Additive Utility Representations of Gambles: Old and New Axiomatizations

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Abstract

A number of classical as well as quite new utility representations for gains are explored with the aim of understanding the behavioral conditions that are necessary and sufficient for various subfamilies of successively stronger representations to hold. Among the utility representations are: additive, weighted, rank-dependent (which includes cumulative prospect theory as a special case), gains decomposition, subjective expected, and independent increments*, where * denotes something new in this article. Among the key behavioral conditions are consequence monotonicity, idempotence, general status-quo event commutativity*, coalescing, gains decomposition, consequence-event substitutability*, and component summing*. The structure of relations is sufficiently simple that certain key experiments are able to exclude entire classes of representations. For example, the class of rank-dependent utility models is very likely excluded because of empirical results about the failure of coalescing. Figures 1-3 summarize all but two of the network of primary results.

Key Words: coalescing, component summing, consequence-event substitutability, event commutativity, gains decomposition, utility representations

Economics Classification: D46, D81

Additive Utility Representations of Gambles

This article explores the relations among a variety of utility representations. Some of them are fairly classical, other axiomatizations are new. This undertaking involves partially summarizing and sharpening some existing results and proving seven new ones.

Section 1 states a number, although not all, of the representations of gambles that have arisen in the literature, and Section 2 summarizes three old results, with some improvements, as well two new one about the interrelations among the representations. These results are summarized as Figure 1 in Section 2.7. Section 3 adds the concept of joint receipts and in terms of that defines a number of additional properties. The distinction between the joint receipt of subjectively independent gambles and of the joint receipt of totally dependent ones is explored. The result in the independent case entails a sharpening of an existing result. A crucial new one in the totally dependent case, presented in Section 3.1.2, uses component summing, a condition which, although apparently innocent, turns out to be surprisingly restrictive. These results are summarized in Figure 2 in Section 3.6. Section 4 introduces a new class of representations, increasing utility increments, relates it to other representations, to the concepts of segregation and distributivity, and axiomatizes it. Some of those results are summarized in Figure 3 in Section 4.5. Finally, Section 5 provides a brief recapitulation and states four open problems.

1 Classes of Representations of Gambles

1.1 Notation

Let X denote the set of pure consequences for which chance plays no role. A distinguished element $e \in X$ is interpreted to mean *no change from the status quo*. We assume a *preference order* $\succsim$ exists over X and that it is a weak order. Let $\sim$ denote the corresponding indifference relation. A typical *first-order gamble of gains* g with n consequences is of the form

$$g = (x_1, C_1; \cdots; x_i, C_i; \cdots; x_n, C_n) \equiv (\cdots; x_i, C_i; \cdots),$$

where $x_i \succsim e$ are consequences (of gains) and $\mathbf{C}_n = (C_1, \cdots, C_i, \cdots, C_n)$ is an ordered partition of some "universal event" $C(n) = \bigcup_{i=1}^{n} C_i$, where the word "order" refers to the order in which (consequence, event) pairs are written in the description of the gamble [see (1) below]. The underlying event $C(n)$ is only "universal" for the purpose of this gamble. Such a gamble is an ordered $2n$-tuple.

We intentionally do not follow the Savage (1954) route of having a common "state space" underlying all gambles, and for that reason some economists have criticized our approach. We contend that there is nothing holy in Savage's formulation and that ours has some distinct advantages if one wishes to account for observed behavior. First, the reader should attempt to explicitly formalize the state space corresponding to the several decisions made during an ordinary day of his or her life. Better yet, consider a year. It is unmercifully complex and hardly anyone approaches decisions in this fashion. Second, and closely related, when running experimental studies, the gambles are presented to the respondent in our form, not Savage's. Third, the state space approach builds in some properties that are quite restrictive. This point is discussed more fully in Section 2.6 where we argue that if one is interested in potentially descriptive theories, one should avoid Savage's formulation.

This article is confined to all gains (or, equally, to all losses). We assume X is so rich that for any first-order gamble g, there exists an $x \in X$ such that $x \sim g$. In that case we denote x by $CE(g)$ and refer to it as the *certainty equivalent* of g. Thus, the preference order $\succsim$ can be extended to the domain of gains $\mathcal{D}_+$ that consists of X and all first-order gambles. For a few results we need to expand $\mathcal{D}_+$ to include *second-order gambles* in which some of the x_i are replaced by first-order gambles. It turns out that under the usual monotonicity assumptions, all second-order gambles can be reduced to first-order ones by using certainty equivalents. We do not bother to write things in that fashion.

We also assume that if ρ is a permutation of the indices $\{1, 2, ..., n\}$, then

$$\begin{aligned} & (x_{\rho(1)}, C_{\rho(1)}; \cdots ; x_{\rho(i)}, C_{\rho(i)}; \cdots ; x_{\rho(n)}, C_{\rho(n)}) \\ \sim \quad & (x_1, C_1; \cdots ; x_i, C_i; \cdots ; x_n, C_n). \end{aligned} \tag{1}$$

Sometimes it is convenient to assume that we have carried out a permutation of the indices such such that the consequences are ordered from the most preferred to the least preferred, in which case we simply use the notation

$$x_1 \succsim \cdots \succsim x_i \succsim \cdots \succsim x_n \succsim e.$$

If this ordering is significant in stating a representation, we give that representation the same name as the corresponding unordered one, but with the prefix "ranked" added, and the abbreviations also prefixed with R. We do not explicitly include the relevant definitions for the ranked cases. In general, when people empirically test theories involving ranked consequences, such as RDU below, they present the gambles in the ordered form. But always keep in mind that so long as (1) is satisfied, the ranked form is nothing but a convenience for writing the representation or for writing an axiom leading to a ranked representation.

Note that the following property is perfectly reasonable: a gamble with $n > 2$ consequences that is degenerate in the sense that one of the events C_i is the null event $\varnothing$ is actually a gamble with $n - 1$ consequences in the following sense:

$$\begin{aligned} & (x_1, C_1; \cdots ; x_{i-1}, C_{i-1}; x_i, \varnothing; x_{i+1}, C_{i+1}; \cdots ; x_n, C_n) \\ & \sim (x_1, C_1; \cdots ; x_{i-1}, C_{i-1}; x_{i+1}, C_{i+1}; \cdots ; x_n, C_n). \end{aligned} \tag{2}$$

Properties (1) and (2) are examples of what have been called accounting indifferences (Luce, 2000). Their basic feature is that the bottom line on the two sides of $\sim$ are identical, and so normatively they should be equal. We will introduce a number of additional ones in Section 2.

We explore utility representations U onto real intervals of the form $I = [0, \kappa[$, where $\kappa \in]0, \infty]$, that meet various, increasingly stronger, restrictions. Two conditions that are common to all representations in this article are:

$$g \succsim h \quad \text{iff} \quad U(g) \geq U(h), \tag{3}$$
$$U(e) = 0. \tag{4}$$

We refer to these as *order-preserving representations.* Note that because I is open on the right, there is no maximal element in the structure.

1.2 Additive utility representations

Definition 1 *An order-preserving representation $U : \mathcal{D}_+ \overset{onto}{\longrightarrow} I$ is an* ***additive utility (AU)*** *one iff, for all $x_i \in X$ ($x_i \succsim e$) and for every ordered partition $\mathbf{C}_n$ of $C(n)$, there exist strictly increasing functions $L^{(i)}_{\mathbf{C}_n} : I \overset{onto}{\longrightarrow} I$, with the following properties:*

$$\begin{aligned} U(x_1, C_1; \cdots; x_i, C_i; \cdots; x_n, C_n) &= \sum_{i=1}^{n} L^{(i)}_{\mathbf{C}_n}(U(x_i)), \qquad (5) \\ L^{(i)}_{\mathbf{C}_n}(0) &= 0, \\ C_i = \varnothing \text{ implies } L^{(i)}_{\mathbf{C}_n}(z) &= 0 \ (z \in X). \end{aligned}$$

It is a ***ranked additive utility (RAU)*** *representation iff the above holds for all $x_1 \succsim \cdots \succsim x_i \succsim \cdots \succsim x_n \succsim e$, rather than for all $x_i \in X$ ($x_i \succsim e$)*

Because of (1), there is no loss of generality in the above restriction of the statement of the ranked representation to the case of partitions ordered by $x_1 \succsim \cdots \succsim x_i \succsim \cdots \succsim x_n \succsim e$.

Throughout the paper, we assume that everything holds for all k, $k \leq n$, where n is the set size mentioned in the relevant statement, such as the above definition. This assumption is stronger than is needed in general, but it does cover all the relevant cases.

Note that in Def. 1, we assume that $C_i = \varnothing$ implies $L^{(i)}_{\mathbf{C}_n}(z) = 0$ ($z \in X$), whereas, for simplicity, we restrict the proofs of the relevant representation theorems to partitions with all nonnull sets. However, we can adjoin (2) to the assumptions to obtain: if $C_i = \varnothing$, and $C_j \neq \varnothing, j = 1, ..., i-1, i+1, ..., n$, then

$$L^{(j)}_{(C_1,..,C_{i-1},\varnothing,C_{i+1},\cdots,C_n)} = L^{(j)}_{(C_1,..,C_{i-1},C_{i+1},\cdots,C_n)} \quad (j \neq i),$$

and so gambles over n consequences with one null event are determined by gambles over $n-1$ consequences with no null event; in this way, the representations can be extended to all gambles with one or more null events. Therefore, we assume (2), and thus such extensions, throughout the paper.

An axiomatization of AU is given in Theorem 8 in Section 2.1.

Given (5), one would like to know of specific behavioral constraints that limit it to the particular forms found in the literature including rank-dependent utility, subjective expected utility, and the several descriptive configural weight models, such as RAM and TAX of M. H. Birnbaum and his colleagues (for summaries, see Birnbaum, 1997, 1999). Some results along these lines are known[1] including the fact that the TAX model is equivalent to the idempotent weighted utility representation of the next subsection.

1.3 Weighted utility representations

A concept of a weighted utility, WU, representation, is defined in Marley and Luce (2001). Earlier use of the term subjectively weighted utility by Karamarkar (1978) is a special case of our usage. Included in the WU class is the general class of rank-dependent utility representations and others as well, such as one based on the concept of gains decomposition first defined in Liu (1995). The form of the WU representation is given by:

Definition 2 *An order-preserving representation* $U : \mathcal{D}_+ \xrightarrow{onto} I \subseteq \mathbb{R}_+$ *is a* ***weighted utility (WU)*** *one iff there exists weights* $S_i(\mathbf{C}_n)$ *assigned to each index* $i = 1, ..., n$ *and possibly dependent on the entire partition* $\mathbf{C}_n$*, where* $0 \leq S_i(\mathbf{C}_n) \leq 1$ *and* $S_i(\mathbf{C}_n) = 0$ *iff* $C_i = \varnothing$*, such that*

$$U(\cdots; x_i, C_i; \cdots) = \sum_{i=1}^{n} U(x_i) S_i(\mathbf{C}_n). \tag{6}$$

Note that the multiplicative weights of (6) depend both on i and on the entire event partition $\mathbf{C}_n$. Other representations will state limitations on the latter dependence.

Consider a gamble with $y_1 = \cdots = y_i = \cdots = y_n = y$ and consider the following property:

Definition 3 ***Idempotence*** *in consequences is satisfied iff, for every* $y \in X$ *and every event ordered partition* $(C_1, \cdots, C_i, \cdots, C_n)$*,*

$$(y, C_1; \cdots; y, C_i; \cdots; y, C_n) \sim y. \tag{7}$$

If we assume idempotence along with WU, we see that

$$\sum_{i=1}^{n} S_i(\mathbf{C}_n) = 1. \tag{8}$$

Most theories of utility, including the WU one described in Marley and Luce (2001), have either explicitly or implicitly assumed idempotence. Luce and

[1] Marley, A. A. J., R. D. Luce (2003). "Independence properties vis-à-vis several utility representations," in preparation.

Marley (2000) present some theoretical results for non-idempotent binary gambles. To our knowledge, no related theoretical work has been done on failures of idempotence in general gambles.

Here we will, insofar as we know how, define concepts and arrive at results without assuming idempotence, and then impose that property separately.

Weighted utility, (6), is a special case of AU. We assume that $S_i(\mathbf{C}_n) = 0$ iff $C_i = \varnothing$. For $C_i \neq \varnothing$, then $S_i(\mathbf{C}_n) > 0, i = 1, ..., n$, which means that the representation is strictly increasing in each consequence. In that case, WU is simply the additive form of (5) in which the functions $L^{(i)}_{\mathbf{C}_n}$ are all linear, i.e., $L^{(i)}_{\mathbf{C}_n}(Z) = ZS_i(\mathbf{C}_n)$.

We discuss axiomatizations of the WU form in Sections 2.2 and Section 3.2.

The ranked WU representation can be rewritten in a form that shows its relation to the rank-dependent form described in Section 1.4, and in another way that shows its relation to the increasing utility increment form discussed in Section 4.

Proposition 4

1. *The following forms are equivalent when $x_1 \succsim \cdots \succsim x_n \succsim e$:*

   (i) *Ranked-weighted utility, (6).*

   (ii)

$$U(\cdots; x_i, C_i; \cdots) = \sum_{i=1}^{n} U(x_i) \left[W_i(\mathbf{C}_n) - W_{i-1}(\mathbf{C}_n)\right], \tag{9}$$

   where

$$W_i(\mathbf{C}_n) = \begin{cases} 0, & i = 0 \\ W_{i-1}(\mathbf{C}_n) + S_i(\mathbf{C}_n), & 0 < i \leq n \end{cases}.$$

2. *In the idempotent case, RWU is equivalent to each of the following:*

   (i) *The form (9) satisfying (8).*

   (ii)

$$U(\cdots; x_i, C_i; \cdots) - U(x_n) = \sum_{i=1}^{n-1} \left[U(x_i) - U(x_n)\right] S_i(\mathbf{C}_n) \tag{10}$$

   (iii)

$$U(\cdots; x_i, C_i; \cdots) - U(x_n) = \sum_{i=1}^{n-1} \left[U(x_i) - U(x_{i+1})\right] W_i(\mathbf{C}_n). \tag{11}$$

The equivalence of idempotent RWU, (6) and (8), to the form of (9) was established in Marley and Luce (2001) but it is easily generalized to the non-idempotent case. The equivalence of idempotent RWU and (10) follows by a simple calculation using (8). And the equivalence of idempotent RWU and (11) is also a simple calculation. The form of (10) is likely to be of interest only if $x_i \succsim x_n$. The forms (9) and (10) are, respectively, related to the representations of rank-dependent utility and increasing utility increments discussed in Section 4.

1.4 Rank-dependent utility

The ranked WU representation, (6), with $x_1 \succsim \cdots \succsim x_n \succsim e$, is of interest because it encompasses several models in the literature including the standard rank-dependent model. In the RDU model the weights are expressed only in terms of weights from the binary case, in which case all of the weights are of the form $W_1(C(i), C(n)\backslash C(i))$ which we chose to write as

$$W_{C(n)}(C(i)) := W_1(C(i), C(n)\backslash C(i))$$

because knowing $C(n)$ and $C(i)$ tells us what the partition is. This notation is similar to notation that is often used. For example, when $n = 3$, and (C, D, E) is an ordered partition of $C \cup D \cup E$, the RDU model, which is assumed to be idempotent, gives

$$\begin{aligned} &U(x, C; y, D; z, E) \\ &\quad = U(x)W_{C\cup D\cup E}(C) + U(y)[W_{C\cup D\cup E}(C \cup D) - W_{C\cup D\cup E}(C)] \\ &\qquad + U(z)[1 - W_{C\cup D\cup E}(C \cup D)]. \end{aligned}$$

Define

$$C(i) := \bigcup_{j=1}^{i} C_j. \tag{12}$$

Definition 5 ***Rank-dependent utility***[2] ***(RDU)*** *is the special case of RWU in the form (9) with weights of the form*

$$W_i(\mathbf{C}_n) = \begin{cases} 0, & i = 0 \\ W_{C(n)}\left(C(i)\right), & 0 < i < n \\ 1, & i = n \end{cases}.$$

[2] In much of the literature since Gilboa and Schmeidler (1989) and Wakker (1990) this is called Choquet expected utility because the weights are of a form studied in Choquet (1953). John Quiggin, the originator of the representation in Quiggin (1982), which he originally called anticipated utility, came to call it rank-dependent expected utility in Quiggin (1993). Several others, including the authors, use the term rank-dependent utility without the adjective "expected." With certain additional assumptions it has also been called cumulative prospect theory for gains (Tversky & Kahneman, 1992).

Given this definition, RDU is idempotent. One can experiment with a non-idempotent generalization by replacing 1 for $i = n$ by $W_{C(n)}\left(C(n)\right)$. We do not pursue this generalization. Since in any idempotent case, we have

$$1 = \sum_{i=1}^{n} S_i(\mathbf{C}_n),$$

then in the RDU case we have

$$1 = \sum_{i=1}^{n} [W_{C(n)}(C_i) - W_{C(n)}(C_{i-1}) = W_{C(n)}(C(n)).$$

Given the RWU representation, a natural next question is what restriction is equivalent to the RDU form. The result, Theorem 12 below, was proved in Luce (1998), but the proof we include here is simpler.

1.5 Gains-decomposition utility

A second class of idempotent ranked WU models was proposed in Marley and Luce (2001).

Definition 6 *Within the domain of second-order gambles of gains, $\mathcal{G}_{2,+}$, a* ***gains-decomposition utility*** *(GDU) representation holds iff ranked WU holds for a family of binary weights $W_{C(i)}, i = 1, ..., n$, with $C(n)$ the universal event, and with the weights $W_i(\mathbf{C}_n)$ in (9) given by:*

$$W_i(\mathbf{C}_n) = \begin{cases} 0, & i = 0 \\ \prod_{j=i}^{n-1} W_{C(j+1)}\left(C(j)\right), & 1 \leq i < n \\ 1, & i = n \end{cases} . \tag{13}$$

As stated, the GDU representation is idempotent. One can consider the non-idempotent generalization mentioned in connection with RDU.

This representation, although it arises quite naturally as we shall see, has received hardly any attention. Because RDU is almost certainly not adequate descriptively (see Section 2.3), this one bears more examination (see Section 2.4 and the article listed in footnote 1).

1.6 Subjective expected utility

The following special case of RWU is perhaps the most thoroughly explored and used utility representation, both normatively and prescriptively, despite the fact it is far from descriptive and is far from the only one that can reasonably contend for the normative title "rational."

Definition 7 ***Subjective expected utility (SEU)*** *is the special case of idempotent WU where*

$$S_i(\mathbf{C}_n) = W_{C(n)}(C_i), \tag{14}$$

with the weights $W_{C(n)}$ finitely additive.

If we begin with the RDU model, we see that (14) actually implies that the weights are finitely additive:

$$\begin{aligned} W_{C(n)}(C_i) &= S_i(\mathbf{C}_n) \\ &= W_i(\mathbf{C}_n) - W_{i-1}(\mathbf{C}_n) \\ &= W_{C(n)}\left(C(i)\right) - W_{C(n)}\left(C(i-1)\right) \\ &= W_{C(n)}\left(C_i \cup C(i-1)\right) - W_{C(n)}\left(C(i-1)\right), \end{aligned}$$

which is equivalent to

$$W_{C(n)}\left(C_i \cup C(i-1)\right) = W_{C(n)}(C_i) + W_{C(n)}\left(C(i-1)\right).$$

Because both C_i and $C(i-1)$ are arbitrary disjoint sets, this means that the binary weights $W_{C(n)}$ are finitely additive.

The SEU utility form was first made very prominent by Savage (1954) who, however, assumed a common universal set (state space) for all gambles (or *acts* as he called them) (see Section 2.6). Many subsequent axiomatizations have been given; for a summary see Fishburn (1988).

1.7 On alternative approaches

A substantial part of the literature follows the tack taken by Anscombe and Aumann (1963) who augmented the underlying structure by supposing that there exists a dense set of events with known probabilities. Indeed, for each $\lambda \in]0,1]$ and each nonnull event C, one assumes there is an event $C_\lambda \subseteq C$ such that $\Pr(C_\lambda | C)$ is defined and is equal to λ. In particular, the second-order binary lotteries so generated all exist and are denoted

$$\lambda f + (1-\lambda)g := (f, C_\lambda; g, C \backslash C_\lambda).$$

Most of the assumptions are formulated within this structure with little direct emphasis on the uncertain or vague events, but the theory induces indirectly weights on such events. Some of the axiomatizations lead to forms we have mentioned, such as rank-dependent (Choquet expected) utility. Others do not. Perhaps the most important variants are those for which the weights are of a minmax character: one assigns to each event a weight that corresponds to the minimum value of the probabilities of that weight from the several distributions in some family of probability distributions. The idea is that the decision maker thinks the distributions in the family are all possible and selects the one that assigns the smallest probability as the weight to use. Some of the relevant papers are Casadesus-Masanell, Klibanoff, Ozdenoren (2000), Ghirardato and

Marinacci (2002), Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2003), Gilboa and Schmeidler (1989), and Schmeidler (1989).

It may be useful to compare briefly the differences in domains postulated in the several approaches to decision making under risk and uncertainty. The seminal work of von Neumann and Morgenstern (1947) explicitly axiomatized the representation of binary gambles, but included an assumption about compound lotteries that extended the representation to lotteries of any size. They also assumed events with known probabilities—risk. They used those probabilities to construct a utility function that satisfied the EU representation, which is SEU where $W_{C(n)}(C_i) = \Pr(C_i\,|C(n)\,)$. The SEU representation of Savage (1954) assumed a (huge) state space for which probabilities were not specified although he was led to assume what amounted to very fine partitions into nearly equal subjective probabilities. The state space was assumed to encompass any conceivable uncertain events over which one might construct gambles. Those following the tradition of Anscombe and Aumann (1963) retained the Savage state space but augmented it with some of the probability structure assumed by von Neumann and Morgenstern (1947). Using that they inferred subjective probabilities on the other events. The present approach as well as that described in Luce (2000) deals with what amounts to conditional gambles constructed from the event space of a specific "experiment," not on the fixed space of states of nature. This is similar in spirit to von Neumann and Morgenstern (1947), but without assuming that chance events have assigned probabilities. We feel that conditional gambles are far more easily identified with empirical situations than are universal state spaces, and avoiding determinate probabilities means that the theory encompasses the many decision makers who do not have a very firm grip on classical probability in any context or lack the relevant information to estimate probabilities.

2 Results on Representations of Gambles

2.1 Axiomatization of additive utility

The standard result on additive conjoint measurement, Krantz, Luce, Suppes, and Tversky (1971), or in the case of ranked consequences [see (1)] that of Wakker (1991) are sufficient to prove:

Theorem 8 *Suppose that there exists* $U : \mathcal{D}_+ \overset{onto}{\longrightarrow} I$ *that is order preserving, (3), and* $U(e) = 0$, *and that the following conditions are satisfied:* $\succsim$ *is a weak order on general gambles with* $n \geq 2$ *consequences; gambles are strictly monotonic increasing over each subset of consequences*[3], *and for* $n = 2$ *the Thomsen condi-*

[3] Let $M \subset \{1, 2, ..., n\} = N$ and let $\succsim_{M,\mathbf{C}_n}$ denote the ordering induced on $\prod_{j \in M} X_j$ by $\succsim$ for fixed $\mathbf{C}_n$ and some choice of the x_i, $i \in N \backslash M$. The property is that, as the notation suggests, this induced order is independent of the latter choice. We explicitly assume that null events are excluded.

tion[4] *also holds. Then U may be selected so that there are increasing functions $L^{(i)}_{\mathbf{C}_n} : I \overset{onto}{\longrightarrow} I$, with $L^{(i)}_{\mathbf{C}_n}(0) = 0$ and if $C_i = \varnothing$, $L^{(i)}_{\mathbf{C}_n}(z) = 0$, such that the AU representation, (5), is satisfied or, if the consequences are ordered, RAU is satisfied. This representation is unique up to positive similarity transformation* $U \to \alpha U$, $\alpha > 0$.

2.2 First axiomatization of WU

2.2.1 Event commutativity

Let us suppose that the domain $\mathcal{D}_+$ is extended to include second-order compound gambles (see Section 1.1). Within that extended domain, one major property of binary gambles that has played a fairly key role is the concept of *binary event commutativity*: For all events C, C', D, D' with $C \cap C' = D \cap D' = \varnothing$, and $x \succ y \succsim e$

$$((x, D; y, D'), C; y, C') \sim ((x, C; y, C'), D; y, D'). \tag{15}$$

One sees that if one reduces this to a first-order gamble, it amounts to saying on each side that x is the consequence if C and D both occur, and otherwise y is the consequence, the only difference being the order in which the experiments (C, C') and (D, D') are conducted. The binary RDU model satisfies this property. If one has $y = e$ in (15) it is called *status-quo, binary event commutativity.* The question is how this concept might be generalized for gambles with more than two consequences. We explore one possible generalization that gives rise to the WU representation.

Definition 9 *Suppose $\mathbf{C}_n, \mathbf{D}_m$ are any two ordered event partitions and $x \succ e, y \succ e$. Then* ***status-quo event commutativity*** *is satisfied iff for every* $i, j \in \{1, 2, \cdots, n\}, k \in \{1, 2, \cdots, m\}$

$$\begin{aligned}&(\cdots; (\cdots; x, D_k; \cdots), C_i; \cdots; (\cdots; y, D_k; \cdots), C_j; \cdots)\\ &\quad \sim (\cdots; (\cdots; x, C_i; \cdots; y, C_j; \cdots), D_k; \cdots),\end{aligned} \tag{16}$$

where all of the other events have e attached to them.

As in the binary case, both sides give rise to x if both C_i and D_k occur, to y if both C_j and D_k occur, and to e, no change in the status quo, otherwise. This seems just as rational as binary event commutativity. In the binary case with $i = k = 1$, $j = 2$, $y = e$ it reduces to binary status-quo event commutativity. It has received empirical attention only in the binary case, where it seems supported (Luce, 2000, pp. 74-76).

Note that if we wish to put such a compound gamble into ranked form, there will be two permutations involved, one for the $\mathbf{C}_n$ partition and another for the $\mathbf{D}_m$ partition.

[4] Focusing on the consequences holding the partition $\{C_1, C_2\}$ fixed and suppressing it in the notation, $(x, v) \sim (u, z)$ and $(u, y) \sim (w, v)$ imply $(x, y) \sim (w, z)$.

A property called *timing indifference* of which status-quo event commutativity is a special case was formulated in Wang (2003). Because time is not an explicit variable in the theory, we prefer our term.

2.2.2 The result

Theorem 10 *Consider the structure $\langle \mathcal{D}_+, \succsim \rangle$ for gambles with $n \geq 2$. The following statements are equivalent:*

1. *The structure has an AU representation, Def. 1, and satisfies status-quo event commutativity, (16).*

2. *The structure has a WU representation, Def. 2.*

2.3 Axiomatization of RDU

2.3.1 Coalescing and the result

The RDU representation exhibits the following property:

Definition 11 ***Coalescing*** *is satisfied iff for all ordered partitions and ordered consequences $x_1 \succsim \cdots \succsim x_n \succsim e$, $n > 2$, with $x_{k+1} = x_k$, $k < n$:*

$$\begin{aligned}&(x_1, C_1; \cdots ; x_k, C_k; x_k, C_{k+1} \cdots ; x_n, C_n) \\ &\quad \sim (x_1, C_1; \cdots ; x_k, C_k \cup C_{k+1}; \cdots ; x_n, C_n) \quad (k = 1, \cdots , n-1). \quad (17)\end{aligned}$$

Note that the gamble on the right has $n-1$ consequences. Observe that if RDU obtains and simplifying the notation to $W := W_{C(n)}$, (17) follows from the fact that

$$\begin{aligned}&W\left(C(k+1)\right) - W\left(C(k)\right) + W\left(C(k)\right) - W\left(C(k-1)\right) \\ &\quad = W\left(C(k+1)\right) - W\left(C(k-1)\right) \\ &\quad = W\left(C_{k+1} \cup C_k \cup C(k-1)\right) - W\left(C(k-1)\right).\end{aligned}$$

The next result shows that coalescing is the key to RDU.

Theorem 12 *For $n > 2$, the following statements are equivalent:*

1. *RWU, (10), idempotence, Def. 3, and coalescing, (17), all hold.*

2. *RDU, Def. 5, holds.*

This was proved by Luce (1998); we give a somewhat simpler proof.

There is a large literature on ways to arrive at idempotent RDU (see, e.g., Luce, 2000, and Quiggin, 1993, for further discussion and references). None of these approaches seem simpler or more straightforward than our assumptions: transitivity, general consequence monotonicity, status-quo event commutativity, and coalescing. Somewhat related is Köbberling and Wakker (2003) who invoke weak ordering, weak monotonicity, co-monotonic Archimedean, and co-monotonic trade-off consistency axioms. These co-monotonic properties are quite different from anything in this article;dropping the co-monotonic requirement forces SEU.

2.3.2 Evidence against coalescing

We begin with one of the famous Ellsberg (1961) paradoxes. Recall that it arises from two pair of choices: A vs. B and A' vs. B' where

	Events		
	R	G	Y
A :	100	0	0
B :	0	100	0

and

	Events		
	R	G	Y
A' :	100	0	100
B' :	0	100	100

and $\Pr(R) = 1/3$, $\Pr(G \cup Y) = 2/3$. The probability of G, and so of Y, is not specified. People typically pick A over B and B' over A'. The usual argument suggesting that this result is paradoxical is to note that one can (rationally) ignore event Y because the consequences in each pair are identical given Y. Once the event Y is ignored, the remaining structure in each pair is identical.

We now show for the separable form with $e = 0$,

$$U(x, C; 0, D) = U(x)W_{C\cup D}(C),$$

that if choices conform to the Ellsberg paradox, then coalescing and a form of event monotonicity cannot both hold. Using coalescing,

$$\begin{aligned} A &\sim (100, R; 0, Y \cup G) \equiv (100, 1/3; 0, 2/3) \\ B &\sim (100, G; 0, R \cup Y) \\ A' &\sim (100, R \cup Y; 0, G) \\ B' &\sim (100, G \cup Y; 0, R) \equiv (100, 2/3; 0, 1/3). \end{aligned}$$

Using the separable form and suppressing the subscript on $W_{R\cup Y\cup G}$,

$$U(A) \gtreqless U(B) \text{ iff } W(R) \gtreqless W(G)$$

and

$$U(A') \gtreqless U(B') \text{ iff } W(R \cup C) \gtreqless W(G \cup C).$$

Thus, there is no paradox iff

$$W(R) \gtreqless W(G) \text{ iff } W(R \cup Y) \gtreqless W(G \cup Y), \tag{18}$$

which is a form of event monotonicity. Thus, as stated above, given the separable form, the Ellsberg paradox means that coalescing, (17), and event monotonicity, (18), cannot both hold. The usual interpretation has been that event monotonicity does not hold, which it need not in some RDU models (of course it does hold in the SEU model because W is finitely additive.) If the paradox continues to hold when the gambles are presented in coalesced form, then either separability or event monotonicity is at fault. We are not aware of such data.

Until recently, no serious attention has been paid to the alternative that coalescing might be the culprit. But in the past few years, Birnbaum has provided empirical evidence pointing to the strong possibility that coalescing is not descriptive. For instance, Birnbaum (2000) provides evidence that its failure underlies the famous Allais paradox. As a further example Birnbaum (1999), considered the trinary lotteries:

$$\begin{aligned} g &= (96, .90; 14, .05; 12, .05) \\ h &= (96, .85; 90, .05; 12, .10) \end{aligned}$$

Of 100 undergraduates, 70 chose h over g.

Now consider

$$\begin{aligned} g' &\sim (96, .85; 96, .05; 14, .05; 12, .05), \\ h' &\sim (96, .85; 90; .05; 12, .05; 12, .05). \end{aligned}$$

Then assuming consequence monotonicity, transitivity, and coalescing, we have

$$\begin{aligned} g' &\sim g \\ h' &\sim h. \end{aligned}$$

Clearly, g' dominates h'. When g' and h' were presented, 15 of the 100 undergraduates chose h'. Thus, if consequence monotonicity and transitivity hold, then coalescing fails.

Birnbaum[5] explores this sort of violation more thoroughly. Coalescing may be, psychologically, asymmetric in the following sense (Luce, 2000). Given the pairs x, C_k and x, C_{k+1} there is but one way to coalesce them. But given a pair x, D_k, usually there are many ways to partition it into x, D and $x, D_k \backslash D$ where $D \subset D_k$. The Ellsberg paradox uses coalescing in the easy direction, whereas many of Birnbaum's examples, such as the one described, involve the more diffuse direction, often called *event splitting*. But, conceptually, an event split from, say, a gamble g to a gamble g', is equivalent to coalescing g' to g. This can be seen in the g, g' pair of the Birnbaum example above

Taken together, these results suggest that coalescing may not be descriptively true thereby falsifying the class of RDU models. Note that this includes as special cases the popular Cumulative Prospect Theory (Tversky & Kahneman, 1992) and Subjective Expected Utility (Savage, 1954). These observations suggest an increasing focus on forms of WU that do not require coalescing. The next subsection discusses one example.

2.4 Axiomatization of gains decomposition utility

We repeat, without proof, the main finding about the GDU representation, Def. 6 above, as given in Marley and Luce (2001). To that end we need another

[5] Birnbaum, M. H. (2003). Gambles represented by natural frequencies: Violations of stochastic dominance, event-splitting, and cumulative independence. Under review.

property. Suppose $g_{\mathbf{C}_n}$ is a gamble with $n > 2$ consequences that is based on the ordered partition $\mathbf{C}_n$ of $C(n) = \bigcup_{i=1}^{n} C_i$. Here, as in Section 2.2, we must extend the domain $\mathcal{D}_+$ to include second-order gambles involving as consequences first-order ones as well as pure consequences. Define the following subgamble of $g_{\mathbf{C}_n}$:

$$g_{\mathbf{C}_{n-1}} := (x_1, C_1; \cdots; x_{n-1}, C_{n-1}). \tag{19}$$

Note that $g_{\mathbf{C}_{n-1}}$ is based on the sub-experiment with the universal event $C(n-1)$ (see (12)).

The following definition modifies slightly the terminology used in Luce (2000, Ch. 5) which, in turn, generalized to more general events a property introduced in Liu (1995) for known probabilities:

Definition 13 *Within the domain of second-order compound gambles of gains,* ***gains decomposition*** *holds iff for $g_{\mathbf{C}_n}$ with $x_1 \succsim \cdots \succsim x_n \succsim e$,*

$$g_{\mathbf{C}_n} \sim \left(g_{\mathbf{C}_{n-1}}, C(n-1); x_n, C_n\right), \tag{20}$$

where $\left(g_{\mathbf{C}_{n-1}}, C(n-1); x_n, C_n\right)$ is a compound binary gamble.

The right side is the compound gamble with universal event $C(n)$. If the outcome lies in the event $C_n = C(n) \backslash C(n-1)$, then the consequence is x_n. If, however, the outcome lies in $C(n-1)$, then the subgamble $g_{\mathbf{C}_{n-1}}$ is the consequence attached to it, and so the experiment with universal event $C(n-1)$ is next run to determine which consequence x_i, $i = 1, \ldots, n-1$, is received.

This is another accounting indifference and so is a rational property in the sense that the bottom lines associated with the two sides are identical, with the difference being whether one or two chance phenomena are carried out. In general, gains decomposition is not consistent with coalescing, although each property seems rational in its own right. Gains decomposition has not, to our knowledge, received any empirical study.

Theorem 14 *Suppose that there is a representation U of second-order compound gambles of gains with $n > 2$. Let $\mathcal{E}_D$ denote the family of subevents of the event D. Then:*

1. *The following are equivalent:*

(i) *Binary ranked WU, idempotence, Def. 3, and gains decomposition, Def. 13, all hold.*

(ii) *GDU for gains, Def. 6, holds.*

2. *Any two of the following imply the third:*

(i) *RDU is satisfied.*

(ii) *GDU is satisfied.*

(iii) *The binary weights satisfy the following* choice property*: For all events with $C \subseteq D \subseteq E$,*

$$W_{\mathbf{E}}(C) = W_{\mathbf{D}}(C) W_{\mathbf{E}}(D), \tag{21}$$

This result follows immediately from the proof of Theorem 3, Luce and Marley (2001), although we failed there to notice that the full statement of part 2, above, follows from that proof.

Equation (21) is just the choice axiom investigated by Luce (1959) under the assumption that the weights are finitely additive probabilities. As noted by Luce (2000, p. 78), the choice property simply says that the weights act somewhat like (subjective) conditional probabilities of the experiment $\mathbf{E}$. This would be fully the case if there were a universal set Ω that includes all experiments as subevents and on which there exists a function $W : \Omega \to [0,1]$ such that

$$W_{\mathbf{E}}(C) = \frac{W(C)}{W(D)}$$

Luce (2000, p. 77) noted also that, in the presence of binary WU, the choice property (21) is equivalent to the behavioral condition

$$((x, C; e, D\backslash C), D; e, E\backslash D) \sim (x, C; e, E\backslash C), \tag{22}$$

which is called *conditionalization.* Although conditionalization has a close family resemblance to gains decomposition, it does not follow from it unless coalescing holds.

No attempt has yet been made to subject the GDU model to empirical study.

We now show that the Ellsberg paradox, discussed relative to RDU and SEU, is surely not a problem for GDU. Apply the gains decomposition property to the various gambles of the Ellsberg paradox:

$$\begin{aligned} A &\sim ((100, R; 0, G), R \cup G; 0, Y) \\ B &\sim ((100, G; 0, R), R \cup G; 0, Y) \\ A' &\sim ((100, R, 100, Y), R \cup Y; 0, G) \\ B' &\sim ((100, G; 100, Y), G \cup Y; 0, R). \end{aligned}$$

By consequence monotonicity and GDU,

$$\begin{aligned} A \succsim B \quad & \text{iff } (100, R; 0, G) \succsim (100, G; 0, R) \\ & \text{iff } U(100) W_{R\cup G}(R) \geq U(100) W_{R\cup G}(G) \\ & \text{iff } W_{R\cup G}(R) \geq W_{R\cup G}(G). \end{aligned}$$

And, using gains decomposition,

$$\begin{aligned} A' \succsim B' \quad & \text{iff } U(A') \geq U(B') \\ & \text{iff } U(100, R; 100, Y) W_{R\cup G\cup Y}(RUY) \geq U(100, G; 100, Y) W_{R\cup G\cup Y}(GUY) \\ & \text{iff } W_{R\cup G\cup Y}(R \cup Y) \geq W_{R\cup G\cup Y}(GUY). \end{aligned}$$

Thus, the paradox does not occur provided

$$W_{R\cup G}(R) \geq W_{R\cup G}(G)$$

iff

$$W_{R\cup G\cup Y}(R\cup Y) \geq W_{R\cup G\cup Y}(G\cup Y).$$

Violation of this condition is not the same as violating (18). One can easily imagine these inequalities might be reversed because the weights are over different conditioning events. So the Ellsberg paradox does not automatically reject GDU.

2.5 Axiomatization of subjective expected utility

Luce (2000, p. 79) formulated sufficient conditions that restrict RDU to SEU; however, they were sufficiently strong that they implied the choice property (21). Here we present a better axiomatization by presenting a necessary and sufficient condition for RDU to reduce to SEU, i.e., for the binary weights to be finitely additive.

To this end, we make the assumption of *consequence solvability:* for any $z \succ e$, and any events C, D, C', D' with $C \cap D = C' \cap D' = \varnothing$, there exists z' such that

$$(z, C; e, D) \sim (z', C'; e, D').$$

In the following definition, x' and y' are the *solutions* of equivalences of the above form.

Definition 15 *Suppose that* $x \succ e, y \succ e$, *and* C, D, E *are mutually disjoint nonnull events as are* C', D', E'. *Consequent-event (C-E) substitutability is satisfied iff*

$$(x, C; e, D) \sim (x', C'; e, D') \quad \textit{and} \quad (y, D; e, C) \sim (y', D'; e, C') \tag{23}$$

imply

$$(x, C; y, D) \sim (x', C'; y', D') \tag{24}$$
$$(x, C; y, D; e, E) \sim (x', C'; y', D'; e, E') \tag{25}$$

Theorem 16 *Suppose that consequence solvability is satisfied. Then the following statements are equivalent:*

1. *RDU is satisfied and consequence-event substitutability holds.*

2. *SEU is satisfied.*

So far as we know, this is a new axiomatization of SEU.

2.6 On Savage's acts over states of nature

Many theoretical economists are quite content to use Savage's (1954) formulation of the space of decision alternatives. This views uncertain alternatives—acts—as functions, with finite support, from a universal set of elementary "states of nature" into a set of pure consequences. And some have been quite critical of the approach taken by most psychologists—us, for example—in which each decision alternative or gamble is defined over its own event structure. Earlier we mentioned the fact that any realistic set of decision alternatives tends to result in very large–billions of–states of nature. Here we make a further and key point.

As indicated above, in the Savage formulation, augmented in our work with a "natural" zero element e, all acts are simply a list of state-consequence pairs, with each state being elementary. In such a representation, there is no distinction between the two sides of the expressions that occur in an accounting indifference, such as event commutativity, coalescing, gains decomposition, and idempotence, and so they are all automatically satisfied in that formulation. But once these four conditions hold, our results force RDU (Def. 5), GDU (Def. 6), and the choice property, (21), provided that enough monotonicity conditions are assumed to get the AU representation of Theorem 9. This is because event commutativity then forces a WU representation (Theorem 10); coalescing forces a RDU representation (Theorem 12); and gains decomposition forces a GDU representation (Theorem 14.1); and the choice property follows (Theorem 14.2). It has not been previously recognized that the Savage structure leads to the restrictive choice property. To get from RDU to SEU, we found the necessary and sufficient condition of consequence-event substitutability (Def. 15, Theorem 16). Observe that the latter property, although a necessary condition of SEU, is not an accounting indifference. To our knowledge, it has not been studied empirically.

These facts strongly argue against Savage's description of decision situations unless one is content to arrive at RDU and GDU. Anyone interested in descriptive theories that deviate from these both holding should either shun the state-space formulation or admit some violations of consequence monotonicity. The argument against the Savage formulation would be even stronger were we able to find an accounting indifference that reduces RDU to SEU.

2.7 Summary

Figure 1 presents a summary of what we know using just properties of gambles.

Insert Fig. 1 here

3 Joint Receipts

In this section we extend the domain $\mathcal{D}_+$ to include the joint receipt of pure consequences and gambles. With X the set of pure consequences, for $x, y \in X$,

$x \oplus y \in X$ represents receiving or having both x and y. When X denotes money, many authors assume that $x \oplus y = x + y$, but as discussed in Luce (2000) this is certainly not necessary and may well be false. When f, g are gambles, then $f \oplus g$ means having or receiving both gambles. In this article, we assume $\oplus$ to be commutative, strictly increasing in each variable, with e its identity. Within the psychophysical context, Luce (2002, 2003a) has studied the non-commutative case.

The following concept of generalized additivity is familiar from the functional equation literature:

Definition 17 *The operation $\oplus$ has a **generalized additive representation** $U : \mathcal{D}_+ \overset{onto}{\longrightarrow} I$ iff (3), (4), and there exists a strictly increasing function φ such that*

$$U(f \oplus g) = \varphi^{-1}\left(\varphi\left(U(f)\right) + \varphi\left(U(g)\right)\right). \tag{26}$$

*It is called **additive** if φ is the identity.*

Note that $V = \varphi(U)$ is additive. Observe also that if $\oplus$ satisfies (26), then it is both commutative and associative and so

$$U(f_1 \oplus \cdots \oplus f_m) = \varphi^{-1}\left(\sum_{i=1}^{m} \varphi\left(U(f_i)\right)\right)$$

iff

$$V(f_1 \oplus \cdots \oplus f_m) = \sum_{i=1}^{m} V(f_i).$$

An important special case is when φ is the identity in which case U is additive over $\oplus$. This is, of course, a strong property. For example, if for money consequences $x \oplus y = x + y$, then additive U implies $U(x) = \alpha x$ and so $\kappa = \infty$, i.e., $I = [0, \infty[= \mathbb{R}_+$. For at least modest amounts of money–"pocket money"–this may not be unrealistic, as M. H. Birnbaum and collaborators [6] have argued by fitting data.

Another important special case of generalized additivity is for some $\delta \neq 0$,

$$U(f \oplus g) = U(f) + U(g) + \delta U(f)U(g), \tag{27}$$

which form has been termed *p-additivity.* This corresponds to the mapping $\varphi(z) = sgn(\delta)\ln(1 + \delta z)$ (see, for instance, Luce, 2000).

3.1 Independent and dependent gambles

In previous work involving joint receipt, it has been assumed, sometimes implicitly, that the gambles involved are based on, in some sense, *independent*

[6] Birnbaum and Beeghley (1997), Birnbaum, Coffey, Mellers, and Weiss (1992), Birnbaum and Sutton (1992) for judgment data, and Birnbaum and Chavez (1997) and Birnbaum and Navarrete (1998) for choice.

realizations of an underlying experiment. So if $\mathbf{C}_n$ and $\mathbf{D}_m$ are two ordered event partitions, and

$$f = (\cdots; x_i, C_i; \cdots), \quad g = (\cdots; y_j, D_j; \cdots),$$

then it is assumed when speaking of $f \oplus g$ that the experiments underlying $\mathbf{C}_n$ and $\mathbf{D}_m$ are run independently. With experiments whose statistical properties are not available, there is no easy formalization of exactly what "run independently" means. However, if what is relevant is the decision maker's behavioral manifestation of independence, then we are able to give the formalization below. But first we need a further restriction.

3.1.1 Decomposability

Most scientific models of utility assume that the utility U is decomposable in certain ways. We now formalize two such assumptions .

Definition 18 *Suppose there exists a representation* U*:* $\mathcal{D}_+ \overset{onto}{\longrightarrow} I = [0, \kappa[$, *with* $U(e) = 0$*. Then* U *is* ***decomposable over gambles*** *iff there exists a function* $G_{\mathbf{C}_n} : I \overset{onto}{\longrightarrow} I$ *that is strictly increasing in each variable such that for any gamble* $(x_1, C_1; \cdots; x_i, C_i; \cdots; x_n, C_n)$,

$$U(\cdots; x_i, C_i; \cdots) = G_{\mathbf{C}_n}(\cdots, U(x_i), \cdots). \tag{28}$$

Note that all of the representations of gambles we have examined are decomposable.

Definition 19 U *is said to be* ***decomposable over joint receipt*** *iff there is a function* $F : I \times I \overset{onto}{\longrightarrow} I$*, with* F *strictly increasing in each variable, such that*

$$U(f \oplus g) = F\left(U(f), U(h)\right) \quad (f, g \in \mathcal{D}_+). \tag{29}$$

Because we are assuming that $\oplus$ is commutative, we have $F(X, Y) = F(Y, X)$, and because e is an identity of $\oplus$, $F(X, 0) = F(0, X) = X$.

All this definition says is that the utility of a joint receipt of gambles depends on the consequences via their individual utilities $U(f)$ and $U(g)$. This is true of most extant theories of utility. In particular, it is true for the concept of generalized additivity given by (26).

3.1.2 Behaviorally independent gambles

In the world of objective probabilities two gambles (in the form of random variables) are independent if, in the distribution of the sum of the random variables, $x_i + y_j$ arises with probability $\Pr(C_i)\Pr(D_j)$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$. In statistics this distribution is called the convolution[7] of the other two. Within the domain of uncertain alternatives, something else must be used. To that end, we define:

[7] There is a theoretical literature that explores analogues of convolution for non-independent random variables. But independence usually underlies the concept of convolution used in statistics.

Definition 20 *Suppose that* $f = (x_1, C_1; ...; x_n, C_m)$ *and* $g = (y_1, D_1; ...; y_m, D_m)$ *are two gambles with, respectively, universal sets* $C(n) = \bigcup_{i=1}^{n} C_i$ *and* $D(m) = \bigcup_{j=1}^{m} D_j$. *Then the* ***reduced joint receipt***[8] $f \circledast g$, *is defined to be the first-order gamble*

$$f \circledast g := (...; x_i \oplus y_j, (C_i, D_j); ...),$$

where the universal set is $C(n) \times D(m)$.

We will be concerned, of course, with conditions when the two related concepts of joint receipt are in fact the same, i.e., when $f \oplus g \sim f \circledast g$ for all gambles f, g based on the given ordered partitions $\mathbf{C}_n$ and $\mathbf{D}_m$. We denote this property by $\oplus \approx \circledast$.

Suppose that there is a utility function U such that the WU representation, (6), holds with weights $S_i(\mathbf{C}_n)$, $S_j(\mathbf{D}_m)$, and $S_{i,j}(\mathbf{C}_n \times \mathbf{D}_m)$ for, respectively, f, g, and $f \circledast g$. Then one possible notion of *behavioral independence*, which is certainly the most conservative, is that

$$S_{i,j}(\mathbf{C}_n \times \mathbf{D}_m) = S_i(\mathbf{C}_n) S_j(\mathbf{D}_m). \tag{30}$$

If all of the gambles under consideration are behaviorally independent, then within the context of weighted utility representations we see that for every f, g, $f \circledast g \sim f \oplus g$. As we note in the next theorem, the converse is not quite true.

This is by no means a fully satisfactory solution to the issue of "independent events," but we do not know how to formulate that concept formally when we do not have probabilities.

The following modifies and improves Marley and Luce (2001, Theorem 5), but is stated here for the WU representation rather than the RWU representation. The general case simplifies both the statement and its proof. In this result we will arrive at several restrictions on the weighting functions. In addition to (30), we will also arrive at

$$S_i(\mathbf{C}_n) = \sum_{l=1}^{m} S_{i,l}(\mathbf{C}_n \times \mathbf{D}_m), \tag{31}$$

$$S_j(\mathbf{D}_m) = \sum_{k=1}^{n} S_{k,i}(\mathbf{C}_n \times \mathbf{D}_m), \tag{32}$$

Note that in the idempotent case (30) implies both (31) and (32).

Theorem 21 *Suppose that* U *is a WU representation onto the real interval* $[0, k[$ *and that* U *is decomposable for joint receipts; that gambles are strictly monotonic in the consequences; that joint receipts are strictly increasing in each variable; that* e *is an identity of* $\oplus$; *and that* $\oplus$ *is commutative. Then:*

[8] This operation $\circledast$ was called "qualitative convolution" in Marley and Luce (2001). We have since concluded that this was a poor choice of term because no form of independence is involved, which is basic in the statistical concept of convolution as the distribution of the sum of two independent random variables. So we withdraw that term, and simply call it reduced joint receipt.

1. *If $\oplus \approx \circledast$, then (31) and (32) both hold.*

2. *If (31) and (32) both hold and U is additive over $\oplus$, then $\oplus \approx \circledast$.*

3. *Suppose that U is non-additive over $\oplus$. Then the following conditions are equivalent.*

1. **(i)** $\oplus \approx \circledast$.

   (ii) *U is p-additive over $\oplus$ and (30) holds.*

As noted above, (30) implies both (31) and (32). That the converse is not true is seen by taking constants $\gamma_{k,l}, k = 1, \cdots, m, l = 1, \cdots, m$, with not all $\gamma_{k,l}$ equal to zero, but with

$$\sum_{l=1}^{m} \gamma_{i,l} = \sum_{k=1}^{n} \gamma_{k,j} = 0,$$

and

$$S_i(\mathbf{C}_n)S_j(\mathbf{C}_m) + \gamma_{i,j} \geq 0.$$

Then let

$$S_{i,j}(\mathbf{C}_n \times \mathbf{D}_m) = S_i(\mathbf{C}_n)S_j(\mathbf{C}_m) + \gamma_{i,j}.$$

3.1.3 Dependent gambles and component summing

There is at the opposite extreme the idea of experimentally totally dependent gambles. This arises when $m = n$ and $\mathbf{D}_n \equiv \mathbf{C}_n$, and it is assumed that a single, commonly realized experiment underlies all three gambles $f, g, f \oplus g$. An example involves two tickets for the same lottery realization, with the tickets possibly differing in their consequences on at least one event of the lottery. In such a case, we define a concept of component-wise summing when the partitions underlying f and g are identical:

Definition 22 *Let*

$$\begin{aligned} f &= (x_1, C_1; \cdots; x_i, C_i; \cdots; x_n, C_n), \\ g &= (y_1, C_1; \cdots; y_i, C_i; \cdots; y_n, C_n) \end{aligned}$$

be gambles having the same ordered event partition $\mathbf{C}_n$ for a single realization of the underlying experiment. Then ***component summing,*** *denoted $f \divideontimes g$, of the gambles f and g is defined by:*

$$f \divideontimes g := (x_1 \oplus y_1, C_1; \cdots; x_i \oplus y_i, C_i; \cdots; x_n \oplus y_n, C_n). \tag{33}$$

A natural question is when is $\oplus \approx \divideontimes$ true? This indifference is a rational property if the chance experiment is run just once for all three gambles, f, g, and $f \divideontimes g$. Indeed, it almost seems to be a triviality and, with money consequences, is often invoked, without explicit comment, using simple addition. However, as we shall see in Theorem 23, it is in fact a powerful assumption.

Note that if we assume the WU representation and that $\circledast \approx \divideontimes$, then $S_{i,i}(\mathbf{C}_n \times \mathbf{C}_n) = S_i(\mathbf{C}_n)$ which means $S_{i,j}(\mathbf{C}_n \times \mathbf{C}_n) = 0$ for $i \neq j$, and so

$$S_{i,j}(\mathbf{C}_n \times \mathbf{C}_n) = \begin{cases} S_i(\mathbf{C}_n), & i = j \\ 0, & i \neq j \end{cases} \tag{34}$$

is a solution to (31).

The property of *duplex decomposition*[9]

$$(x, C; y, C') \sim (x, C; e, C') \oplus (e, C; y, C') \quad (x \succsim e \succsim y, C \cap C' = \varnothing), \tag{35}$$

which has been applied to binary gambles where $x \succsim e \succsim y$ in Luce (2000), is also a special case of component summing if we assume that all three gambles involve the ordered partition (C, C'), and not independent replications, and we extend the definition to cover losses as well as gains.

A version of segregation, Section 4.2.3 below, and also of duplex decomposition for mixed binary gambles was studied experimentally in Cho, Luce, and Truong (2002). They interpreted the ordered partitions as representing independent realizations of the chance events in the sense that they established certainty equivalents for each term and then asked if, for example,

$$CE(x, C; y, C') = CE(x, C; e, C') + CE(e, C; y, C') \quad (x \succsim e \succsim y).$$

At most half of their respondents satisfied either duplex decomposition or segregation generalized to the mixed case. Perhaps the results would have been different if Cho et al. (2002) had held the events C, C' fixed and asked whether the certainty equivalents satisfy

$$CE(x, C; y, C') = CE(x, C; e, C') \oplus CE(e, C; y, C') \quad (x \succsim e \succsim y).$$

with C, C' fixed in the three terms instead of C arising from independent experiments in the three terms.

3.2 Second axiomatization of WU

Our next issue is an axiomatization of the WU form, (6), within the context of joint receipts as well as gambles.

Theorem 23 *Suppose that U is an order-preserving representation on the real interval $[0, k[$ that is decomposable for gambles, Def. 18, that gambles are strictly increasing in the consequences, that joint receipts are strictly increasing in each variable, and that $\oplus$ is commutative. Then the following statements are equivalent:*

[9] In the cited sources, the notation $\overline{C}$ is used instead of C'. This is probably unwise when there is no universal set from which the events are drawn.

1. *The operation $\oplus$ is decomposable in U, Def. 19, e is an identity of $\oplus$, and for gambles with the same event partition $\oplus \approx \circledast$ is satisfied, where $\circledast$ denotes component summing, Def. 22.*

2. *U forms a WU representation, Def. 2, of gambles, and U is additive over $\oplus$, Def. 17.*

This result is surprisingly strong and a bit disquieting. The property of component summing of gambles on the same ordered partition seems, on its face, highly innocent. Yet, in the presence of decomposability and $\oplus \approx \circledast$, it implies not only WU, which is fine, but also additive joint receipts, which is not so fine. Much of the empirical literature, with the exception of some of Birnbaum's model fitting, strongly suggests that with money gambles and assuming that WU holds, then U is non-linear with money. But if for money $x \oplus y = x + y$, then U has to be proportional to money. This result seems inconsistent with various interpretations of empirical data unless we are prepared to abandon one of the assumptions—all of which seem plausible. For example, many decision scientists invoke component summing in monetary contexts without comment because it seems so natural, and no one has yet seriously contemplated dropping decomposability for theories that assign utility to individual gambles.

3.3 Summary

The results of Theorem 23 are summarized in Figure 2. The results of Theorem 21 about behavioral independence and its relation to reduced joint receipt does not lend itself to graphical form. Suffice it to say that if U is non-additive over joint receipt, then behavioral independence is equivalent to joint receipt agreeing with reduced joint receipt.

Insert Fig. 2 about here

4 Increasing Utility Increments

In this section, the following two concepts will play a major role.

4.1 Definition

A great deal of data (for a summary, see Birnbaum, 1997) suggest that at least binary gambles of the form $(x, C; e, D)$ are dealt with differently from $(x, C; y, D)$, $y \succ e$, in the sense that the utility of the latter does not approach that of the former as $U(y)$ approaches 0. Also Slovic, Finucane, Peters, and MacGregor (2002) have made a similar observation. In particular, this rules out using segregation (see Section 4.2.3 below) as a tool in arriving at $U(x, C; y, D)$, $y \succ e$. Luce (2003b) has shown for binary gambles how to bypass segregation by using the property of binary distributivity (see Section 4.2.3). This gives rise to a representation involving differences of utility which he called increasing utility increments. The natural generalization of that form to general gambles is:

Definition 24 *An order-preserving utility representation U is an* ***increasing utility increment (IUI)*** *one iff there exist functions $M_{\mathbf{C}_n}^{(i)}$ that are strictly increasing with $M_{\mathbf{C}_n}^{(i)}(0) = 0$ such that, for all $x_i \succsim x_n \succ e$,*

$$U(\cdots; x_i, C_i; \cdots) - U(x_n) = \sum_{i=1}^{n-1} M_{\mathbf{C}_n}^{(i)} \left(U(x_i) - U(x_n) \right), \tag{36}$$

and $U(\cdots; x_i, C_i; \cdots; x_n, C_n)$ is not constant on any interval of x_n with the other terms $x_i, i \neq n$, fixed.

Note that the IUI representation implies that idempotence must hold.

So far as we know, this representation is new and not a great deal is yet known about it. It arose naturally in the binary case in a psychophysical interpretation of the primitives (Luce, 2003a), and the current form is a straightforward generalization of that case. Note that for each $i < n$, the gambles must be strictly monotonically increasing because, by the fact that U is strictly increasing, $x_i' \succsim x_i \succ x_n$ is equivalent to $U(x_i') - U(x_n) \geq U(x_i) - U(x_n)$ which in turn makes the gamble strictly increasing because each $M_{\mathbf{C}_n}^{(i)}$ is, by assumption, strictly increasing. However, the dependence on the nth component need not be strictly increasing because $U(x_n)$ appears with both positive and negative signs. In the psychophysical context with $n = 2$ it was useful to assume that the representation is not constant with respect to the 2nd component over any nontrivial interval—a concept that is coyly referred to in the functional equation literature as "philandering."

Two other generalizations of the WU form (Proposition 4) may also be worth considering. The first generalizes (9) and assumes that

$$U(\cdots; x_i, C_i; \cdots) = \sum_{i=2}^{n} P_{\mathbf{X}}^{(i)} \left(W_i(\mathbf{C}_n) - W_{i-1}(\mathbf{C}_n) \right),$$

where $\mathbf{X} = (U(x_1), ..., U(x_n))$ and the functions $P_{\mathbf{X}}^{(i)}$ are strictly increasing with $P_{\mathbf{X}}^{(i)}(0) = 0$. The second generalizes (11) as follows:

$$U(\cdots; x_i, C_i; \cdots) - U(x_n) = \sum_{i=1}^{n-1} Q_{\mathbf{C}_n}^{(i)} \left(U(x_i) - U(x_{i+1}) \right),$$

where the functions $Q_{\mathbf{C}_n}^{(i)}$ are strictly increasing with $Q_{\mathbf{C}_n}^{(i)}(0) = 0$. Neither of these has been studied to our knowledge, and we do not do so here.

4.2 AU and IUI

4.2.1 Case $n > 2$

The next result shows that for $n > 2$ the simultaneous imposition of the general AU and the IUI representations force the WU representation over all gambles with $x_i \succsim x_n \succ e$.

Theorem 25 *. Suppose that an order-preserving, utility structure U of $\langle \mathcal{D}_+, \succsim \rangle$ forms an AU representation, (5), and an order-preserving utility structure U^* of $\langle \mathcal{D}_+, \succsim \rangle$ forms an IUI representation, (36). For gambles with $n > 2$, then $U^* = rU$, where $r > 0$, and the idempotent WU representation, (6), holds. Conversely, an idempotent WU representation is both an AU and an IUI representation.*

This theorem does not satisfactorily axiomatize the WU model because we do not know of a direct axiomatization of IUI. The difficulty in axiomatizing IUI using only gambles lies in formulating what underlies such terms as $U(x_i) - U(x_n)$. As we shall see in Section 4.3, it is simple to do using joint receipts, but to do so without that concept may require some version of additive conjoint measurement.

4.2.2 Case n=2

The above result on AU and IUI representations for $n > 2$ is now examined for $n = 2$, but also using joint receipt.

Theorem 26 *Suppose that an order-preserving utility structure U of $\langle \mathcal{D}_+, \succsim \rangle$ forms an AU representation, (5), and a utility structure U^* of $\langle \mathcal{D}_+, \succsim \rangle$ forms an IUI representation, (36), for $n = 2$, and the structure is augmented with $\oplus$. Then U forms a RDU representation, U is p-additive, and U^* is additive over $\oplus$. n. Conversely, if one has an RDU representation U with U p-additive and a mapping U^* that is additive over $\oplus$, then U forms an AU representation and U^* forms an IUI one.*

4.2.3 Segregation and distributivity

Both Luce (2000) and earlier papers cited there made heavy use of the following property in arriving at the RDU representation:

Definition 27 ***Segregation*** *is said to hold iff for every integer n, $x_i \succsim e$, $y \succsim e$, and every partition $\mathbf{C}_n$ of $C(n)$,*

$$\begin{aligned}&(x_1, C_1; x_2, C_2; \cdots; x_{n-1}, C_{n-1}; e, C_n) \oplus y \\ &\quad \sim (x_1 \oplus y, C_1; x_2 \oplus y, C_2; \cdots; x_{n-1} \oplus, C_{n-1}; e \oplus y, C_n). \qquad (37)\end{aligned}$$

Luce (2003b) looked at the following property in the binary case which is closely related to segregation, Def. (37):

Definition 28 *(Right)* ***distributivity*** *is said to hold iff for every integer n, every $x_i \succsim x_n \succ e$, $y \succsim e$, and every ordered partition $\mathbf{C}_n$ of $C(n)$,*

$$\begin{aligned}&(x_1, C_1; \cdots; x_i, C_i; \cdots; x_n, C_n) \oplus y \\ &\quad \sim (x_1 \oplus y, C_1; \cdots; x_i \oplus y, C_i; \cdots; x_n \oplus y, C_n) \quad (x_n \succ e). \qquad (38)\end{aligned}$$

Obviously, distributivity and segregation are conceptually very similar concepts except that in the former $x_n = e$, whereas that case is excluded from the latter. A detailed argument exists for keeping the two concepts distinct (Luce, 2003b).

Proposition 29 *Idempotence, Def. 3, and component summing, Def. 22, imply segregation and distributivity.*

The proof is trivial because, for $y \succsim e$, using idempotence and component summing

$$\begin{aligned} & (x_1, C_1; ...; x_i, C_i; ...; x_n, C_n) \oplus y \\ \sim \quad & (x_1, C_1; ...; x_i, C_i; ...; x_n, C_n) \oplus (y, C_1; ...; y, C_i; ...; y, C_n) \\ \sim \quad & (x_1 \oplus y, C_1; ...; x_i \oplus y, C_i; ...; x_n \oplus, C_n). \end{aligned}$$

An interesting property of IUI when U is additive over $\oplus$ is:

Theorem 30 *Suppose that U is additive over $\oplus$, Def. 17, and an order- preserving representation of gambles.*

1. *If IUI, Def. 24, is satisfied, then distributivity, Def. 28, holds.*

2. *Suppose U is also an AU representation, Def. 1. Then segregation holds iff U is an IUI representation for $x_n \succ e$. Under these conditions, U is an idempotent WU representation.*

4.3 An axiomatization of IUI

We know of no purely gambling axiomatization of IUI; it would be interesting to have one, but here we give one using joint receipts as well as gambles. As we noted earlier, not a great deal is known about the general case of IUI although the binary one is better understood (Luce, 2003b).

For the following, it is useful to have the qualitative concept of subtraction, $\ominus$, that is defined for $x \succsim y$ by $x \ominus y \sim z \Leftrightarrow x \sim z \oplus y$. As noted earlier, it appears difficult, if not impossible, to axiomatize IUI without such a concept.

Theorem 31 *Suppose that $n > 2$ and that the set of gambles satisfies all of the conditions of Theorem 8, i.e., there is an order-preserving representation U and $\succsim$ is a weak order; gambles are strictly monotonic increasing over each subset of consequences and for $n - 1 = 2$ the Thomsen condition also holds over the first two components; $\oplus$ is a commutative, strictly increasing operation for which e is the identity; and segregation, Def. 27, holds. Moreover, suppose that U is decomposable over gambles and $\oplus$ with, respectively, the combining functions G and F. Then:*

1. *There exist strictly increasing functions $L^{(i)}_{\mathbf{C}_n}$ with $L^{(i)}_{\mathbf{C}_n}(0) = 0$, $i = 1, \cdots, n-1$, such that for all gambles with $x_i \succsim x_n$,*

$$U(\cdots; x_i, C_i; \cdots) = F\left(\sum_{i=1}^{n-1} L^{(i)}_{\mathbf{C}_n}\left(U(x_i \ominus x_n)\right), U(x_n)\right).$$

2. *If the representation of* $\oplus$ *is generalized additive, Def. 17, this representation reduces, with* $V = \varphi(U)$, *to:*

$$V(\cdots;x_i,C_i;\cdots) = \varphi\left(\sum_{i=1}^{n-1} M_{\mathbf{C}_n}^{(i)}\left(V(x_i) - V(x_n)\right)\right) + V(x_n).$$

3. *If the representation of* $\oplus$ *is additive, i.e.,* φ *in Part 2 is the identity, and* $x_n \succ e$, *then* U *has the IUI form.*

We do not know what happens when U is not strictly increasing in two or more of the components.

4.4 Summary

Figure 3 provides a graphical summary of the two results about AU and IUI.

Insert Fig. 3 here

The results of Theorem 31, about IUI and variants on it, does not lend itself to graphical representation.

5 Conclusions

The article includes seven new results plus slightly sharpened versions of three older ones about additive representations of gambles. One of the new, Theorem 10, is an axiomatization of weighted utility. Another, Theorem 16, gives a new axiomatization of SEU. The old ones, Theorems 8, 12, 14 give axiomatizations, respectively, of additive utility of gambles, rank-dependent utility, and gains-dependent utility, and are summarized in Figure 1 of Section 2.7. Two more new ones are in Section 3. Theorem 23 axiomatizes weighted utility using properties of joint receipt; it is summarized in Figure 2 of Section 4.4. The other result in this section, Theorem 21, formulates a behavioral equivalent to subjectively independent gambles, improving an earlier result. Section 4 includes four new results about structures with both additive utility and increasing increment utility, IUI, representations which is graphed in Figure 3 of Section 4.4. The cases of $n = 2$ and $n > 2$ are somewhat different. Theorem 30, a new result, explores some relations of IUI with segregation and distributivity. The last new result, Theorem 31, explores axioms leading to a IUI representation and generalizations of it. The result does not lend itself to being graphed.

Perhaps the most striking findings of the article are the following two. First, once consequence monotonicity is assumed, all of the other additive utility representations, save for SEU, involve adding accounting indifferences. This led us to conclude that the Savage formulation of decision making in terms of acts over state space goes a very long way to driving both RDU and GDU, and SEU follows from RDU by a simple necessary and sufficient condition. Second, is the remarkable fact of how strong component summing is in the presence of fairly weak axioms in that it forces not just WU but that U must also be additive over joint receipt. This flies in the face of the fact that component summing is often used with little or no comment when dealing with monetary gambles.

The following four unsolved problems are worthy of note. First: exactly what do we mean by independent realizations of gambles? We have given a definition that corresponds to subjective independence, but this is not what one usually means. Second: find a behavioral property not involving joint receipt that in conjunction with the existence of a continuous utility representation is equivalent to an IUI representation. We are not convinced that this is possible because of the presence of the terms $U(x_i) - U(x_n)$, which seem to be difficult to understand without the concept of joint receipt although they do arise in an alternative form for WU, (10). Third: either find an accounting indifference which with RDU is equivalent to SEU or prove that no such condition exists. The interest in this lies in whether the Savage formulation really forces SEU as well as RDU and GDU from the AU representation. Fourth: find some conditions that are necessary and sufficient to reduce ranked, idempotent WU to, at least, the special cases of the configural weight model that Birnbaum and his co-worker's have successfully fit to data (see Birnbaum, 1997, 1999). As mentioned in Section 1.2, we have shown that his general TAX model is equivalent to idempotent WU. But, in practice, Birnbaum has worked with very special cases and these have yet to be axiomatized.

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Appendix: Proofs

Theorem 8
Proof: 1. For $x_i \succsim e$, we establish that there is an additive decomposition of gambles of the form

$$U(x_1, C_1; \cdots; x_n, C_n) = \sum_{i=1}^{n} L_{\mathbf{C}_n}^{(i)} \left(U(x_i) \right),$$

where each $L_{\mathbf{C}_n}^{(i)}$ is strictly increasing and unique up to positive affine transformations with a common unit. This is then sufficient, given our notation, to simultaneously handle both the unranked and ranked cases. To obtain this representation, we use either the standard result of additive conjoint measurement, Krantz et al. (1971), or if we are in the ordered case $x_1 \succsim \cdots \succsim x_n \succ e$, we invoke Wakker (1991, Theorem 4). Both lead to the following form

$$U(x_1, C_1; \cdots; x_n, C_n) = \sum_{i=1}^{n} U_{\mathbf{C}_n}^{(i)}(x_i). \tag{39}$$

Observe that the desired representation follows by defining $L^{(i)}_{\mathbf{C}_n} = U^{(i)}_{\mathbf{C}_n} U^{-1\cdot}$.

The following conditions are sufficient for the representation: weak order, essentialness of each coordinate, general strict monotonicity and for $n = 2$ the Thomsen condition, solvability, Archimedeanness, and no maximal or minimal consequences. We have assumed weak ordering and general strict monotonic increasing for each subset of alternatives which, of course, insures essentialness, i.e., that each component affects the preference ordering. The minimal element e corresponding to $U(e) = 0$ is excluded and there is no maximal element because I is open on the right. Solvability holds because U is onto I. To show Archimedeanness, consider two vectors of consequences over all save the i^{th} component. Denote these vectors by $\overrightarrow{y}_{\bar{i}}$ and $\overrightarrow{z}_{\bar{i}}$, and the corresponding real vectors under U by $\overrightarrow{Y}_{\bar{i}}, \overrightarrow{Z}_{\bar{i}} \in I^{n-1}$. Consider those for which

$$\min\{Y_{i-1}, Z_{i-1}\} > \max\{Y_{i+1}, Z_{i+1}\}.$$

For some $X_i = U(x_i)$ in that interval, the vectors $(X_i, \overrightarrow{Y}_{\bar{i}})$ and $(X_i, \overrightarrow{Z}_{\bar{i}})$ are appropriately rank ordered and, by strict monotonicity, $(x_i, \overrightarrow{y}_{\bar{i}}) \prec (x_i, \overrightarrow{z}_{\bar{i}})$. So, a standard sequence $\{X_j\}$ is defined by

$$(x_{i,j+1}, \overrightarrow{y}_i) \sim (x_{i,j}, \overrightarrow{z}_i).$$

Such a sequence is certainly bounded by the fact $X_{i,j} \leq \min\{Y_{i-1}, Z_{i-1}\}$. If it is not finite, then by the fact it is an increasing bounded sequence, it converges to a value, say A_i, and so

$$(A_i, \overrightarrow{Y}_i) \sim (A_i, \overrightarrow{Z}_i).$$

But this contradicts strict monotonicity. So Archimedeanness is satisfied, and therefore (5) is satisfied. As noted earlier, the choice of $L^{(i)}_{\mathbf{C}_n}$ is unique up to positive affine transformations with a common unit. So with no loss of generality, we may choose the unit to be 1 and the additive constants so that $\lim_{X\to 0} L^{(i)}_{\mathbf{C}_n}(X) = 0$ and so define $L^{(i)}_{\mathbf{C}_n}(0) = 0$. □

Theorem 10

Proof: Suppose that, for $n \geq 2, m \geq 2$, both AU, Def. 1, and status-quo event commutativity, Def. 9, hold. Apply the former to the latter to get:

$$L^{(i)}_{\mathbf{C}_n} L^{(k)}_{\mathbf{D}_m}(U(x)) + L^{(j)}_{\mathbf{C}n} L^{(k)}_{\mathbf{D}_m}(U(y)) = L^{(k)}_{\mathbf{D}m}\left(L^{(i)}_{\mathbf{C}n}(U(x)) + L^{(j)}_{\mathbf{C}_n}(U(y))\right).$$

Introduce the abbreviations $X = U(x), Y = U(y), \theta = L^{(k)}_{\mathbf{D}_m}, \varphi = L^{(i)}_{\mathbf{C}_n}, \psi = L^{(j)}_{\mathbf{C}_n}$, then the functional equation becomes

$$\varphi(\theta(X)) + \psi(\theta(Y)) = \theta(\varphi(X) + \psi(Y)).$$

Observe that by setting $X = 0$ and then, separately, $Y = 0$, we have $\psi(\theta) = \theta(\psi)$ and $\varphi(\theta) = \theta(\varphi)$, and so

$$\theta(\varphi(X)) + \theta(\psi(Y)) = \theta(\varphi(X) + \psi(Y)),$$

which is a Cauchy equation with strictly increasing θ and so $\theta(Z) = aZ$ (Aczél, 1966). Thus, for any $k = 1, \cdots, m$, we have $L^{(k)}_{\mathbf{D}_m}(Z) = S_k(\mathbf{D}_m)Z$. So the WU representation holds.

The converse is trivial. $\square$

Theorem 12

Proof: The proof is trivial in going from RDU.

To prove the converse, consider, first, $n = 3$, and so by coalescing we have

$$
\begin{aligned}
(x_1, C_1; x_1, C_2; x_3, C_3) &\sim (x_1, C_1 \cup C_2; x_3, C_3) \\
(x_1, C_1; x_2, C_2; x_2, C_3) &\sim (x_1, C_1; x_2, C_2 \cup C_3)
\end{aligned}
$$

Thus, by RWU in the form of (9), and using coalescing on the first two outcomes, we have:

$$
\begin{aligned}
&U(x_1, C_1; x_1, C_2; x_3, C_3) \\
&\quad = U(x_1)W_1(C_1, C_2, C_3) + U(x_1)[W_2(C_1, C_2, C_3) - W_1(C_1, C_2, C_3)] \\
&\qquad\quad + U(x_3)[1 - W_2(C_1, C_2, C_3)] \\
&\quad = U(x_1)W_2(C_1, C_2, C_3) + U(x_3)[1 - W_2(C_1, C_2, C_3)] \\
&\quad = U(x_1, C_1 \cup C_2; x_3, C_3) \\
&\quad = U(x_1)W_1(C_1 \cup C_2, C_3) + U(x_3)[1 - W_1(C_1 \cup C_2, C_3)].
\end{aligned}
$$

Because $x_1 \geq x_3$ can be selected independently, it follows from the final three equations that we may define the binary weight

$$W_{C(3)}(C_1 \cup C_2) := W_1(C_1 \cup C_2, C_3) = W_2(C_1, C_2, C_3).$$

Again, by RWU in the form of (9), and using coalescing on the last two outcomes, we have

$$
\begin{aligned}
&U(x_1, C_1; x_2, C_2; x_2, C_3) \\
&\quad = U(x_1)W_1(C_1, C_2, C_3) + U(x_2)[W_2(C_1, C_2, C_3) - W_1(C_1, C_2, C_3)] \\
&\qquad\quad + U(x_2)[1 - W_2(C_1, C_2, C_3)] \\
&\quad = U(x_1)W_1(C_1, C_2, C_3) + U(x_2)[1 - W_1(C_1, C_2, C_3)] \\
&\quad = U(x_1, C_1; x_2, C_2 \cup C_3) \\
&\quad = U(x_1)W_1(C_1, C_2 \cup C_3) + U(x_2)[1 - W_1(C_1, C_2 \cup C_3)].
\end{aligned}
$$

Thus, we introduce the following notation for the binary weights

$$W_{C(3)}(C_1) := W_1(C_1, C_2 \cup C_3) = W_1(C_1, C_2, C_3).$$

Therefore

$$
\begin{aligned}
&U(x_1, C_1; x_2, C_2; x_2, C_3) \\
&\qquad = U(x_1)W_1(C_1, C_2, C_3) + U(x_2)[W_2(C_1, C_2, C_3) - W_1(C_1, C_2, C_3)] \\
&\qquad\quad + U(x_3)[1 - W_2(C_1, C_2, C_3)] \\
&\qquad = U(x_1)W_{C(3)}(C_1) + U(x_2)[W_{C(3)}(C_1 \cup C_2) - W_{C(3)}(C_1)] \\
&\qquad\quad + U(x_3)[1 - W_{C(3)}(C_1 \cup C_2)],
\end{aligned}
$$

which is RDU for $n = 3$.

We proceed by induction. Suppose the result is true for $n-1$, then by coalescing for $k = 1$,

$$(x_1, C_1; x_1, C_2; \cdots x_n, C_n) \sim (x_1, C_1 \cup C_2; \cdots ; x_n, C_n),$$

whence by the induction hypothesis

$$\begin{aligned} 0 &= U(x_1, C_1; x_1, C_2; \cdots x_n, C_n) - U(x_1, C_1 \cup C_2; \cdots ; x_n, C_n) \\ &= U(x_1)[W_1(\mathbf{C}_n) + W_2(\mathbf{C}_n) - W_1(\mathbf{C}_n)] + \sum_{i=3}^{n} U(x_i)\left[W_i(\mathbf{C}_n) - W_{i-1}(\mathbf{C}_n)\right] \\ &- U(x_1)W_{C(n)}(C_1 \cup C_2) - \sum_{i=3}^{n} U(x_i)\left(W_{C(n)}\left(C\left(i\right)\right) - W_{C(n)}\left(C\left(i-1\right)\right)\right), \end{aligned}$$

so, as in the case of $n = 3$, we have

$$\begin{aligned} W_2(\mathbf{C}_n) &= W_{C(n)}(C_1 \cup C_2), \\ W_i(\mathbf{C}_n) - W_{i-1}(\mathbf{C}_n) &= W_{C(n)}(C(i)) - W_{C(n)}(C(i-1)) \quad (i = 3, \cdots, n). \end{aligned}$$

and so we can define

$$W_{C(n)}(C(i)) := W_i(\mathbf{C}_n) = W_i(C_1, ..., C_n) \quad (i = 2, ..., n)$$

In like manner, using the coalescing

$$(x_1, C_1; ...; x_1, C_{n-1}; x_n, C_n) \sim (x_1, \cup_{j=1}^{n-1} C_j; x_n)$$

allows us to define

$$W_{C(n)}(C_1) := W_1(\mathbf{C}_n) = W_1(C_1, ..., C_n).$$

Thus, RDU holds. □

Theorem 16

Proof: (ii) implies (i) is trivial.

(i) implies (ii). Idempotence follows from RDU. Also, under RDU, the hypothesis (23) is equivalent to

$$U(x)W_{C\cup D}(C) = U(x')W_{C'\cup D'}(C') \tag{40}$$

$$U(y)W_{C\cup D}(D) = U(y')W_{C'\cup D'}(D'). \tag{41}$$

And if $x \succsim y, x' \succsim y'$, $(x, C; y, D) \sim (x', C'; y', D')$ is equivalent to

$$U(x)W_{C\cup D}(C) + U(y)[1 - W_{C\cup D}(C)] = U(x')W_{C'\cup D'}(C') + U(y')[1 - W_{C'\cup D'}(C')]$$

which by (40) is equivalent to

$$U(y)[1 - W_{C\cup D}(C)] = U(y')[1 - W_{C'\cup D'}(C')].$$

Using this and (41),Using this and (41),

$$\begin{aligned}\frac{U(y)}{U(y')} &= \frac{1-W_{C'\cup D'}(C')}{1-W_{C\cup D}(C)} = \frac{W_{C'\cup D'}(D')}{W_{C\cup D}(D)} \\ \Leftrightarrow \frac{W_{C\cup D}(D)}{1-W_{C\cup D}(C)} &= \frac{W_{C'\cup D'}(D')}{1-W_{C'\cup D'}(C')}. \end{aligned} \tag{42}$$

But this holds for any events C' and D' with C and D fixed, so for some constant $\alpha > 0$

$$W_{C\cup D}(D) = \alpha[1 - W_{C\cup D}(C)]$$

If, on the right of (42), we let $C' = D$ and $D' = C$, we have

$$W_{C\cup D}(C) = \alpha[1 - W_{C\cup D}(D)].$$

So, subtracting

$$W_{C\cup D}(D) - W_{C\cup D}(C) = \alpha[W_{C\cup D}(D) - W_{C\cup D}(C)],$$

whence $\alpha = 1$ and so

$$W_{C\cup D}(C) + W_{C\cup D}(D) = 1,$$

i.e.,

$$1 - W_{C\cup D}(C) = W_{C\cup D}(D).$$

Thus the binary RDU weights become

$$\begin{aligned} S_1(C,D) &= W_{C\cup D}(C), \\ S_2(C,D) &= 1 - W_{C\cup D}(C) = W_{C\cup D}(D), \end{aligned}$$

which, since idempotence follows from RDU, gives the binary case of SEU.

Next, $(x,C;y,D;e,E) \sim (x',C';y',D';e,E')$ is equivalent to

$$\begin{aligned} &U(x)W_{C\cup D\cup E}(C) + U(y)[W_{C\cup D\cup E}(C\cup D) - W_{C\cup D\cup E}(C)] \\ &\quad = U(x')W_{C'\cup D'\cup E'}(C') + U(y')[W_{C'\cup D'\cup E'}(C'\cup D') - W_{C'\cup D'\cup E'}(C')]. \end{aligned}$$

Note that by coalescing $(x,C;e,D;e,E) \sim (x,C;e,D\cup E)$, and so the hypothesis holds with D replaced by $D\cup E$ and with D' replaced by $D'\cup E'$, and so the left term on each side of = cancel, leaving

$$\begin{aligned} \frac{U(y)}{U(y')} &= \frac{W_{C'\cup D'\cup E'}(C'\cup D') - W_{C'\cup D'\cup E'}(C')}{W_{C\cup D\cup E}(C\cup D) - W_{C\cup D\cup E}(C)} = \frac{W_{C'\cup D'\cup E'}(D')}{W_{C\cup D\cup E}(D)} \\ &\Leftrightarrow \frac{W_{C\cup D\cup E}(C\cup D) - W_{C\cup D\cup E}(C)}{W_{C\cup D\cup E}(D)} \\ &\qquad = \frac{W_{C'\cup D'\cup E'}(C'\cup D') - W_{C'\cup D'\cup E'}(C')}{W_{C'\cup D'\cup E'}(D')}. \end{aligned}$$

So,

$$\frac{W_{C\cup D\cup E}(C\cup D) - W_{C\cup D\cup E}(C)}{W_{C\cup D\cup E}(D)} = \gamma.$$

Letting $C' = D$ and $D' = C$ yields

$$\frac{W_{C\cup D\cup E}(C\cup D) - W_{C\cup D\cup E}(D)}{W_{C\cup D\cup E}(C)} = \gamma.$$

Therefore, eliminating $W_{C\cup D\cup E}(C\cup D)$,

$$W_{C\cup D\cup E}(C)(1-\gamma) = W_{C\cup D\cup E}(D)(1-\gamma).$$

and so $\gamma = 1$. Thus,

$$W_{C\cup D\cup E}(C\cup D) = W_{C\cup D\cup E}(C) + W_{C\cup D\cup E}(D),$$

which establishes that the weights are finitely additive. Using this fact in the RDU weights, we obtain: for $i = 1, .., n$,

$$S_i(\mathbf{C}_n) = W_{C(n)}(C(i)) - W_{C(n)}(C(i-1)) = W_{C(n)}(C_i),$$

i.e., since idempotence follows from RDU, we have SEU. □

Theorem 21

Proof: 1. From U satisfies WU, e is the zero element of $\oplus$, and $f \oplus g \sim f \circledast g$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} U(x_i) S_i(\mathbf{C}_n) &= U(f) \\
&= U(f \oplus e) \\
&= U(f \circledast e) \\
&= \sum_{i=1}^{n}\sum_{l=1}^{m} U(x_i) S_{i,l}(\mathbf{C}_n \times \mathbf{C}_m) \\
&= \sum_{i=1}^{n} U(x_i) \sum_{l=1}^{m} S_{i,l}(\mathbf{C}_n \times \mathbf{C}_m).
\end{aligned}$$

It then follows from the (constrained) independence of the choices of the x_i that (31) holds. A parallel argument using the gamble g shows that (32) holds.

2. From U additive, and (31), (32), we have

$$\begin{aligned}
U(f \oplus g) &= U(f) + U(g) \\
&= \sum_{i=1}^{n} U(x_i) S_i(\mathbf{C}_n) + \sum_{j=1}^{m} U(y_j) S_j(\mathbf{D}_m) \\
&= \sum_{i=1}^{n} U(x_i) \sum_{l=1}^{m} S_{i,l}(\mathbf{C}_n \times \mathbf{C}_m) + \sum_{j=1}^{m} U(y_j) \sum_{k=1}^{n} S_{k.j}(\mathbf{C}_n \times \mathbf{C}_m) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{m} [U(x_k) S_{k,l}(\mathbf{C}_n \times \mathbf{C}_m) + U(y_l) S_{k.l}(\mathbf{C}_n \times \mathbf{C}_m)] \\
&= \sum_{k=1}^{n} \sum_{l=1}^{m} [U(x_k) + U(y_l)]\, S_{k,l}(\mathbf{C}_n \times \mathbf{C}_m) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{m} U(x_k \oplus y_l) S_{k,l}(\mathbf{C}_n \times \mathbf{C}_m) \\
&= U(f \circledast g).
\end{aligned}$$

3. (ii) implies (i). This is simply a matter of substitution. We have that U is p-additive with $\delta \neq 0$, and (30), so

$$\begin{aligned}
&1 - \delta U(f \circledast g) \\
&= 1 - \delta \sum_{k=1}^{n} \sum_{l=1}^{m} U(x_k \oplus y_l) S_{k,l}(\mathbf{C}_n \times \mathbf{C}_m) \\
&= 1 - \delta \sum_{k=1}^{m} \sum_{l=1}^{n} U(x_k \oplus y_l) S_k(\mathbf{C}_m) S_l(\mathbf{C}_n) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{m} [1 - \delta U(x_k \oplus y_l)]\, S_k(\mathbf{C}_n) S_l(\mathbf{C}_m) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{m} [1 - \delta U(x_k)]\, [1 - \delta U(y_l)]\, S_k(\mathbf{C}_n) S_l(\mathbf{C}_m) \\
&= \left(\sum_{k=1}^{n} [1 - \delta U(x_k)]\, S_k(\mathbf{C}_n) \right) \left(\sum_{l=1}^{m} [1 - \delta U(y_l)]\, S_l(\mathbf{C}_m) \right) \\
&= \left(1 - \delta \sum_{k=1}^{n} U(x_k) S_k(\mathbf{C}_n) \right) \left(1 - \delta \sum_{l=1}^{m} U(y_l) S_l(\mathbf{C}_m) \right) \\
&= [1 - \delta U(f)][1 - \delta U(g)] \\
&= 1 - \delta U(f \oplus g).
\end{aligned}$$

(i) implies (ii). We proceed in a manner very similar to, but not identical to, parts of the proof of Marley and Luce (2001, Theorem 5). As in that proof, using the function F, (29), define

$$H(\zeta, \eta) := F(\zeta, \eta) - \zeta - \eta, \tag{43}$$

Since $F(\zeta, 0) = F(0, \eta) = 0$ for all ζ and η, we have $H(\zeta, 0) = H(0, \eta) = 0$.

Using the definition of H, we have

$$\begin{aligned} U(f \oplus g) &= F\left(U(f), U(g)\right) \\ &= U(f) + U(g) + H\left(U(f), U(g)\right), \end{aligned} \tag{44}$$

and

$$\begin{aligned} U(f \circledast g) &= \sum_{k=1}^{n} \sum_{l=1}^{m} U(x_k \oplus y_l) S_{k,l}(\mathbf{C}_n \times \mathbf{C}_m) \\ &= \sum_{k=1}^{n} \sum_{l=1}^{m} \left[(U(x_k) + U(y_l) + H\left(U(x_k), U(y_l)\right)) \right] S_{k,l}(\mathbf{C}_n \times \mathbf{C}_m) \end{aligned} \tag{45}$$

When $g \sim e$, (44) and (45) give

$$U(f) = \sum_{k=1}^{n} \sum_{l=1}^{m} U(x_k) S_{k,l}(\mathbf{C}_m \times \mathbf{C}_n),$$

and when $f \sim e$, (44), (45) give

$$U(g) = \sum_{k=1}^{n} \sum_{l=1}^{m} U(y_l) S_{k,l}(\mathbf{C}_n \times \mathbf{C}_m).$$

Substituting these expressions for $U(f)$ and $U(g)$ in the right-hand-side of (44), and equating (44) and (45), gives

$$H[U(f), U(g)] = \sum_{k=1}^{n} \sum_{l=1}^{m} H[U(x_k), U(y_l)] S_{k,l}(\mathbf{C}_n \times \mathbf{C}_m). \tag{46}$$

Substituting the WU representations in the left hand side of (46), we obtain

$$\begin{aligned} &H\left(\sum_{k=1}^{n} U(x_k) S_k(\mathbf{C}_n), \sum_{l=1}^{m} U(y_l) S_l(\mathbf{D}_m) \right) \\ &\quad = \sum_{k=1}^{n} \sum_{l=1}^{m} H\left(U(x_k), U(y_l)\right) S_{k,l}(\mathbf{C}_m \times \mathbf{D}_m). \end{aligned} \tag{47}$$

With $x_1 = x, y_1 = y, x_i = e, i > 1, y_j = e, j > 1$, and using the fact that $H(\zeta, 0) = H(0, \eta) = 0$ for all ζ, η in its range, we obtain that

$$H\left(U(x_1) S_1(\mathbf{C}_n), U(y_1) S_1(\mathbf{D}_m)\right) = H\left(U(x_1), U(y_1)\right) S_{1,1}(\mathbf{C}_n, \mathbf{D}_m)$$

i.e., with $U(x_1) = \zeta, U(y_1) = \eta, S_1(\mathbf{C}_m) = \varpi, S_1(\mathbf{D}_n) = \rho$,

$$H(\zeta \varpi, \eta \rho) = H(\zeta, \eta) S_{1,1}(\mathbf{C}_n, \mathbf{D}_m). \tag{48}$$

Setting $\zeta = \eta$=1, this gives

$$H(\varpi, \rho) = H(1,1)S_{1,1}(\mathbf{C}_n, \mathbf{D}_m). \tag{49}$$

We next show that $S_{1,1}(\mathbf{C}_n, \mathbf{D}_m) \neq 0$ when $S_1(\mathbf{C}_n) = \varpi \neq 0$, $S_1(\mathbf{D}_m) = \rho \neq 0$. Note that with $x_1 = x, y_1 = y, x_i = e, i > 1, y_j = e, j > 1$, we have

$$\begin{aligned}&(x_1, C_1; e, C_2; \cdots; e, C_n) \oplus (y_1, D_1; e, D_2; \cdots; e, D_m)\\ &\quad = (x_1 \oplus y_1, (C_1, D_1); x_1, (C_1, D_2); \cdots; x_1, (C_1, D_m); y_1, (C_2, D_1);\\ &\qquad \cdots; y_1, (C_n, D_1)).\end{aligned}$$

Since U is a WU representation, if $S_{1,1}(\mathbf{C}_n, \mathbf{D}_m) = 0$, we then have

$$\begin{aligned}U(x \oplus y) &= \sum_{l=2}^{m} U(x_1) S_{1,l}(\mathbf{C}_n \times \mathbf{D}_m) + \sum_{k=2}^{n} U(y_1) S_{k,1}(\mathbf{C}_n \times \mathbf{D}_m)\\ &= U(x_1) \sum_{l=2}^{m} S_{1,l}(\mathbf{C}_n \times \mathbf{D}_m) + U(y_1) \sum_{k=2}^{n} S_{k,1}(\mathbf{C}_n \times \mathbf{D}_m).\end{aligned}$$

With $y_1 = e$, this gives

$$\sum_{l=2}^{m} S_{1,l}(\mathbf{C}_n \times \mathbf{D}_m) = S_1(\mathbf{C}_n),$$

and with $x_1 = e$, it gives

$$\sum_{k=2}^{n} S_{k,1}(\mathbf{C}_n \times \mathbf{D}_m) = S_1(\mathbf{D}_m),$$

which substituted back in the original equation gives

$$U(x \oplus y) = U(x_1)S_1(\mathbf{C}_n) + U(y_1)S_1(\mathbf{D}_m) = U(x) + U(y),$$

which contradicts the assumption that U is non-additive. So, $S_{1,1}(\mathbf{C}_n, \mathbf{D}_m) \neq 0$ when $S_1(\mathbf{C}_n) = \varpi \neq 0$,$S_1(\mathbf{D}_m) = \rho \neq 0$.

We now show that $H(1,1) \neq 0$. Suppose, on the contrary, $H(1,1) = 0$. First consider the case $\zeta < 1, \eta < 1$. Then since $\varpi, \rho \in]0,1[$, setting $\varpi = \zeta, \rho = \eta$ in (49) gives $H(\zeta, \eta) = 0$ for $\zeta < 1, \eta < 1$. For $\zeta > 1, \eta > 1$, set $\varpi = 1/\zeta, \rho = 1/\eta$ and so by (48)

$$0 = H(1,1) = \frac{H(\zeta, \eta)}{S_{11}(\mathbf{C}_m, \mathbf{C}_n)},$$

and so $H(\zeta, \eta) = 0$ for $\zeta > 1, \eta > 1$. Now suppose $\zeta \geq 1 \geq \eta$, then we may choose ϖ and ρ so that $\zeta\varpi < 1, \eta\rho < 1$, whence by the first part and (48),

$$0 = H(\zeta\varpi, \eta\rho) = H(\zeta, \eta)S_{1,1}(\mathbf{C}_n, \mathbf{D}_m),$$

and so $H(\zeta, \eta) = 0$ for $\zeta \geq 1 \geq \eta$. Since $\oplus$ is commutative, H is symmetric and so the conclusion holds for the other inequality. And so $H(\zeta, \eta) \equiv 0$, i.e., U is additive, contrary to assumption. Therefore, $\delta = H(1, 1) \neq 0$.

Now retracing our steps, (49) gives

$$S_{1,1}(\mathbf{C}_n, \mathbf{C}_m) = \frac{H(\varpi, \rho)}{H(1,1)} = \frac{H(\varpi, \rho)}{\delta},$$

which substituted in (48) gives

$$H(\zeta\varpi, \eta\rho) = H(\zeta, \eta)\frac{H(\varpi, \rho)}{\delta},$$

i.e.,

$$\frac{H(\zeta\varpi, \eta\rho)}{\delta} = \frac{H(\zeta, \eta)}{\delta}\frac{H(\varpi, \rho)}{\delta},$$

It then follows from Aczél (1997, Theorem 1) that for all a, b in the domain of H,

$$\frac{H(a, b)}{\delta} = ab,$$

i.e.,

$$H(a, b) = \delta ab.$$

Thus, U is p-additive.

We can now proceed as in the proof of Marley and Luce (2001, Theorem 5.1b.) to derive that

$$S_{i,j}(\mathbf{C}_n \times \mathbf{D}_m) = S_i(\mathbf{C}_n)S_j(\mathbf{D}_m)$$

for $i = 1, \cdots, m, j = 1, \cdots, n$ iff

$$F(\zeta + \alpha, \eta) - F(\zeta, \eta) \neq F(\zeta + \alpha, \eta') - F(\zeta, \eta')$$

when $\eta \neq \eta'$. The latter is the case since U is p-additive, and hence $F(\zeta, \eta) = \zeta + \eta + \delta\zeta\eta$. □

Theorem 23

Proof: 1 implies 2. Let $G_{\mathbf{C}_n}$ and F be the functions for the decomposability of gambles and $\oplus$, respectively. From that and the assumption that $\oplus \approx \circledast$,

$$F[G_{\mathbf{C}_n}(\cdots, X_i, \cdots), G_{\mathbf{C}_n}(\cdots, Y_i, \cdots)] = G_{\mathbf{C}_n}[\cdots, F(X_i, Y_i), \cdots], \tag{50}$$

where $X_i = U(x_i), Y_i = U(y_i)$. For $n = 2$, the form is the generalized bisymmetric functional equation:

$$F\left(G(x, y), G(u, v)\right) = G\left(F(x, u), F(y, v)\right).$$

This functional equation was studied by Aczél (1997, Eq. (6), p. 230) and he reports the solution worked out by Aczél and Maksa (1996), namely,

$$F(X, Y) = \varphi^{-1}\left(\varphi_1(X) + \varphi_2(Y)\right),$$

where $\varphi, \varphi_1, \varphi_2$ are each strictly increasing. The result was later proved in Maksa (1999) under weaker conditions. Using the commutativity of $\oplus$ and that e is an identity, this solution for F reduces to there being a strictly increasing $\varphi : \mathbb{R}_+ \overset{onto}{\longrightarrow} \mathbb{R}_+$ such that F has a generalized additive representation, Def. 17, with respect to U^*. Let $U := \varphi(U^*)$, then by induction

$$U(g_1 \oplus \cdots \oplus g_m) = \sum_{i=1}^{m} U(g_i).$$

For the ordered partition $\mathbf{C}_n$, let $g_{\mathbf{C}_n}^{(i)}$ denote the gamble with x_i associated with C_i and e with all of the other events of the ordered partition. By induction on component summation

$$g \sim (\cdots; x_i, C_i; \cdots) \sim g_{\mathbf{C}_n}^{(1)} \oplus \cdots \oplus g_{\mathbf{C}_n}^{(i)} \oplus \cdots \oplus g_{\mathbf{C}_n}^{(n)},$$

and so

$$U(g) = \sum_{i=1}^{m} U(g_{\mathbf{C}_n}^{(i)}) = \sum_{i=1}^{m} L_{\mathbf{C}_n}^{(i)}\left(U(x_i)\right),$$

where $L_{\mathbf{C}_n}^{(i)}\left(U(x_i)\right) = G_{\mathbf{C}_n}\left(0, \cdots, 0, U(x_i), 0, \cdots, 0\right)$. Observe that by $\oplus \approx \divideontimes$, $g_{\mathbf{C}_n}^{(i)} \oplus h_{\mathbf{C}_n}^{(i)} \sim (e, C_1; \cdots; e, C_{i-1}; x_i \oplus y_i, C_i; e, C_{i+1}; \cdots; e, C_n)$, and so

$$L_{\mathbf{C}_n}^{(i)}\left(U(x_i \oplus y_i)\right) = L_{\mathbf{C}_n}^{(i)}\left(U(x_i) + U(y_i)\right) = L_{\mathbf{C}_n}^{(i)}\left(U(x_i)\right) + L_{\mathbf{C}_n}^{(i)}\left(U(y_i)\right),$$

and since the functions are onto $\mathbb{R}_+$ and strictly monotonic, we have that $L_{\mathbf{C}_n}^{(i)}(Z) = ZS_i(\mathbf{C}_n)$. So,

$$U(x \oplus y) = U(x) + U(y) \tag{51}$$

$$U(\cdots; x_i, C_i; \cdots) = \sum_{i=1}^{n} U(x_i) S_i(\mathbf{C}_n). \tag{52}$$

2 implies 1. An additive $\oplus$ is, of course, decomposable. The calculation for $\oplus \approx \divideontimes$ is routine. □

Theorem 25

Proof: Define f by $U = f(U^*)$. Note that f is strictly increasing. Now set $x_i = e$ for all $i = 1, \cdots, n$ except for $i = k < n$, and let $x_k = z$. Then with $Z = U(z)$, (5) and (36) give

$$L_{\mathbf{C}_n}^{(k)}(Z) = f\left(M_{\mathbf{C}_n}^{(k)}\left(f^{-1}(Z)\right)\right), \tag{53}$$

Now set $x_i = x_n$ for all $i = 1, .., n$, except for $i = k < n$ and $i = l < n$. Recall that $L_{\mathbf{C}_n}^{(i)}(0) = M_{\mathbf{C}_n}^{(i)}(0) = 0$, and let $X_k = U^*(x_k)$, $X_l = U^*(x_l), X_n = U^*(x_n)$. Then because $U = f(U^*)$, the assumption that AU and IUI both hold yields the functional equation

$$L_{\mathbf{C}_n}^{(k)}\left(f(X_k)\right) + L_{\mathbf{C}_n}^{(l)}\left(f(X_l)\right) + \sum_{i=1,\neq k,l}^{n} L_{\mathbf{C}_n}^{(i)}\left(f(X_n)\right)$$
$$= f\left(M_{\mathbf{C}_n}^{(k)}(X_k - X_n) + M_{\mathbf{C}_n}^{(l)}(X_l - X_n) + X_n\right),$$

which is continuous because each $L_{\mathbf{C}_n}^{(i)}(X_n)$ is strictly increasing on an interval. By idempotence,

$$\sum_{i=1,\neq k,l}^{n} L_{\mathbf{C}_n}^{(i)}\left(f(X_n)\right) = f(X_n) - L_{\mathbf{C}_n}^{(k)}\left(f(X_n)\right) - L_{\mathbf{C}_n}^{(l)}\left(f(X_n)\right),$$

so the original equation becomes

$$L_{\mathbf{C}_n}^{(k)}\left(f(X_k)\right) + L_{\mathbf{C}_n}^{(l)}\left(f(X_l)\right) + f(X_n) - L_{\mathbf{C}_n}^{(k)}\left(f(X_n)\right) - L_{\mathbf{C}_n}^{(l)}\left(f(X_n)\right)$$
$$= f\left(M_{\mathbf{C}_n}^{(k)}(X_k - X_n) + M_{\mathbf{C}_n}^{(l)}(X_l - X_n) + X_n\right). \tag{54}$$

By (53),

$$L_{\mathbf{C}_n}^{(i)}(X) = f\left(M_{\mathbf{C}_n}^{(i)}\left(f^{-1}(X)\right)\right) = f\left(h_i\left(f^{-1}(X)\right)\right),$$

where h_i stands for $M_{\mathbf{C}_n}^{(i)}$. Substituting this into (54) gives

$$f\left(h_k(X_k)\right) + f\left(h_l(X_l)\right) + f(X_n) - f\left(h_k(X_n)\right) - f\left(h_l(X_n)\right)$$
$$= f\left(h_k(X_k - X_n) + h_l(X_l - X_n) + X_n\right),$$

i.e., defining $Y = X_n, X = X_k, Z = X_l$,

$$f\left(h_k(X)\right) + f\left(h_l(Z)\right) + f(Y) - f\left(h_k(Y)\right) - f\left(h_l(Y)\right)$$
$$= f\left(h_k(X - Y) + h_l(Z - Y) + Y\right). \tag{55}$$

In this functional equation, set $Y = 0$ to get

$$f\left(h_k(X) + h_l(Z)\right) = f\left(h_k(X)\right) + f\left(h_l(Z)\right),$$

which is Cauchy's equation and so $f(R) = cR$ with $c > 0$ because f is strictly increasing. Using that in (55) and collecting terms yields

$$h_k(X - Y) - h_k(X) + h_k(Y) = -h_l(Z - Y) + h_l(Z) - h_l(Y).$$

Because X, Z, k, and l may be chosen independently, the common value is a function of Y only, say $K(Y)$. Then, setting $X = Y$ shows that $K(Y) \equiv 0$. Thus, h_k satisfies a Cauchy equation and so

$$M_{\mathbf{C}_n}^{(k)}(X) = h_k(X) = S_k(\mathbf{C}_n)X.$$

Because k can be any index $< n$, we see that the WU representation is satisfied. $\square$

Theorem 26

Proof: Using the notation presented earlier in the proof of Theorem 25, for $n = 2$ we have for the AU representation, $M_C(z) = h(z)$, for the IUI representation, $L_C(z) = f M_C\left(f^{-1}(z)\right) = f\left(h\left(f^{-1}(z)\right)\right)$, and for the mapping between the AU representation and the IUI representation, $U = f(U^*)$. This yields the functional equation

$$f(h(x)) - f(h(y)) + f(y) = f(h(x-y) + y) \quad (x \geq y). \tag{56}$$

There are the following three classes of solutions[10] to (56):

Solution 1

$$h(z) = z,\ f \text{ arbitrary.} \tag{57}$$

is of no interest because it means there is no dependence on the variable y, which is the utility corresponding to consequences.

Solution 2 asserts that there exist constants r, s such that

$$f(y) = ry \quad (r > 0), \tag{58}$$

$$h(z) = sz \quad (s \in]0,1]), \tag{59}$$

which corresponds to U being an RDU representation with each of U and U^* additive over $\oplus$.

Solution 3 asserts the existence of constants $\alpha \neq 0, \gamma > 0, a \in]0,1]$ such that

$$f(y) = \frac{1}{\alpha\gamma}(e^{\alpha y} - 1), \tag{60}$$

$$h(z) = \frac{1}{\alpha}\ln\left(\frac{ae^{\alpha z} + 1}{a+1}\right). \tag{61}$$

Using (60) and $U(z) = f[U^*(z)]$, it is routine to check that U is p-additive over $\oplus$ with $U(x \oplus y) = U(x) + U(y) + \frac{1}{\alpha\gamma}U(x)U(y)$ iff U^* is additive over $\oplus$, and that

$$f^{-1}(y) = \frac{1}{\alpha}\ln(1 + \alpha\gamma y)$$

Using this and (61), we have $L_C(z) = f\left(M_C\left(f^{-1}(z)\right)\right) = f\left(h\left(f^{-1}(z)\right)\right)$. Because h actually depends on the event C, and f does not, we replace a by a_C. Then a routine calculation gives that

$$L_C\left(U(\widetilde{x})\right) = \frac{a_C}{a_C + 1}U(\widetilde{x}),$$

that is, U is a RDU representation.

[10] This was first solved by Aczél, Luce, and Marley (2003) on the assumption that f and h are once differentiable. Later Ng showed that the result is unchanged if differentiability is dropped: Ng, C.T. (2003). "Monotonic solutions of a functional equation arising from simulataneous utility representations." Submitted.

The converse follows from a trivial rewriting of the idempotent WU form.□

Theorem 30

Proof: By assumption, U is additive over $\oplus$.

1. Suppose IUI and so $x_n \succ e$. Using the additivity of U freely,

$$\begin{aligned} U\left((\cdots; x_i, C_i; \cdots) \oplus y\right) &= U(\cdots; x_i, C_i; \cdots) + U(y) \\ &= \sum_{i=1}^{n-1} M_{\mathbf{C}_n}^{(i)} \left(U(x_i) - U(x_n)\right) + U(x_n) + U(y) \\ &= \sum_{i=1}^{n-1} M_{\mathbf{C}_n}^{(i)} \left(U(x_i \oplus y) - U(x_n \oplus y)\right) + U(x_n \oplus y) \\ &= U(\cdots; x_i \oplus y, C_i; \cdots), \end{aligned}$$

whence (38).

2. Recall that we are assuming that U is additive both over joint receipt $\oplus$ and over gambles. For $x \succsim y$, define $x \ominus y \sim z$ iff $x \sim z \oplus y$. Then because U is additive over joint receipts we have $U(x \ominus y) = U(x) - U(y)$ and $(x \ominus y) \oplus y \sim x$. Thus,

$$U(\cdots; x_i, C_i; \cdots) = U(\cdots; (x_i \ominus x_n) \oplus x_n, C_i; \cdots; e \oplus x_n, C_n).$$

Now suppose that segregation holds. Then, from this expression and using that U forms an additive representation of gambles, (5),

$$\begin{aligned} U(\cdots; x_i, C_i; \cdots) &= U\left((\cdots; x_i \ominus x_n, C_i; \cdots; e, C_n) \oplus x_n\right) \\ &= U(\cdots; x_i \ominus x_n, C_i; \cdots; e, C_n) + U(x_n) \\ &= \sum_{i=1}^{n-1} L_{\mathbf{C}_n}^{(i)} \left(U(x_i) - U(x_n)\right) + U(x_n), \end{aligned}$$

which is the IUI form (24).

Conversely, suppose the IUI form, then

$$\begin{aligned} U(\cdots; x_i \oplus x_n; \cdots; e \oplus x_n) &= \sum_{i=1}^{n-1} M_{\mathbf{C}_n}^{(i)} \left(U(x_i \oplus x_n) - U(x_n)\right) + U(x_n) \\ &= \sum_{i=1}^{n-1} M_{\mathbf{C}_n}^{(i)} \left(U(x_i)\right) + U(x_n) \\ &= U(\cdots; x_i, C_i; \cdots; e) + U(x_n) \\ &= U\left((\cdots; x_i, C_i; \cdots; e) \oplus x_n\right), \end{aligned}$$

whence, taking U^{-1}, segregation follows.

Thus, we have

$$\sum_{i=1}^{n} L_{\mathbf{C}_n}^{(i)} \left(U(x_i)\right) = \sum_{i=1}^{n-1} M_{\mathbf{C}_n}^{(i)} \left(U(x_i) - U(x_n)\right) + U(x_n),$$

i.e., there is both an additive and IUI representation, and so, for $n > 2$, Theorem **??** implies that idempotent WU holds. □

Theorem 31.

Proof: 1. By segregation, decomposability of gambles using G and of joint receipt using F, the assumption that gambles with $x_n = e$ satisfy the conditions of Theorem 8, and the assumption that $x_i \succsim x_n$,

$$\begin{aligned}
&U(\cdots ; x_i, C_i; \cdots) \\
&\quad = U\left((x_1 \ominus x_n, C_1; \cdots ; x_{n-1} \ominus x_n, C_{n-1}; e, C_n) \oplus x_n\right) \\
&\quad = F\left(U(x_1 \ominus x_n, C_1; \cdots ; x_{n-1} \ominus x_n, C_{n-1}; e, C_n), U(x_n)\right) \\
&\quad = F\left(\sum_{i=1}^{n-1} L_{\mathbf{C}_n}^{(i)}\left(U(x_i \ominus x_n)\right), U(x_n)\right).
\end{aligned}$$

2. Suppose that U is generalized additive over $\oplus$, (26), then:

$$\begin{aligned}
U(\cdots ; x_i, C_i; \cdots) &= F\left(\sum_{i=1}^{n-1} L_{\mathbf{C}_n}^{(i)}\left(U(x_i \ominus x_n)\right), U(x_n)\right) \\
&= \varphi^{-1}\left(\varphi\left(\sum_{i=1}^{n-1} L_{\mathbf{C}_n}^{(i)}\left(U(x_i \ominus x_n)\right)\right) + \varphi\left(U(x_n)\right)\right) \\
&= \varphi^{-1}\left(\varphi\left(\sum_{i=1}^{n-1} L_{\mathbf{C}_n}^{(i)}\left(\varphi^{-1}\left(\varphi(U(x_i) - \varphi U(x_n)\right)\right)\right) + \varphi\left(U(x_n)\right)\right),
\end{aligned}$$

which, if we set $V = \varphi\left(U\right)$, becomes:

$$\begin{aligned}
V(\cdots ; x_i, C_i; \cdots) &= \varphi\left(\sum_{i=1}^{n-1} L_{\mathbf{C}_n}^{(i)} \varphi^{-1}\left(V(x_i) - V(x_n)\right)\right) + V(x_n) \\
&= \varphi\left(\sum_{i=1}^{n-1} M_{\mathbf{C}_n}^{(i)}\left(V(x_i) - V(x_n)\right)\right) + V(x_n)
\end{aligned}$$

where $M_{\mathbf{C}_n}^{(i)} = L_{\mathbf{C}_n}^{(i)} \varphi^{-1}$.

3. Suppose that φ is the identity. Then the representation U of Part 2 over gambles with $x_n \succ e$ reduces to the IUI form, Def. 24. □

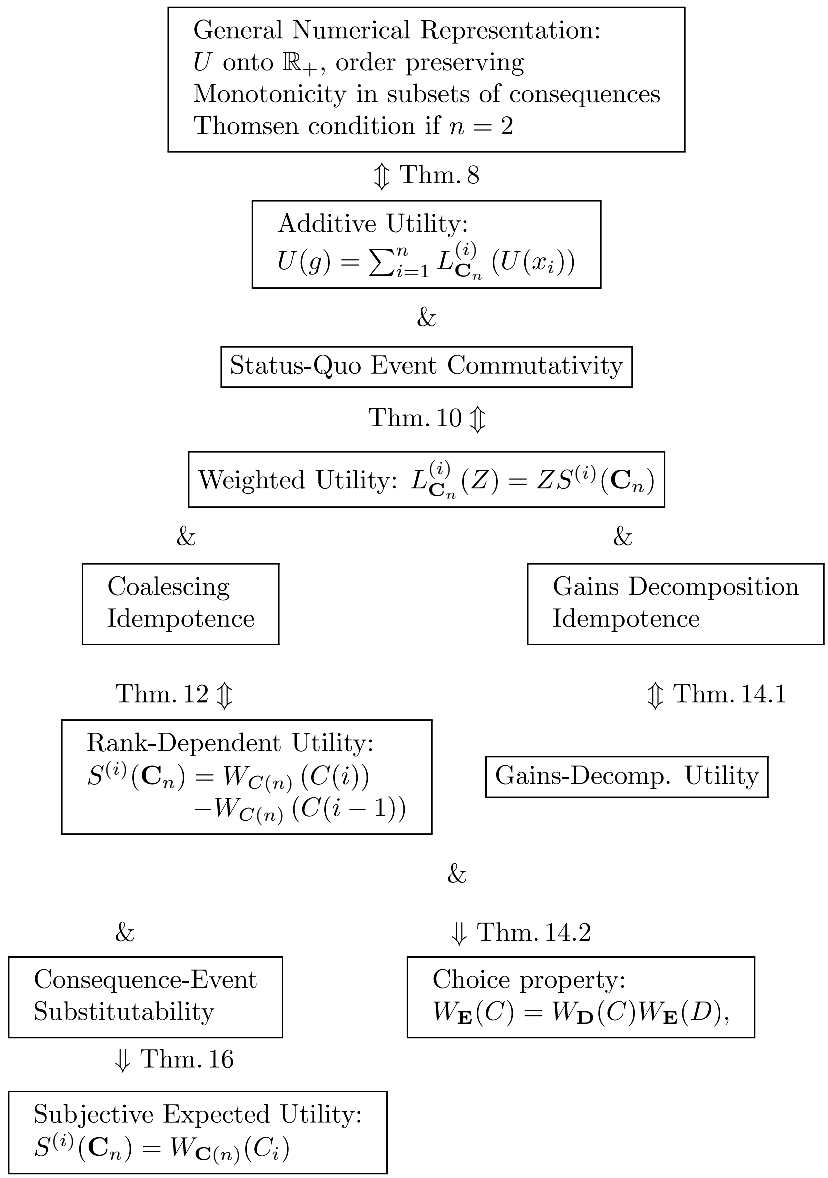

Figure 1. The network of theorems having to do with the additive representations of gambles (Section 2). The background assumptions are an order-preserving utility representation U, gambles are strictly increasing in the consequences, and joint receipts are strictly increasing in each variable. Conventions: When a pointed end of an arrow points to a box, all of its implications are in that box, and when it points to & then both properties are implied.

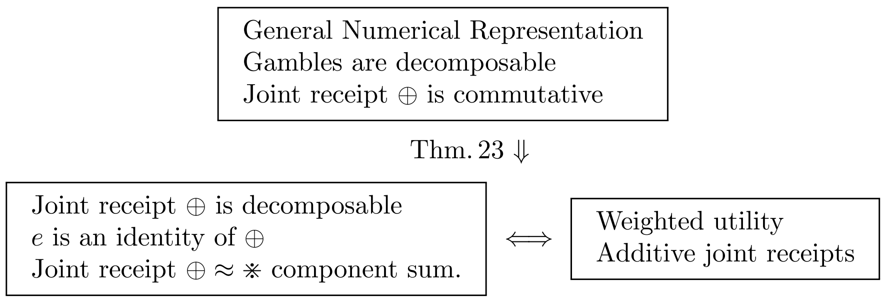

Figure 2. A graphical representation of Theorem 23. The notational conventions are those of Fig. 1.

Increasing Utility Increments:
$\sum_{i=1}^{n-1} M_{\mathbf{C}_n}^{(i)} \left(U(x_i) - U(x_n)\right) + U(x_n)$

&

Additive Utility:
$U(g) = \sum_{i=1}^{n} L_{\mathbf{C}_n}^{(i)} \left(U(x_i)\right)$

$n > 2 \Updownarrow$ Thm. 25 $\qquad$ $n = 2 \Updownarrow$ Thm. 26

WU $\qquad$ RDU
p-additive $\oplus$

Figure 3. The results from assuming that both IUI and AU representations hold, Theorems 26 and 25. The notational conventions are those of Fig. 1.