MEASUREMENT, THEORY OF

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Most mathematical sciences rest upon quantitative models, and the theory of measurement is devoted to making explicit the qualitative assumptions that underlie them. This is accomplished by first stating the qualitative assumptions – empirical laws of the most elementary sort – in axiomatic form and then showing that there are structure preserving mappings, often but not always isomorphisms, from the qualitative structure into a quantitative one. The set of such mappings forms what is called a "scale of measurement".

A theory of the possible numerical scales plays an important role throughout measurement – and therefore throughout science. Just as the qualitative assumptions of a class of structures narrowly determine the nature of the possible scales, so also the nature of the underlying scales greatly limits the possible qualitative structures that give rise to such scales. Two major themes of this entry reflect research results of the 1970s and 80s: (*i*) the possible scales that are useful in science are necessarily very limited and, (*ii*) once a type of scale is selected (or assumed to exist) for a qualitative structure, then a great deal is known about that structure and its quantitative models. A third theme concerns applications of these ideas to the behavioral sciences, especially to utility theory and psychophysics from 1980 onward.

There are several general references to the axiomatic theory. Perhaps the most elementary and the one with the most examples is Roberts (1979). Pfanzagl (1968) and Krantz et al. (1971/1989) are on a par, with the latter more comprehensive. Narens (1985), which is the mathematically most sophisticated, covers much of the basic material mentioned here. Later additions are: Luce, et al. (1990), which has much in common with Narens (1985); Suppes, et al. (1989), which is focused on geometric representations and probability generalizations; Narens (in press), which is a more narrowly focused introductory book with examples mainly from psychophysics; and Suppes (2002). Mostly, we cite only references not included in one of these surveys.

1. Axiomatizability

The Qualitative Setup. The qualitative situation is usually conceptualized as a relational structure $\mathcal{X} = \langle X, S_0, S_1, \dots \rangle$, where the S_0, S_1, $\dots$ are finitary relations on X. The number of relations can be either finite or infinite, but in applications almost always finite. X is called the domain of the structure and the S_i its primitive relations. In most applications, S_0 will be some type of ordering relation that is usually written as $\succsim$. The following are some examples of qualitative structures used in measurement situations.

The first, goes back to Helmholtz (see section 4). It has for its domain a set X of objects with the properties like those of mass. There are two primitive relations. The first, $\succsim$, is a binary ordering according to mass (which may be determined, for example, by using an equal-arm pan balance so that $x \succsim y$ means that the pans either remain level or the one containing x drops). The second is a binary operation $\circ$, which formally is a ternary relation. For mass it is empirically defined as follows: if x and y are placed in the same pan and are exactly balanced by z, then we write $x \circ y \sim z$, where $\sim$ means equivalence in the attribute. Other interpretations of the primitives of $\langle X, \succsim, \circ\rangle$ can be found in the above references. Axiomatic treatments of the structure $\langle X, \succsim, \circ\rangle$ are discussed in section 4.

A second example is from economics. Suppose that $C_1, \ldots, C_n$ are sets each consisting of different amounts of a commodity, and $\succsim$ is a preference ordering exhibited by a person or an institution over the set of possible commodity bundles $C = \prod_i C_i$. $\langle C, \succsim\rangle$ is called a *conjoint structure*, and axioms about it are given that among other things induce an ordering, $\succsim_i$, of an individual's preferences for the commodities associated with each component i.

A third example, due to B. de Finetti, has as its domain an algebra of subsets, called "events", of some non-empty set Ω. The primitives of the structure consist of an ordering relation $\succsim$ of "at least as likely as", the events Ω and $\varnothing$ and the set theoretical operations of union $\cup$, intersection $\cap$, and complementation $\neg$. The relational structure

$$\mathcal{P} = \langle \mathcal{E}, \succsim, \Omega, \varnothing, \cup, \cap, \neg\rangle \tag{1}$$

is intended to characterize qualitatively probability-like situations. The primitive $\succsim$ can arise from many different processes, depending upon the situation. In one, which is of considerable importance to Bayesian probability theorists and statisticians, $\succsim$ represents a person's ordering of events according to how likely they seem, using whatever basis he or she wishes in making the judgements. In such a case, $\mathcal{P}$ is thought of as a *subjective or personal probability structure.* In another, $\succsim$ is an ordering of events based on some probability model for the situation (possibly one coupled with estimated relative frequencies), as in much of classical probability theory.

2. Ordered Structures

Weak Order, Dedekind Completeness, and Unboundedness. Two types of "quantitative" representations have played a major role in science: systems of coordinate geometry and the real number system (the latter being the one-dimensional specialization of the former). Results about the former are in Suppes et al. (1989), but our focus here is the latter. The absolutely simplest case, included in all of the above examples, is the order-preserving representation ϕ of $\langle X, \succsim\rangle$ into $\langle \mathbb{R}, \geq\rangle$, where $\mathbb{R}$ denotes the real numbers. An immediate implication is that $\succsim$ must be transitive, reflexive, and connected (for all x and y, either $x \succsim y$ or $y \succsim x$). Such relations are given many different names including *weak order*. An antisymmetric weak order is called a *total* or *simple order*. There has been much empirical controversy about the transitivity of $\succsim$, with the most recent Bayesian analyses favoring transitivity of $\succ$ but not of $\sim$ (Myung, Karabatsos, & Iverson, 2005). Some doubt has been expressed about completeness. Nevertheless, most of the well-developed measurement-theoretic techniques assumes both the completeness and transitivity of $\succsim$ as idealizations.

G. Cantor showed that for $\langle X, \succsim \rangle$ to be so represented, necessary and sufficient conditions are that $\succsim$ be a weak order and that there be a finite or countable subset Y of X that is order dense in X (i.e., for each $x \succ z$ there exists a y in Y such that $x \succsim y \succsim z$. For many purposes, this subset plays the same role as do the rational numbers within the system of real numbers.

In order for the representation to be onto either $\langle \mathbb{R}, \geq \rangle$ or $\langle \mathbb{R}^+, \geq \rangle$, where $\mathbb{R}^+$ denotes the positive real numbers, which often happens in physical measurement, two additional conditions are necessary and sufficient: *Dedekind completeness* (each non-empty bounded subset of X has a least upper bound in X) and *unboundedness* (there is neither a least nor a greatest element).

In measurement axiomatizations, one usually does not postulate a countable, order-dense subset, but derives it from axioms that are intuitively more natural. For example, with a binary operation of combining objects, order density follows from a number of properties including an *Archimedean axiom* which states in some fashion that no object is either infinitely larger than or infinitesimally close to another object. When the structure is Dedekind complete and the operation is monotonic, it is also Archimedean. Dedekind completeness and Archimedeaness are what logicians call "second order axioms," and in principle they are incapable of direct empirical verification.

The most fruitful and intensively examined measurement structures are those with a weak ordering $\succsim$ and an associative, positive binary operation $\circ$ that is strictly monotonic ($x \succsim y$ iff $x \circ z \succsim y \circ z$). They have been the basis of much physical measurement. However, for much of the 20th century, they played little role in the behavioural and social sciences but, as seen in sections 6 and 7, in the past 20 years such operations have come to be useful. The development of a general non-associative and non-positive ($x \succsim x \circ y$ for some x and y) theory began in 1976, and it is moderately well understood in certain situations having many symmetries[1]. This, and its specialization to associative structures, is the focus of section 3.

Representations and Scales. A key concept in the theory of measurement is that of a *representation*, which is defined to be a structure preserving map ϕ of the qualitative, weakly ordered relational structure $\mathcal{X}$ into a quantitative one, $\mathcal{R}$, in which the domain is a subset of the real numbers. Representations are either isomorphisms or homomorphisms. The latter are used in cases where equivalences play an important role (e.g., conjoint structures where trade-offs between components are the essence of the matter), in which case equivalence classes of equivalent elements are assigned the same number. We say ϕ is a *$\mathcal{R}$-representation for $\mathcal{X}$*.

From 1960 through 1990, measurement theorists were largely focused on certain types of qualitative structures for which numerical representations exist. The questions faced are two: The first, the "existence" problem, is to establish that the set of $\mathcal{R}$-representations is non-empty for $\mathcal{X}$. Cantor's conditions above establish existence of a numerical representation of any weak order. The

[1]Technically, automorphisms of the structure are isomorphic transformations of the structure onto itself.

second, the "uniqueness", problem is to describe compactly the set of all $\mathcal{R}$-representations. Several examples are cited. Since 1990, the focus has been increasingly on applying these insights to behaviour. We cite aspects of utility theory, global psychophysics, and probability.

For the qualitative mass structure $\mathcal{X} = \langle X, \succsim, \circ \rangle$ described previously, the qualitative representing structure is taken to be $\mathcal{R} = \langle \mathbb{R}^+, \geq, + \rangle$ where $\geq$ and $+$ have their usual meanings in $\mathbb{R}^+$. The set of $\mathcal{R}-$representations of $\mathcal{X}$ consist of all functions ϕ from X into $\mathbb{R}^+$ such that for each x and y in X,

(i) $x \succsim y$ iff $\phi(x) \geq \phi(y)$, and

(ii) $\phi(x \circ y) = \phi(x) + \phi(y)$.

Such a function is called a *homomorphism for* $\mathcal{X}$, and the set of all of them is called a *scale (for* $\mathcal{X}$*)*. In addition to Helmholtz, others – including O. Hölder, P. Suppes, Luce and A. A. J. Marley, and J.-C. Falmagne – have stated axioms about the primitives that are sufficient to show the existence of such homomorphisms and to show the following uniqueness theorem: Any two homomorphisms ϕ and ψ are related by positive multiplication, that is, there is some real $r > 0$ such that $\psi = r\phi$. In the language introduced by S. S. Stevens (1946), such a form of measurement is said to form a "ratio scale". For cases where $\circ$ is an operation (defined for all pairs), F. S. Roberts (1979) and Luce and Narens (1985) gave necessary and sufficient conditions for such a representation. Such a complete characterization as this one is rather unusual in measurement; sufficient conditions are far more the norm. Often they entail structural assumptions, such as a solvability condition, as well as necessary ones.

Representations of the structure $\mathcal{C} = \langle \prod_i C_i, \succsim \rangle$ of commodity bundles are usually taken in economics to be n-tuples $\langle \phi_1, ..., \phi_n \rangle$ of functions, where ϕ_i maps C_i into $\mathbb{R}^+$, such that for each x_i and y_i in C_i, $i = 1, \ldots, n$,

$$(x_1, ..., x_n) \succsim (y_1, ..., y_n) \ \text{ iff } \ \sum_i \phi_i(x_i) \geq \sum_i \phi_i(y_i)\,. \tag{2}$$

In the measurement literature such a conjoint representation is called "additive". G. Debreu, Luce and J. W. Tukey, D. Scott, A. Tversky and others, gave axioms about $\mathcal{C}$ for which existence of an additive representation can be shown, and such that any two representations $\langle \phi_1, ..., \phi_n \rangle$ and $\langle \psi_1, ..., \psi_n \rangle$ are related by affine transformations of the form $\psi_i = r\phi_i + s_i, i = 1, ..., n, r > 0$. Note that r is common to all components. In Stevens' nomenclature, the set of such representations ψ_i for each fixed i are said to form an "interval scale".

In the example of the subjective probability structure, Equation **1**, the usual sort of representation is a probability function P from $\mathcal{E}$ into $[0, 1]$, such that, for all A, B in $\mathcal{E}$,

(i) $P(\Omega) = 1$ and $P(\varnothing) = 0$,

(ii) $A \succsim B$ iff $P(A) \geq P(B)$, and

(iii) if $A \cap B = \varnothing$, then $P(A \cup B) = P(A) + P(B)$.

Unlike the previous two examples, here any two representations are identical, which scales Stevens called "absolute". Such a scale might be appropriate for representing a qualitative structure describing a relative frequency approach to probability. However, for subjective probability, it is better to view P as being a representation of the bounded ratio scale $\{rP \,|\, r > 0\}$ that is normalized by setting the bound, Ω, to be $1 = rP(\Omega)$.

A number of authors have given sufficient conditions in terms of the primitives for P to exist. Fine (1973) gave the first good, early summary of a variety of approaches to probability. Additional approaches to qualitative and subjective probability can be found in Narens (submitted).

Interlocked Measurement Structures. A very common, and fundamentally important, feature of measurement is the existence of two or more ways to manipulate the same attribute. Again, mass measurement is illustrative. The mass order $\langle X, \succsim \rangle$ is determined as above. Mass can be manipulated in at least two ways by varying volumes and/or substances. Let $\langle V, \succsim', \circ_V \rangle$ be a structure for combining volumes, where V is a set of volumes and $\circ_V$ is a strictly monotonic, positive, and associative operation over V, and let $X = V \times S$ be a structure of masses, where S is a set of homogeneous substances of various densities. (v, s) is interpreted as an object of volume v filled with substance s and that, therefore, has mass. By definition, $\circ_V$ is the operation on $V \times \{s\}$ such that $(v, s) \circ_V (v', s) = (v \circ_V v', s)$. The first manipulation is to vary $\succsim$ via volume concatenation of a single homogeneous material s, $\langle V \times \{s\}, \succsim, \circ_V \rangle$. The second is to manipulate the conjoint trade-off between volumes and substances, $\langle V \times S, \succsim \rangle$. Let m and m^* be the resulting representations of mass which, because they both preserve $\succsim$, they must be strictly monotonically related. The ordering interlock alone is insufficient to develop measurement as was done in classical physics and as reflected in the familiar structure of physical units. Comparable developments are now beginning to appear in the behavioral and social sciences. The two structures must be interlocked beyond $\succsim$. Such interlocks are often types of distribution laws. In the mass case, the *distributive interlock* is: For $u, v \in V$ and $r, s \in S$,

$$(u, r) \sim (v, s) \text{ and } (u', r) \sim (v', s) \text{ imply } (u \circ_V u', r) \sim (v \circ_V v', s)\,.$$

For much more detail, see Luce et al. (1990). Such laws are the source of the structure reflected in the units of physical measurement that are used and underlie dimensional analysis (Krantz et al., 1971; Luce et al., 1990; Narens, 2002).

Typically, one is able to use the two separate numerical representations to reduce the interlock to solving a functional equation.[2]

[2] A functional equation resembles a differential one in that its solutions are the unknown functions satisfying the equation. It is unlike a differential equation in that no derivatives are involved; rather the equation relates the value of the function at several values of the independent variable. See Aczél (1966, 1987) for a general introduction and classical examples of functional equations. Some arising in the behavioural and social sciences were novel and have required the aid of specialists to solve.

Behavioural examples of interlocked structures are cited in sections 6 and 7.

Empirical Usefulness of Axiomatic Treatments. One, seemingly under appreciated, advantage of a measurement approach to some scientific questions is that it offers an alternative way of testing quantitative models other than attempting to fit the representation to data and to evaluate it by a measure of goodness of fit. Because representations, such as utility and subjective probability, in general have free parameters and often free functions, estimation is necessary. In contrast, the axioms underlying such representations are (usually) parameter free. Testing the axioms often makes clear the source of a problem, thereby giving insight into what must be altered. Not everyone values the overall axiomatic (as compared to an analytic mathematical) approach to scientific questions; in particular, Anderson (1981, pp. 347–56) has sharply attacked it.

A familiar economic example arose in the theory of subjective expected utility (Fishburn, 1970; Savage, 1954). In its simplest form the domain is gambles of the form $x \circ_A y$, meaning that x is the consequence attached to the occurrence of the chance event A, whereas y is the consequence when the chance outcome is $\neg A$. The x and y may be pure consequences or may be themselves gambles, and the theory postulates a preference ordering $\succsim$ over the pure consequences and gambles constructed from pure consequences and gambles. Classical axiomatizations establish conditions on preferences over gambles so that there exists a probability measure P on the algebra of events, as in a probability structure, and a "utility function" U over the gambles such that U preserves $\succsim$ and

$$U(x \circ_A y) = P(A)U(x) + [1 - P(A)]U(y)\,. \tag{3}$$

A series of early empirical studies (for summaries see Allais and Hagen, 1979, and Kahneman and Tversky, 1979) made clear that this representation, which can be readily defended on grounds of rationality, fails to describe human behaviour. Among its axioms, the one that appears to be the major source of difficulty is the "extended sure-thing principle". It may be stated as follows: For events A, B and C, with C disjoint from A and B,

$$x \circ_A y \succsim x \circ_B y \;\text{ iff }\; x \circ_{A\cup C} y \succsim x \circ_{B\cup C} y\,. \tag{4}$$

It is easy to verify that Equation 3 implies equation Equation 4, but people seem unwilling to abide by Equation **4**. Any attempt at a descriptive theory must abandon it (see below).

Non-Uniqueness of Axiom Systems. The isolation of properties in the axiomatic approach has an apparently happenstance quality because the choice of qualitative axioms is by no means uniquely determined by the representation. Any infinite structure has an infinity of equivalent axiom systems, and it is by no means clear why we select the ones that we do. It is entirely possible for a descriptive failure to be easily described in one axiomatization and to be totally obscure in another. Thus, some effort is spent on finding alternative but equivalent axiomatizations.

A related use of axiomatic methods, including the notion of scale (see *Representations and Scales* above and section 3) is to study scientific meaningfulness, which is treated under *Meaningfulness and Invariance.*

3. Scale Types

Classification. As was noted in the examples, scale type has to do with the nature of the set of maps from one numerical representation of a structure into all other equally good representations, in a particular numerical structure such as the multiplicative real numbers. For some fixed numerical structure $\mathcal{R}$, a scale of the structure $\mathcal{X}$ is the collection of all $\mathcal{R}$-representations of $\mathcal{X}$. Much the simplest case, the one to which we confine most of our attention, occurs when $\mathcal{X}$ is totally ordered, the domain of $\mathcal{R}$ is either $\mathbb{R}$ or $\mathbb{R}^+$, and the $\mathcal{R}$-representations are all onto the domain and so are isomorphisms. Such scales are then usually described in terms of the (mathematical) group of real transformations that take one representation into another. As Stevens noted, four distinct groups of transformations have appeared in physical measurement: any strictly monotonic function, any linear function $rx + s, r > 0$, any similarity transformation $rx, r > 0$, and the identity map. The corresponding scales are called *ordinal*, *interval*, *ratio*, and *absolute*. (Throughout this entry, although not in all of the literature, ratio scales are assumed to be onto $\mathbb{R}^+$ thereby ruling out cases were an object maps to 0.)

A property of the first three scale types, called *homogeneity*, is that for each element x in the qualitative structure and each real number r in the domain of $\mathcal{R}$, some representation maps x into r. Homogeneity, which is typical of physical measurement, plays an important role in formulating many physical laws. Two general questions are: What are the possible groups associated with homogeneous scales, and what are the general classes of structures that can are represented by homogeneous scales?

It is easiest to formulate answers to these questions in terms of automorphisms (= symmetries), that is, isomorphisms of the qualitative structure onto itself. The representations and the automorphisms of the structure are in one-to-one correspondence, because if ϕ and ψ are two representations and juxtaposition denotes function composition, then $\psi^{-1}\phi$ is an automorphism of the structure, and if ϕ is a representation and α is an automorphism, then $\psi = \phi\alpha$ is a representation.

It is not difficult to see that homogeneity of a scale simply corresponds to there being an automorphism that takes any element of the domain of the structure into any other element. To make this more specific, for M a positive integer, $\mathcal{X}$ is said to be *M-point homogeneous* if and only if each strictly ordered set of M points can be mapped by an automorphism onto any other strictly ordered set of M points. A structure that fails to be homogeneous for $M = 1$ is said to be *0-point homogeneous*; one that is homogeneous for every positive integer M is said to be *∞-point homogeneous*.

A second important feature of a scale is its degree of redundancy, formulated as follows: a scale is said to be *N-point unique*, where N is a non-negative integer if and only if for every two representations ϕ and ψ in the scale that agree at N distinct points, then $\phi = \psi$. By this definition, ratio scales are 1-point unique, interval scales are 2-point unique, and absolute scales 0-point unique. Scales, such as ordinal ones, that take infinitely many points to determine a representation are said to be ∞-*point unique.* Equally, we speak of the structure being N-*point unique* if and only if every two automorphisms that agree at N distinct points are identical.

The abstract concept of scale type can be given in terms of these concepts. The *scale type* of $\mathcal{X}$ is the pair (M, N) such that M is the maximum degree of homogeneity and N is the minimum degree of uniqueness of $\mathcal{X}$. For the types of cases under consideration, it can be shown that $M \leq N$. Ratio scales are of type $(1, 1)$ and interval scales of type $(2, 2)$. Narens (1981a,b) showed that the converses of both statements are true. And Alper (1987) showed that if $M > 0$ and $N < \infty$, then $N = 1$ or 2. The group in the $(1, 2)$ case consists of transformations of the form $rx + s$, where s is any real number and r is in some non-trivial, proper subgroup of the multiplicative group $\langle \mathbb{R}^+, \cdot \rangle$. One example is $r = k^n$, where $k > 0$ is fixed and n ranges over the integers. So a structure is homogeneous if and only if it is of type $(1, 1), (1, 2), (2, 2)$, or (M, ∞). The (M, ∞) case is not fully understood. Ordinal scalable (∞, ∞) structures appear frequently in science, and a $(1, \infty)$ structure for threshold measurement appears in psychophysics. We focus here on the $(1, 1), (1, 2), (2, 2)$ cases. For detailed references, see Luce et al. (1990), Narens (1985), or Narens (in press).

Unit Representations of Homogeneous Concatenation Structures. The next question is: Which structures have scales of these types? Although the full answer is unknown, it is completely understood for ordered structures with binary operations. This is useful because, as was noted, the associative form of these operations play a central role in much physical measurement and, as we shall see below, both associative and non-associative forms arise naturally in two distinct ways of interest to behavioural and social scientists.

Consider real concatenation structures of the form $\mathcal{R} = \langle \mathbb{R}^+, \geq, *' \rangle$ where $\geq$ has its usual meaning and we have replaced $+$ by a general binary, numerical operation $*'$ that is strictly monotonic in each variable. The major result is that if $\mathcal{X}$ satisfies $M > 0$ and $N < \infty$ – a sufficient condition for finite N is that $*'$ be continuous (Luce and Narens, 1985) – then the structure can be mapped canonically into an isomorphic one of the form $\langle \mathbb{R}^+, \geq, * \rangle$, with a function f from $\mathbb{R}^+$ onto $\mathbb{R}^+$ such that

(i) f is strictly increasing,

(ii) $f(x)/x$ is strictly decreasing, and

(iii) for all x, y in $\mathbb{R}^+$, $x * y = yf(x/y)$

(Cohen and Narens, 1979).

This type of canonical representation, which is called a *unit representation,* is invariant under the similarities of a ratio scale, that is, for each positive real r,

$$rx * ry = ryf(rx/ry) = ryf(x/y) = r(x * y) .$$

The two most familiar examples of unit representations are ordinary additivity, for which $f(z) = 1 + z$ and so $x * y = x + y$, and bisymmetry, for which $f(z) = z^c$,

$c \in (0, 1)$, and so $x * y = x^c y^{1-c}$. Situations where such representations arise are discussed later.

A simple invariance property of the function f corresponds to the three finite scale types (Luce and Narens, 1985). Consider the values of $\rho > 0$ for which $f(x^\rho) = f(x)^\rho$ for all $x > 0$. The structure is of scale type $(1, 1)$ if and only if $\rho = 1$; of type $(1, 2)$ if and only if for some fixed $k > 0$ and all integers n, $\rho = k^n$; and of type $(2, 2)$ if and only if there are constants c and d in $(0, 1)$ such that

$$f(z) = \begin{cases} z^c, & z \geq 1 \\ z^d, & z < 1\,. \end{cases}$$

If, as is the usual practice in the social sciences (see subjective expected utility, section 6), but not in physics, the above representation is transformed by by taking logarithms, it becomes a weighted additive form on $\mathbb{R}$:

$$x * y = \begin{cases} cx + (1-c)y, & x \geq y \\ dx + (1-d)y, & x < y\,. \end{cases}$$

That representation is called *dual bilinear* and the underlying structures are called dual bisymmetric (when $c = d$, the "dual" is dropped). For references see Luce et al. (1990).

4. Axiomatization of Concatenation Structures

Given this understanding of the possible representations of homogeneous, finitely unique concatenation structures, it is natural to return to the classical question of axiomatizing the qualitative properties that lead to them. Until the 1970s, the only two cases that were understood axiomatically were those leading to additivity and averaging (see below). We now know more, but our knowledge remains incomplete.

Additive Representations. The key mathematical result underlying extensive measurement, due to O. Hölder, states that when a group operation and a total ordering interlock so that the operation is strictly monotonic and is Archimedean in the sense that sufficient copies of any positive element (i.e., any element greater than the identity element) will exceed any fixed element, then the group is isomorphic to an ordered subgroup of the additive real numbers. Basically, the theory of extensive measurement restricts itself to the positive subsemigroup of such a structure. Extensive structures can be shown to be of scale type $(1, 1)$. Various generalizations involving partial operations (defined for only some pairs of objects) have been developed. (For a summary, see Krantz et al., 1971, chs 2, 3, and 5; Luce et al., 1990, ch 19). Not only are these structures with partial operations more realistic, they are essential to an understanding of the partial additivity that arises in such cases as probability structures. They can be shown to be of scale type $(0, 1)$. Michell (1999) gives an alternative perspective on measurment in the behavioural sciences and a critique of axiomatic measurement approaches.

The representation theory for extensive structures not only asserts the existence of a numerical representation, but provides a systematic algorithm (involving the Archimedean property) for constructing one to any preassigned degree of accuracy. This construction, directly or indirectly, underlies the extensive scales used in practice.

The second classical case, due to J. Pfanzagl, leads to weighted average representations. The conditions are monotonicity of the operation, a form of solvability, an Archimedean condition, and bisymmetry, $(x \circ u) \circ (y \circ v) \sim (x \circ y) \circ (u \circ v)$, which replaces associativity. One method of developing these representations involves two steps: first the bisymmetric operation is recoded as a conjoint one (see section 5) as follows: $(u, v) \succsim (x, y)$ iff $u \circ v \succsim x \circ y$; and second, the conjoint structure is recoded as an extensive operation on one of its components. This reduces the proof of the representation theorem to that of extensive measurement, that is to Hölder's theorem, and so it too is constructive.

Non-Additive Representations. The most completely understood generalization of extensive structures, called positive concatenation structures or PCSs for short, simply drops the assumption of associativity. Narens and Luce (see Narens, 1985; Luce et al., 1990, ch 19) showed that this was sufficient to get a numerical representation and that, under a slight restriction which has since been removed, the structure is 1-point unique, but not necessarily 1-point homogeneous. Indeed, Cohen and Narens (1979) showed that the automorphism group is an Archimedean ordered group and so is isomorphic to a subgroup of the additive real numbers; it is homogeneous only when the isomorphism is to the full group. As in the extensive case, one can use the Archimedean axiom to construct representations, but the general case is a good deal more complex than the extensive one and almost certainly requires computer assistance to be practical.

For Dedekind complete PCSs that map onto $\mathbb{R}^+$, a nice criterion for 1-point homogeneity is that for each positive integer n and every x and y, $n(x \circ y) = nx \circ ny$, where by definition $1x = x$ and $nx = (n-1)x \circ x$. The form of the representations of all such homogeneous representations was described earlier.

The remaining broad type of concatenation structures consists of those that are idempotent, i.e. for all x, $x \circ x = x$. The following conditions have been shown to be sufficient for idempotent structures to have a numerical representation (Luce and Narens, 1985): $\circ$ is an operation that is strictly monotonic and satisfies an Archimedean condition (for differences) and a solvability condition that says for each x and y, there exist u and v such that $u \circ x = y = x \circ v$. If, in addition, such a structure is Dedekind complete, it can be shown that it is N-point unique with $N \leq 2$.

5. Axiomatization of Conjoint Structures

Binary structures. A second major class of measurement structures, widely familiar from both physics and the social sciences, are those involving two or more independent variables exhibiting a trade-off in the to-be-measured dependent variable. Their commonness and importance in physics is illustrated by familiar physical relations among three basic attributes, such as kinetic energy $= mv^2/2$, where m is the mass and v the velocity of a moving body. Such conjoint trade-off structures are equally common in the behavioural and social sciences: preference between commodity bundles or between gambles; loudness of pure tones as a function of signal intensity and frequency; trade-off between delay and amount of a reward etc. Although there is some theory for more than two independent variables in the additive case, with the general representation given by Equation 2, for present purposes we confine attention to the two variable case $\langle X \times S, \succsim \rangle$. Michell (1990) gives detailed analyses of a number of behavioral examples.

As with concatenation structures, the simplest case to understand is the additive one in which the major non-structural properties are:

(i) *Independence (monotonicity):* if $(x, s) \succsim (x', s)$ holds for some s, then it holds for all s in S, and the parallel statement for the other component. Note that this property allows us to induce natural orderings, $\succsim_X$ on X and $\succsim_S$ on S.

(ii) *Thomsen condition:* if $(x, r) \sim (y, t)$ and $(y, s) \sim (z, r)$, then $(x, s) \sim (z, t)$.

(iii) *An Archimedean condition* which says that if $\{x_i\}$ is a bounded sequence and if for some $r \nsim s$ it satisfies $(x_i, r) \sim (x_{i+1}, s)$, then the sequence is finite. A similar statement holds for the other component.

These properties, together with some solvability in the structure, are sufficient to prove the existence of an interval scale, additive representation (for a summary of various results, see Krantz et al., 1971, chs 6, 7 and 9). The result has been generalized to non-additive representations by dropping the Thomsen condition, which leads to the existence of a non-additive numerical representation (Luce et al., 1990, chs 19, 20). The basic strategy is to define an operation, say $\circ_X$ on component X, that captures the information embodied in the trade-off between components. The induced structure can be shown to consist of two PCSs pieced together at an element that acts like a natural zero of the concatenation structure. The results for PCSs are then used to construct the representation. As might be anticipated, $\circ_X$ is associative if and only if the conjoint structure satisfies the Thomsen condition.

6. Interlocked Structures and Applications to Utility Theory

Interlocked conjoint/extensive structures. The next more complex structure has the form $\mathcal{D} = \langle X \times S, \succsim, \circ \rangle$, where $\circ$ is an operation on S. Such structures appear in the construction of the dimensional structure of physical units. The key qualitative axioms for physical measurement are that $\langle X \times S, \succsim \rangle$ is a conjoint structure satisfying independence, $\langle S, \succsim_S, \circ \rangle$ is an extensive structure, where $\succsim_S$ is the induced ordering on S, and $\mathcal{D}$ is distributive, that is,

$$\text{if } (x, p) \sim (yq) \text{ and } (x, s) \sim (y, t), \text{ then } (x, p \circ s) \sim (y, q \circ t).$$

These axioms yield the following representation for $\mathcal{D}$: The exists a ratio scale S for the extensive structure $\langle S, \succsim_S, \circ \rangle$ such that for each $\phi \in S$ there exists ψ from X into the positive reals such that for all x, y in X and all s, t, p, q in S, there exists a representation ϕ on S that is part of a multiplicative representation of the conjoint structure and additive over the concatenation operation, that is,

(i) $(x, s) \succsim (y, t)$ iff $\psi(x)\phi(s) \geq \psi(y)\phi(t)$, and

(ii) $\phi(p \circ q) = \phi(p) + \phi(q)$.

Discussions of how to construct the full algebra of physical dimensions using distributive structures and how to generalize these algebras to situations where there are not primitive associative operations are discussed in Luce et al. (1990) and Narens (2002).

Rationality Assumptions in Traditional Utility Theory. As was noted earlier, an extensive literature exists on preferences among uncertain alternatives, often called "gambles." The first major theoretical development was the axiomatization of subjective expected utility (SEU), which is a representation satisfying, in the binary case, Equation 3. Although such axiomatizations are defensible theories in terms of principles of rationality, they fail as descriptions of human behaviour. The rationality axioms invoked are of three quite distinct types.

First, preference is assumed to be transitive. This assumption has been shown to fail in various empirical contexts (especially multifactor ones), with perhaps the most pervasive and still ill-understood example being the "preference reversal phenomenon," discovered by P. Slovic and S. Lichtenstein and investigated extensively by others, most famously by Grether and Plott (1979), and several later references given in Luce (2000, pp. 39-45). Nevertheless, transitivity is the axiom that is least ease to give up. Even subjects who violate it are not inclined to defend their "errors". A few attempts have been made to develop theories without it, but so far they are complex and have not received much empirical scrutiny (Bell, 1982; Fishburn, 1982, 1985; Suppes et al., 1989, chs 16 and 17).

The second type of rationality postulates so-called "accounting" principles in which two gambles are asserted to be equivalent in preference because when analyzed into their component outcomes they are seen to be identical. For example, if $x \circ_A y$ is a gamble and $(x \circ_A y) \circ_B y$ means that if the event B occurs first and then, independent of it, A occurs, then on accounting grounds $(x \circ_A y) \circ_B y \sim (x \circ_B y) \circ_A y$ is rational because, on both sides, x is the outcome when A and B both occur (although in opposite orders) and y otherwise. One of the first "paradoxes" of utility theory, that of M. Allais, is a violation of an accounting equation which assumes that certain probability calculations also take place.

The third type of rationality condition is the extended sure-thing principle, Equation 4. Its failure, which occurs regularly in experiments, is substantially the "paradox" pointed out earlier by D. Ellsberg. Subjects have insisted on the reasonableness of their violations of this principle (MacCrimmon, 1967).

Some Generalizations of SEU. Kahneman and Tversky (1979) proposed a binary modification of the expected utility representation designed to accommodate the last two types of violations, and Tversky and Kahneman (1992) generalized it to general finite gambles. During the 1980s and 90s a great deal of attention was devoted to this general class of so-called rank- (and sometimes sign-) dependent representations (RDU or RSDU) (also called cumulative and Choquet, 1953, representations). Summaries of this work, much of it of an axiomatic character for both risky cases, where probabilities are assumed known, and uncertain cases, where a subjective probability function is constructed, can be found in Quiggin (1993) and Luce (2000). These developments rests very heavily on modifying the distribution laws that are assumed. A far more general survey of utility theory, covering many aspects of it from an economic but not primarily an axiomatic measurement-theoretic perspective, is Barberà et al. (1998, 2004).

Returning to an axiomatic approach, suppose in what follows that $x_1 \succsim x_2 \succsim \cdots \succsim x_n$ and their associated event partition is $(E_1, E_2, ..., E_n)$. Define $E(i) = \bigcup_{j=1}^{i} E_j$. The class of RDU representations involve proving from the axioms the existence of an order-preserving, utility function U over pure consequences and gambles and, in general, non-additive weighting function S over the chance events such that

$$\begin{aligned} &U(x_1, E_1; x_2, E_2; ...; x_n, E_n) \\ &\qquad = \sum_{i=1}^{n} U(x_i) \left[S\left(E_i \cup E(i-1)\right) - S\left(E(i-1)\right) \right] . \end{aligned} \tag{5}$$

Note that the weighting function is essentially the incremental impact of adding E_i to $E(i-1)$. When S is finitely additive, i.e., for disjoint A and B, $S(A \cup B) = S(A) + S(B)$, then Equation 5 reduces to subjective expected utility (SEU).

If there is a unique consequence e, sometimes called a reference level and sometimes taken to be no change from the status quo, then the consequences and gambles can be partitioned into gains, where $x_i \succsim e$, and the remainder, losses. In such cases, usually it follows from the assumptions made that $U(e) = 0$ and, usually, the weighting functions are sign dependent (i.e., their form depends on whether their consequences are positive respect to e or negative). Also, the RSDU representation includes cumulative prospect theory (Tversky & Kahneman, 1992) as a special case having added restrictions on both U and S.

Other interesting developments involving different patterns of weighting are cited in Luce 2000.

A great deal of attention has been paid to issues of accounting for empirical phenomena discovered over the years that have discredited SEU and EU as descriptive models of human behaviour. For some summaries see Luce (2000) and Marley and Luce (2005). M. H. Birnbaum (numerous citations are given to his articles in the last reference) has discovered experimental designs that discredit a major feature of Equation 5 called *coalescing* or, equally, *event splitting*: Suppose $x_k = x_k = y$, then

$$(x_1, E_1; ...; y, E_k; y, E_{k+1}; ...; x_n, E_n) \sim (x_1, E_1; ...; y, E_k \cup E_{k+1}; ...; x_n, E_n) . \tag{6}$$

The left hand side of Equation 6 is called "split" because y is attached to each of two events, E_k and E_{k+1}. The right hand side is called "coalesced" because y is attached to the single coalesced event $E_k \cup E_{k+1}$. Birnbaum has vividly demonstrated that experimental subjects often fail to split gambles in ways that help facilitate rational decisions. The other direction, coalescing, is effortless because no choice is involved. Indeed, Birnbaum (submitted) has shown that splitting the branch (x_1, E_1), which has the best consequence, x_1, enhances the apparent worth of a gamble, whereas splitting (x_n, E_n), the branch with the poorest consequence, diminishes it. Long ago, he proposed a modified representation, called TAX, because it "taxes" the poorest consequence in favor of the best one, that accommodates many empirical phenomena, including this one, but neither he nor anyone else has offered a measurement axiomatization of TAX. This remains an open problem.

Joint receipt. Begining in 1990, Luce and collaborators have investigated an operation $\oplus$ of joint receit in gambling structures and ways that it may interlock with gambling structures. Its interpretation is suggested by its name, having two goods at once which, because $\oplus$ is assumed to be associative and commutative, can be extended to any finite number of goods. Several possible interlocking laws have been studied and improved axiomatizations involving them have been given for a number of classical representations (for a summary, see Luce, 2000). The representation that has arise naturally is called p-additive[3], namely, for some real δ,[4]

$$U(x \oplus y) = U(x) + U(y) + \delta U(x)U(y).$$

*Lack of Idempotence and the Utility of Gambling***.** A feature of very many utility models, in particular, of all RDU or RSDU ones, is *idempotence:*

$$(x, E_1; ...; x, E_i; ...; x, E_n) \sim x\,.$$

Among other things, this has been thought to be a way to connect gambles to pure consequences, but that feature is redundant with the certainty principle $(x, E(n)) \sim x$. Further, if there is an inherent utility or disutility to risk or gambling, as widespread behaviour suggests there is – witness Las Vegas and mountain climbing – violations of idempotence assess it. Luce and Marley (2000) proposed partitioning a gamble g into a pure consequence, called a *kernel equivalent* of g, $KE(g)$, with the joint receipt, $\oplus$, of its unrewarded event structure $(e, E_1; ...; e, E_i; ...; e, E_n)$, which is called an *element of chance.* Although they found properties of such a decomposition based on the assumption that utility is additive over joint receipt, $\oplus$, they did not discover much about the form of the utility of an element of chance. In the case of risk, further work has led to a detailed axiomatic formulation of that leads either to EU plus a Shannon entropy term, or to a linear weighted form plus entropy of some degree

[3]So named, at the suggestion of A. J. Marley, because it is the only polynomial form that can be transformed into an additive one.

[4]By rescaling U there is no loss of generality in assuming δ is either $-1, 0, 1$.

different from 1. In the case of uncertainty, the form for elements of risk is much less restrictive (Luce et al., submitted). This risky form was first arrived at by Meginniss (1976) using a non-axiomatic approach. Because of the symmetry of entropy, this representation is unable to account for Birnbaum's differential event splitting data. This approach needs much more work.

7. Other Applications of Behavioural Interest

A Psychophysical One. A modified version of one of the RDU axiomatizations has been reinterpreted as a theory of global psychophysics, meaning that the focus is on the full dynamic range of intensity dimensions (for example, in audition the range 5 dB to 130 dB SPL[5]), not just local ranges as in discrimination studies. An example of the primitives are sound intensities x and y to the left and right ears, respectively, denoted (x, u), about which the respondent makes loudness judgments. Given two such stimuli (x, x) and (y, y), $x > y$, and a positive number p, the respondent also can be requested to judge which stimulus (z, z) makes the subjective "interval" from (y, y) to (z, z) seem to be p times the "interval" from (y, y) to (x, x). The data are z, which we may denote in operator notation as $(x, x) \circ_p (y, y) := (z, z)$. Luce (2002, 2004) (for a summary of theory, tests, and references, see Luce and Steingrimsson, 2006, in press) provided testable axioms, it is shown that there is a real valued mapping Ψ, called a psychophysical function, and a numerical distortion function W such that

$$\Psi(x, u) = \Psi(x, 0) + \Psi(0, u) + \delta\Psi(x, 0)\Psi(0, u) \quad (\delta \geq 0)\,, \tag{7}$$

$$W(p) = \frac{\Psi\left[(x, x) \circ_p (y, y)\right] - \Psi(y, y)}{\Psi(x, x) - \Psi(y, y)} \quad (x > y \geq 0)\,. \tag{8}$$

The axioms have been empirically tested by Steingrimsson and Luce in four papers. The 2005a focused on each structure, the conjoint one and the operator; the 2005a focused on the interlocks between them for audition. The results are supportive of the theory. Possible mathematical forms for Ψ and W have been reduced to testable conditions that, with one exception[6], were evaluated with considerable, but not perfect, support, for power functions (2006, in press). Narens (1966) earlier proposed a closely related theory that included an axiom that forced $W(1) = 1$. Empirical data of Ellermeier and Faulhammer (2000) and Zimmer (2005) soundly rejected the joint hypothesis that W is a power function with $W(1) = 1$. Subsequent theory and experiments found considerable support for power functions with $W(1) \neq 1$.

Foundations of Probability. Today, the usual approach to probability theory is the classical one due to Kolmogorov. It assumes that probability is a σ-additive[7] measure function P with a sure event having probability 1. It defines

[5] Contemporary IRBs restrict the top of the range to 85 dB.

[6] It is the cases where $\delta \neq 0$, $\Psi(x, 0)$ and $\Psi(0, x)$ are both power functions but with different exponents.

[7] The countable extension of finite additivity.

the important concepts of independence and conditional probability in terms of P.

There are many objections to this approach as a foundation for probability. A summary of most of them can be found in Fine (1973) and Narens (submitted). In particular, independence and conditional probability appear to be more basic concepts than unconditional probability, e.g., one often needs to know the independence of events in order to estimate probabilities. Also in most empirical situations one cannot exactly pin down the probabilities, that is, there are many probability functions consistent with the data. This suggests that in such situations the underlying probabilistic concept should be a family of probability functions instead of a single probability function. Obviously, with many consistent probability functions explaining the data, the Kolmogorov method defining independence by $P(A \cap B) = P(A)P(B)$ really does not work. These and other difficulties disappear with measurement-theoretic approaches to probability (e.g., see Krantz, et al., 1971; Fine, 1973; Narens, 1985; Narens, submitted). The qualitative approach provides richer and more flexible methods than Kolmogorov's for formulating and investigating the foundations of probability.

Both the Kolmogorov and the measurement-theoretic approaches assume an event space that is a boolean algebra of subsets. This assumptions works for most applications in science and is routinely assumed in theoretical and empirical studies of subjective probability. A major exception to it is quantum mechanics, where a different event space is needed (von Neumann, 1955).

Algebras of events correspond to the classical propositional calculus of logic. This calculus captures deductions for propositions that are either true or false. It is not adequate for capturing various concepts of "vagueness," "ambiguity," or "incompleteness based on lack of knowledge." For these, logicians use non-classical propositional calculi. In general, these non-classical calculi cannot be interpreted as the classical propositional calculus with "true" and "false" replaced with probabilities. It is plausible that some of the just mentioned concepts are relevant to how individuals make judgments and decisions. Their incorporation into formal descriptions of behavior requires the event space to be changed from the usual algebra of events used in the Kolmogorov approach to probability to a different kind of event space. This issue and proposals for alternative event spaces are discussed in detail in Narens (submitted).

In summary, the Kolmogorov approach to probability is flawed at a foundational level and is too narrow to account for many important scientific phenomena. The measurement-theoretic approach is one alternative for providing a better foundation and generalizations for the kind of probability theory described by Kolmogorov. One should also consider the possibility of developing probabilistic theories for event spaces different from algebras of events, especially for phenomena that fall outside of usual forms of observation, including various phenomena arising from mentation.

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See also dimensions of economic quantities; meaningfulness and invariance;

mean value; transformations and invariance.

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