants.pdf).

2.1.2 *Estimating one-ear and two-ear matches*

The three types of matches used are listed in Assumption 1. To indicate the precise form of trial types, let $\langle A, B\rangle$ denote a presentation of A followed by a temporally displaced presentation of B. We used a temporal delay between A and B of 500 ms. Three trial types, corresponding to (1), were used.

$$\langle (x, u), (z_l, 0)\rangle, \tag{28}$$

$$\langle (x, u), (0, z_r)\rangle, \tag{29}$$

$$\langle (x, u), (z_s, z_s)\rangle. \tag{30}$$

Following a tone presentation, participants used key presses to adjust the intensity of z_i, $i = l, r, s$, to repeat the previous trial, or to indicate satisfaction with the loudness match. After an adjustment, the altered tone sequence was played. Intensity could be adjusted up and down in step-sizes of 0.5, 1, 2 or 4 dB. At the end, z_i was recorded as the response.

In verbal instructions to participants, the task was explained as that of making the second stimulus pair equal in loudness to the first one.

2.1.3 *Estimating the $\circ_p$ operation*

The basic trial form was $\langle\langle A, B\rangle, \langle A, C\rangle\rangle$ where $\langle A, B\rangle$ and $\langle A, C\rangle$ represent the first and the second intensity interval respectively. The temporal delay between $\langle A, B\rangle$ and $\langle A, C\rangle$ was 750 ms. and between A and B (and A and C), the delay

[5] We judged this acceptable because no knowledge of the design could affect behavior and because it was very useful in fine tuning the procedures.

was 500 ms. The longer delay was introduced to create a subjective sense of two distinct intervals.

An estimate of $x \circ_{p,i} y = v_i$, in the case of $i = s$, was obtained using the trial type

$$\langle\langle(y, y), (x, x)\rangle, \langle(y, y), (v_s, v_s)\rangle\rangle. \tag{31}$$

The value of p was displayed on a monitor. Other aspects of this process were identical to that for matching.

Using verbal description and graphical examples, participants were instructed to find an intensity for v_i such that the first loudness interval stood in proportion p to the second interval.

The special case of $y = 0$ with $i = s$ was estimated using the trial type

$$\langle(x, x), (v_s, v_s)\rangle. \tag{32}$$

Trial forms in the case of $i = r, l$ are constructed in a manner analogous to (31) and 32).

2.1.4 Stimuli and Procedure

The experiments were carried out using a 1,000 Hz sinusoidal tone presented for 100 ms, which included 10 ms on and off ramps.

Experimental sessions lasted no more than one hour. Rest periods were encouraged but their frequency and duration were under the participants' control.

In multi-step experiments, we used averages of estimates in one step as input in a consequent step. Later we used the improved procedure of using individual estimates as input for subsequent estimates. This method was partially incorporated into Experiment 3.

2.1.5 Equipment

Stimuli were generated digitally using a personal computer and played through a 16-bit digital-to-analog converter (Quikki; Tucker-Davis Technology), at the converting rate of 40μ's per sample. Presentation level was controlled by manual and programmable attenuators, and stimuli were presented over Sennheiser HD265L headphones to listeners seated in individual, single-walled, IAC sound booths.

Dr. Bruce G. Berg, UC Irvine, generously made his laboratory and equipment available for the conducting of these experiments for which we are most grateful.

2.1.6 Statistical method and result presentation

The properties stated are of the form $A = B$, where the estimates of A and of B are variable. Thus, to test the theory, we are examining a number of null hypotheses, and the theory will be judged (tentatively) as supported if these are not rejected. Because we have no theoretical prediction for the distributions for these expressions, we used the nonparametric Mann-Whitney U-test with a significance level of .05.

We use averages and standard deviations in reporting the data. Although medians would be preferable, we could not accurately estimate them due to the discrete steps of intensity adjustments.

2.2 Experiment 1: Bisymmetry

We ran this experiment first because if bisymmetry, (27), is sustained, we then know that $\delta = 0$, which greatly simplifies running the experiments on simple joint-presentation decomposition and segregation. The property of bisymmetry can be tested using either one- or two-ear matching (Appendix B). For generality, both matching approaches were employed with the single ear matching done in the left ear.[6]

2.2.1 *Method*

To study (27) requires one to obtain six matches made in two steps. The first consists of

$$w_i = x \oplus_i y \;\text{ and }\; w_i' = u \oplus_i v, \;\text{[right side of (27),]}$$

and

$$z_i = x \oplus_i u \;\text{ and }\; z_i' = y \oplus_i v \;\text{[left side of (27)].}$$

Using the averages of w_i, w_i', z_i, and z_i', the second step consists of the two matches

$$t_i = w_i \oplus_i w_i' \;\text{ and }\; t_i' = z_i \oplus_i z_i'.$$

The property is said to hold if t_i and t_i' are found to be statistically equivalent.

For the left ear match ($i = l$) and the two-ear matches ($i = s$), the six matches were obtained using the trial forms given in Expressions 28 and 30, respectively.

One instantiation of signals was used: $x = 58$ dB, $y = 64$ dB, $u = 70$ dB, and $v = 76$ dB. The four estimates in step one and the two in step two gave rise to six trials types. These were grouped into two blocks. The first block contained two instances of the trial types in step one, eight trials in all; the second block contained three of each of the two trial types in step two, six trials in all. The two blocks were run in separate sessions.

2.2.2 *Results*

Six people participated. Their data are presented in Table 1, where T_1 and T_2 stand for the mean of t_i and t_i' respectively; standard deviations are given in parentheses. The number of observations in each mean is indicated by n (n for

[6] An even more general approach would have been to collect estimates for the right ear as well. We felt that there is no obvious reason to suspect that right ear matching results should be qualitatively different from left ear matching, and so we did not do that.

Part.	i	Mean (s.d.)		p_{stat}	n	Stat.
		T_1	T_2		(Step 1)	Trend
P2	s	68.90 (2.24)	69.15 (1.48)	.464	40 (46)	$T_1 = T_2$
P3	s	62.36 (1.14)	61.90 (1.39)	.144	45 (60)	$T_1 = T_2$
P11	s	61.56 (3.26)	62.20 (3.41)	.261	32 (34)	$T_1 = T_2$
P14	s	68.31 (3.90)	68.34 (2.20)	.208	45 (60)	$T_1 = T_2$
P16	l	77.24 (1.27)	76.83 (1.25)	.106	60 (60)	$T_1 = T_2$
P17	l	76.04 (1.16)	75.73 (0.60)	.064	60 (60)	$T_1 = T_2$

Table 1: Experiment 1: Bisymmetry

step one is given in parenthesis). Left ear and two-ear matching is marked with l and s respectively. Results of statistical tests is indicated by corresponding p-values.

The bisymmetry property was not rejected for any of the six participants.

2.2.3 *Discussion*

These results provide good initial support for the bisymmetric property. Within the context of the theory that means either $\gamma = 1$ or $\delta = 0$. Because we have considerable evidence that mostly $\gamma \neq 1$ (Experiments 1 and 2 of Steingrimsson and Luce 2003a) we are justified in assuming that $\delta = 0$ in designing experiments 2 and 3.

2.3 Experiment 2: Simple joint-presentation decomposition

Simple joint-presentation decomposition was stated in (25), namely,

$$(x \oplus_i u) \circ_{p,i} 0 = (x \circ_{p,i} 0) \oplus_i (u \circ_{p,i} 0) \quad (i = l, r, s).$$

The production judgments needed are of the general form $v_l = x \circ_{p,i} 0$, which are special cases ($y = 0$) of general production judgments that was treated for $\circ_p$ in Section 2.1.3. That procedure has to be modified a bit when dealing with the defined operations $\circ_{p,i}$. On the assumption that $\delta = 0$, which was sustained in Experiment 1, the simple joint-presentation decomposition property can be tested using either one- or two-ear matching—see Appendix C. As explained in Section 2.1.3, it is desirable that estimates such as $t_i = (x \oplus_i u) \circ_{p,i} 0$ be done either in two steps or by using a two-ear ($i = s$) version. Because the latter is experimentally more efficient, we used it.

2.3.1 *Method*

The property has two levels of estimation which were done in two steps. First, the three estimates

$$t_s = (x \oplus_s u) \circ_{p,s} 0, \quad w_s = x \circ_{p,s} 0, \quad s_s = u \circ_{p,s} 0,$$

were made. The averages of w_s and s_s were used in the second step, which consisted of the match

$$t'_s = w_s \oplus_s s_s.$$

The property is considered to hold if t_s and t'_s are found to be statistically equivalent.

We used one pair of intensities, $x = 64$ dB and $u = 70$ dB, resulting in three trial types in the first step. An estimate for t_s was obtained using the trial type $\langle (x, u), (t_s, t_s) \rangle$, and estimates for w_s and s_s used trial types given by expression (31), and in the second step, t' was estimated using the trial type given by expression (30).

A theoretical prediction is that the property holds for both $p < 1$ and for $p \geq 1$. Two production conditions meeting these constraints were chosen, namely $p = \frac{2}{3}$ and $p = 2$.

With two production conditions for each trial type, steps one and two consisted of six and two trial conditions, respectively. These were separated into blocks, where the first contained the six estimates in the first step; the second block contained three of each of the two trial conditions in step two. The blocks were run in separate sessions.

2.3.2 *Results*

Four participants completed this experiment and their data are presented in Table 2.

In the table, T_1 and T_2 stand for the mean of t_s and t'_s respectively; standard deviations are given in parentheses. The number of observations of t_s and t'_s are indicated by n_{T_1} and n_{T_2} respectively. By design, n_{T_1} also indicates the number of observations used for estimating w_s and s_s. Statistical results are indicated by corresponding p-values labelled p_{stat}.

The property held in both conditions for two of four participants with the other two failing one of the two conditions, $p = \frac{2}{3}$ and $p = 2$ respectively. That is, the property was accepted in six out of eight tests.

In the $p = 2$ condition, marked by *, participant P5 sought a few times to produce a t_s and a t'_s with intensity above the procedural safety limit. Hence, the possibility cannot be excluded that without this limit, the two values would have been found statistically different.

A fifth person, P15, completed the first step of the experiment but was observed to violate monotonicity in step two. Specifically, with $w_s < s_s$, the result of the matching task $(w_s, s_s) \sim (t'_s, t'_s)$ was that $w_s, s_s < t'_s$. In the step one results, $w_s < t_s < s_s$ for both values of p, as would be expected (matching is

Part.	p	Mean (s.d.)		p_{stat}	n_{T_1}, n_{T_2}	Stat. Trend
		T_1	T_2			
P4	2	74.0 (1.34)	72.63 (1.79)	.000	60, 60	$T_1 \neq T_2$
	2/3	64.53 (1.94)	64.28 (2.95)	.823		$T_1 = T_2$
P5	2*	87.58 (3.43)	88.98 (1.06)	.257	60, 60	$T_1 = T_2$
	2/3	48.64 (5.99)	50.25 (5.20)	.096		$T_1 = T_2$
P10	2	78.95 (1.13)	78.97 (0.71)	.480	30, 28	$T_1 = T_2$
	2/3	62.42 (1.71)	61.13 (1.38)	.003		$T_1 \neq T_2$
P23	2	78.59 (2.58)	79.58 (0.80)	.065	60, 56	$T_1 = T_2$
	2/3	63.54 (1.54)	63.43 (1.47)	.779		$T_1 = T_2$

Table 2: Experiment 2: Simple joint-presentation decomposition

done in both ears, hence, the intensity of the produced tone should fall between the intensities of the presented tones). Note that P15 participated in Experiment 2 of Steingrimsson and Luce (2003a), which preceded the current experiment, and those data did not show any violation of monotonicity. We do not have any firm hypothesis to account for P15's violation of monotonicity in step two. The fact that monotonicity was repeatedly observed in the participant's responses prior to the current task leads one to suspect that some time-order effect on matching may be created by performing production judgments in the way done here. Yet, absent further information, this case remains a mystery.

2.3.3 *Discussion*

With simple joint-presentation decomposition holding in six out of eight tests, we feel that there exists reasonable initial support of this property.

2.4 Experiment 3: Segregation

Recall that, for $i = l, r, s$, left segregation is given by

$$u \oplus_i (x \circ_{p,i} 0) \sim (u \oplus_i x) \circ_{p,i} (u \oplus_i 0)$$

and right segregation by

$$(x \circ_{p,i} 0) \oplus_i u \sim (x \oplus_i u) \circ_{p,i} (0 \oplus_i u).$$

Although right and left segregation are equivalent in the unbiased case, the results of Experiments 1 and 2 of Steingrimsson and Luce (2003a) suggested that the biased case where the equivalence does not hold.

As explained in Section 1.3, because 0 is a right identity of $\oplus_i$, testing left segregation is easier for $i = l$ and right segregation is easier for $i = r$. For $i = s$ both need to be tested.

The experiment was carried out before Luce (2003a) derived the case for $i = s$ so it was not tested here, but the two other cases were tested. However, at the time we did not fully understand the theory and we believed an interlock to exist between the nature of the bias and the type of segregation needed for $i = l, r$. For this reason, we determined for each participant the nature of the bias, left or right, and studied the corresponding form of segregation, left or right. This meant two things: (i) there was an unnecessary step in determining the nature of the bias and (ii) for each participant (except P22), we studied only one form of segregation, either left or right. In the following, the description of the bias determination is omitted as it was redundant and identical to the procedure used in Experiment 2 of the earlier article. Those data were reported there.

In the following, methodological issues are discussed for right segregation with $i = r$—the other cases follow readily from that discussion.

The compound production $(x \oplus_r u) \circ_{p,r} u$ on the left side of right segregation can be obtained with a trial of the form

$$\langle\langle(0, u), (x, u)\rangle\rangle, \langle(0, u), (0, v_r)\rangle\rangle,$$

This saved the experimenter from first estimating $x \oplus_r u$. However, at least as long as $|x - u|$ is not large, the presentation $(u, 0)$ is subjectively experienced in the right ear whereas the subsequent joint-presentation (x, u) is a head-centered location. Then, when $(0, u)$ is presented again the presentation can create the sensation of rapid movement of a sound from the right ear to a somewhat centered location and back to the right ear. This experience was reported to be distracting. For this reasons, in the case of $i = l, r$ it is beneficial to estimate $x \oplus_r u$ first.[7]

2.4.1 *Method*

Because of the number and variety of estimated values, testing segregation was quite challenging with respect to experimental design and procedures. All estimation steps had to be done with utmost care and additional practice was provided for all participants.

Four estimates must be made

$$\begin{aligned} w_r &= x \circ_{p,r} 0, && (33) \\ t_r &= w_r \oplus_r u, && (34) \\ z_r &= x \oplus_r u, && (35) \\ t'_r &= z_r \circ_{p,r} u. && (36) \end{aligned}$$

The property is said to hold if t_r and t'_r are not found statistically different.

Note that in (34) and (36), the intensities w_r and z_r were presented in the left ear although estimated in the right ear (the converse is the case for left segregation).

[7] Statistically this has some support: Two participants made estimates using both a one and a two step version of the task. The statistical results were closer to the expected outcome in the two step case.

One intensity pair, $x = 72$ dB and $u = 68$ dB, was used except for P23 where it was dropped to $x = 68$ dB and $u = 64$ dB to avoid productions limited by the safety limit. A theoretical predication is that the property hold for both $p < 1$ and $p \geq 1$, hence $p = \frac{2}{3}$ and $p = 2$ were used.

The experiment was carried out using three block types (additional comments follow the list).

B1 The estimates for z_r was done in a block containing two dummy trials to ensure stimulus variety within a block, namely $x \oplus_r u$ and $u \oplus_r u$. Each block contained two instances of each trial type or six in all. The trial type is given by expression (29).

B2 The estimates for w_r and t'_r, were carried out together in a block containing sub-blocks for each estimate. Each sub-block contained three trials of each of the two trials forms, or six in all. Participant were given the textual alerts of "One interval" for a sub-block estimating of w_r and "Two intervals" for t'_r. The trial type for estimating t'_r is given by

$$\langle\langle(0, u), (0, z_r)\rangle, \langle(0, u), (0, v_r)\rangle\rangle,$$

where an average is used for z_r, and for w_r by

$$\langle(0, x), (0, v'_r)\rangle,$$

B3 Two instances, corresponding to the two p-values of the estimate for t_r, were needed. These were done using a block containing three trials of each of the two trials forms for a total of six trials. The trial types are given by expression (29) with an average for w_r used as input.

In B2, all production judgment conditions were run together within a session as recommended in Steingrimsson and Luce (2003a). These are estimated using procedurally different trial types, and participants strongly preferred knowing in advance which type of task they were about to experience. Additionally, participants preferred having only one kind of trial form in a block. Hence, these two types were placed in two sub-blocks within a block and the participants were alerted to the upcoming task type.

Sessions were blocked on these block types. Even though w_r and t'_r were both estimated within a block, pilot data from B2 showed substantial inter-session variation in the relative relationship for the two production tasks. Extensive practice, consisting of at least three practice sessions with B2, seemed to counter this problem.

For P22 with $i = r$, an improved version of the experiment was employed. In this version, all trial conditions were run within a session and individual estimates, rather than averages, were propagated through the process. Within the session, 20 instances of each trial condition was collected before moving to the next step. Hence, a session consisted of 80 trials total.

Part.	Seg.	p	Mean (s.d.)		p_{stat}	n	Stat.
			T_1	T_2		(n_z)	Trend
P10	l	2	80.32 (0.61)	79.21 (0.55)	.000	30	$T_1 \neq T_2$
		2/3	73.60 (1.04)	73.86 (1.09)	.656	(42)	$T_1 = T_2$
P22	l	2	78.95 (1.13)	78.27 (1.63)	.197	30	$T_1 = T_2$
		2/3	71.98 (1.27)	72.22 (2.53)	.573	(30)	$T_1 = T_2$
	r	2	69.18 (1.10)	69.73 (1.73)	.121	40	$T_1 = T_2$
		2/3	62.30 (1.78)	64.33 (1.82)	.000	(40)	$T_1 \neq T_2$
P23	l	2	81.18 (1.14)	80.62 (1.44)	.202	30	$T_1 = T_2$
		2/3	71.10 (2.26)	70.55 (1.69)	.110	(30)	$T_1 = T_2$
P24	r	2	81.30 (1.22)	82.01 (1.42)	.052	30	$T_1 = T_2$
		2/3	73.10 (1.74)	73.37 (1.98)	.594	(30)	$T_1 = T_2$

Table 3: Experiment 3: Segregation

2.4.2 *Results*

Four participants successfully completed this experiment. Their results are reported in Table 3. In that table, the form of segregation is given by l and r for left and right segregation respectively (i has, by design, the same value).

Furthermore, T_1 and T_2 stand for the means of t_i and t'_i respectively; standard deviations are given in parentheses. The value n is the number of observations of t_i, t'_i, and w_i; n_z is the number of observations made of z_i. Statistical results are indicated by corresponding p-values.

The property held in four out of five cases for both $p = \frac{2}{3}$ and $p = 2$, although the $p = 2$ case is almost significant for P24 as well. That is, binary segregation was not rejected in eight (or seven) out of 10 tests.

2.4.3 *Discussion*

An open question is whether people perform production judgments of $v_i = x \circ_{p,i} y$ in, qualitatively, the same fashion for $y > 0$ as for $y = 0$. Substantial intersession variation in the relationship between the two production types was observed. Although practice seemed to alleviate this problem, the observation supports the notion that participants do not perform the two production judgments equivalently. As previously mentioned, this situation has been remedied in Luce (2003b), but those results came too late to influence most of our experiments (concerning this issue, see the “Conclusions” Section of Steingrimsson and Luce 2003a).

Meanwhile, given the challenges the testing of this property provides, reasonable initial support has been established for segregation.

Ex. #	Name	#P	#Tests	#Fail
1	Bisymmetry	6	6	0
2	JP decomp.	4	8	2
3	Segregation	4	10	2

Table 4: Summary of experimental results

3 Conclusions

The topic has been a theory of global psychophysical judgments leading to the two representation classes. For asymmetric matches, the following three properties are satisfied:

$$\begin{aligned}\Psi(x,u) &= \Psi(x,0)+\Psi(0,u)+\delta\Psi(x,0)\Psi(0,u) \quad (\delta \geq 0)\\ \Psi(x,0) &= \gamma\Psi(0,x) \quad (\gamma > 0),\\ \Psi\left[(x,u)\circ_p(y,v)\right]-\Psi(y,v) &= \left[\Psi(x,u)-\Psi(y,v)\right]W(p).\end{aligned}$$

For symmetric matches, the first equation with $\delta = 0$ and the third equation both hold, but that case does not predict the constant bias of the second equation.

These representations have a number of necessary consequences (behavioral properties) that in turn are sufficient under the structural assumptions made to give rise to the representations. These properties were tested. Our central conclusion from this empirical evaluation of Luce's (2002; 2003a) theory is that it has received reasonable initial support in the auditory domain. We are more specific in the next subsection.

An important message of this and the earlier article is that one can study the adequacy of a representation in which there are both free functions and free parameters without estimating either the unspecified functions or the parameters and then carrying out some form of goodness-of fit. The latter approach is commonly followed, with considerable difficulty, in much of cognitive modeling. The behavioral properties we tested are all parameter free. The third article will continue in this vein where we attempt to arrive at the mathematical form of psychophysical and weighting functions, up to a few parameters. Again, we find behaviorally equivalent properties that do not entail estimating or fitting anything. In the light of this, we develop a relatively straightforward test of the constant-ratio property (5)

3.1 Summary of the linking results

The test results are summarized in Table 4.

Within the framework of the theory, the property $\delta = 0$ is equivalent to bisymmetry holding for all three induced operations. That prediction was strongly supported (Experiment 1). Accepting this made it possible to assume $\delta = 0$ in designing tests for simple joint-presentation decomposition (Experiment 2) and segregation (Experiment 3). Both properties were partially sustained with 2 failures in 8 and 10 tests, respectively.

3.2 Further work

Sufficient support for the theory has been found in the two articles to suggest that further theoretical and experimental work be undertaken. Some was described in the earlier article (Steingrimsson and Luce, 2003a) including the possibility of testing the theory in other domains such as vision; here we cite an additional direction suggested by the present experiments, by some informal observations, and by parallel evidence from utility theory.

The observation is that participants seem to process $(x,x) \circ_{p,i} (y,y)$, $y > 0$, differently from $(x,x) \circ_{p,i} (0,0)$. Similar observations have been made in the context of utility theory, e.g. Birnbaum (1997). One can imagine that this might take the form of the participants distorting p differently in the two cases. So, for example, this might lead to replacing (6) by

$$\frac{\Psi\left[(x,x) \circ_p (y,y)\right] - \Psi(y,y)}{\Psi(x,x) - \Psi(y,y)} = \begin{cases} W(p) & y > 0 \\ W_0(p) & y = 0 \end{cases} .$$

This representation and a generalization of it are the topics of Luce (2003b). A major feature of the case of $y > 0$ is that the property of segregation is replaced by the following two forms of distribution:

Left distributivity:

$$z \oplus_i (x \circ_{p,i} u) \sim (z \oplus_i x) \circ_{p,i} (z \oplus_i u) \quad (u > 0). \tag{37}$$

Right distributivity:

$$(x \circ_{p,i} u) \oplus_i z \sim (x \oplus_i z) \circ_{p,i} (u \oplus_i z) \quad (u > 0). \tag{38}$$

Segregation is the special case where $u = 0$ is permitted. Again, in the unbiased case no distinction exists between the left and right version. For $i = l, r$, one can deduce them from the representation. For $i = s$ and with both versions of segregation holding for $\oplus_s$, one can show that both left and right distributivity are satisfied.

These theoretical results need to be explored more carefully in the psychophysical context and additional empirical studies, including left distributivity for $\oplus_l$ and right distributivity for $\oplus_r$ and both for $\oplus_s$, conducted.

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Appendices

A Commutativity and associativity

In the unbiased case, one can easily show from (18) that the following two conditions are met:

Commutativity (or symmetry) of $\oplus_i$:

$$x \oplus_i u = u \oplus_i x. \tag{39}$$

Associativity of $\oplus_i$:

$$x \oplus_i (y \oplus_i z) = (x \oplus_i y) \oplus_i z. \tag{40}$$

Although, as we saw in Steingrimsson and Luce (2003a), a substantial proportion of participants exhibited bias as, in effect, measured by commutativity, for three of them we did not reject the hypothesis of being unbiased and so commutativity appears to hold. However, as Zimmer et al. (2001) showed in their work, associativity was rejected in those cases when commutativity was not rejected. Thus, associativity is an important further test to carry out when trying to decide if a person is actually unbiased.

B Testing bisymmetry using two-ear matches

The object is to show that bisymmetry can be tested using two-ear matching. Bisymmetry is given as

$$(x \oplus_i y) \oplus_i (u \oplus_i v) \sim (x \oplus_i u) \oplus_i (y \oplus_i v).$$

We suppose that $\delta = 0$, which must be true if bisymmetry holds. Define

$$\begin{array}{rcl} (x, y) & \sim & (w, w),\ (u, v) \sim (w', w'),\ (w, w') \sim (t, t), \\ (x, u) & \sim & (z, z),\ (y, v) \sim (z', z'),\ (z, z') \sim (t', t'). \end{array}$$

We show that bisymmetry of $\oplus_i$ is equivalent to $t = t'$.

Using (26), we see that with $\eta(i) := \mu_l(i) / [\mu_l(i) + \mu_r(i)]$,

$$\begin{array}{rcl} \psi_i(w) & = & \eta(i)\psi_i(x) + [1 - \eta(i)]\psi_i(y), \\ \psi_i(w') & = & \eta(i)\psi_i(u) + [1 - \eta(i)]\psi_i(v), \\ \psi_i(t) & = & \eta(i)\psi_i(w) + [1 - \eta(i)]\psi_i(w'). \end{array}$$

Starting at the bottom and substituting

$$\psi_i(t) = \eta(i)^2\psi_i(x) + \eta(i)[1 - \eta(i)][\psi_i(y) + \psi_i(u)] + [1 - \eta(i)]^2\psi_i(v).$$

The analogous expression is

$$\psi_i(t') = \eta(i)^2\psi_i(x) + \eta(i)[1 - \eta(i)][\psi_i(u) + \psi_i(y)] + [1 - \eta(i)]^2\psi_i(v).$$

So, by the commutativity of + and the strict monotonicity of ψ_i, $t = t'$.

C Using two-ear matching to test simple joint-presentation decomposition when $\delta = 0$

The object is to show that when $\delta = 0$ we may test simple joint-presentation decomposition using a two-ear matching and production judgments. For the sake of simplifying the exposition, the proof is demonstrated with the use of the $\oplus_l$ and $\circ_{p,l}$ operators only; the argument using $\oplus_r$ is similar.

The simple joint-presentation decomposition hypothesis is for $i = l, r, s$,

$$(x \oplus_i u) \circ_{p,l} 0 \sim (x \circ_{p,i} 0) \oplus_l (u \circ_{p,i} 0).$$

Define

$$\begin{aligned} x \oplus_i u &\sim z \oplus_i z, \\ (v \oplus_i v) \circ_{p,i} (0 \oplus_i 0) &\sim t \oplus_i t, \\ (x \oplus_i x) \circ_{p,i} (0 \oplus_i 0) &\sim w \oplus_i w, \\ (u \oplus_i u) \circ_{p,i} (0 \oplus_i 0) &\sim s \oplus_i s, \\ w \oplus_i s &\sim t' \oplus_i t'. \end{aligned}$$

We use (26) and the separable representation

$$\psi_i(x \circ_{p,i} 0) = W(p)\psi_i(x).$$

Using these yields for $i = l, r, s$,

$$\begin{aligned} \psi_i(x \oplus_i u) &= [\mu_l(i) + \mu_l(i)]\, \psi_i(z), \\ \psi_i(z)W(p) &= \psi_i(t), \\ \psi_i(x)W(p) &= \psi_i(w), \\ \psi_i(u)W_i(p) &= \psi_i(s), \\ \psi_l(w \oplus_i s) &= [\mu_l(i) + \mu_l(i)]\, \psi_l(t'). \end{aligned}$$

So,

$$\begin{aligned} \psi_i(t) &= \psi_i(z)W(p) \\ &= \frac{\psi_i(x \oplus_i u)}{\mu_l(i) + \mu_r(i)} W(p) \\ &= \frac{\mu_l(i)\psi_i(x) + \mu_r(i)\psi_i(u)}{\mu_l(i) + \mu_r(i)} W(p) \\ &= \frac{\mu_l(i)\psi_i(x)W(p) + \mu_r(i)\psi_i(u)W(p)}{\mu_l(i) + \mu_r(i)} \\ &= \frac{\mu_l(i)\psi_i(w) + \mu_r(i)\psi_i(w)}{\mu_l(i) + \mu_r(i)} \\ &= \frac{\psi_i(w \oplus_i s)}{\mu_l(i) + \mu_r(i)} \\ &= \psi_i(t'), \end{aligned}$$

whence $t = t'$, proving that a two-ear method may be employed.