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The Irony of Measurement by Subjective Estimations

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In 1948 S. S. Stevens, in his famous *Science* article, proposed a theory of measurement that radically differed from the dominate theory of the time. The dominate theory held that all strong forms of scientific measurement—for example, those that yielded ratio scales—had to be based on an observable ordering and an observable commutative and associative operation. Stevens proposed different criteria and introduced his method of magnitude estimation. Stevens as well as measurement theorists considered his method to be radically different from those based on commutative and associative operations. Although his method was controversial, it became a standard tool in the behavioral sciences. This article argues that Stevens' method, together with implicit assumptions he made about the scales of measurement it generated, is from a mathematical perspective the same as the measurement process based on commutative and associative operations. The article also provides a theory of qualitative numbers and shows an interesting relationship between qualitative numbers and Stevens' method.

1. INTRODUCTION

In 1946, the psychologist S. S. Stevens published an article in the journal *Science* that has had an enormous impact on how behavioral scientists thought about and used measurement. The article was Stevens' response to a widely held view of the time that justifiable forms of measurement that were stronger than counting or numerical ordering necessarily relied on the existence of an observable, empirical form of addition. Because psychological phenomena generally lacked such forms of addition, many scientists believed that psychology was not—and could never be—a quantitative science founded on philosophically sound principles. Stevens and other psychologists thought otherwise. This led the British Association for the Advancement of Science to appoint a committee to look into the matter. Stevens (1946) comments,

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> For seven years a committee of the British Association for the Advancement of Science debated the problem of measurement. Appointed in 1932 to represent Section A (Mathematical and Physical Sciences) and Section J (Psychology), the committee was instructed to consider and report upon the possibility of "quantitative estimates of sensory events"—meaning simply: Is it possible to measure human sensation? Deliberation led only to disagreement, mainly about what is meant about the term measurement. An interim report in 1938 found one member complaining that his colleagues "came out by the same door as they went in," and in order to have another try at agreement, the committee begged to be continued for another year.
>
> For its final report (1940) the committee chose a common bone for its contentions, directing its arguments at a concrete example of a sensory scale. This was the Sone scale of loudness (S. S. Stevens and H. Davis. *Hearing*. New York: Wiley, 1938), which purports to measure the subjective magnitude of an auditory sensation against a scale having the formal properties of basic scales, such as those used to measure length and weight. Again 19 members of the committee came out by the routes they entered, and their view ranged widely between two extremes. One member submitted "that any law purporting to express a quantitative relation between sensation intensity and stimulus intensity is not merely false but is in fact meaningless unless and until a meaning can be given to the concept of addition as applied to sensation" (Final Report, p. 245). (p. 667)

In his *Science* article and his much cited *Handbook of Experimental Psychology* chapter (Stevens, 1951), Stevens presented a radically different theory of measurement and argued for a new measuring method based on subjective estimations. He also had a penchant for naming things and introduced much terminology. However, by current standards Stevens was not very sophisticated mathematically or philosophically, and much of his theory was ill-founded.

In this article I will present what I consider to be the soundest version of the ideas inherent in the relevant parts of Steven's theory of measurement. These ideas will often be formulated in a manner a little different from his, including a different terminology. This should present little problem, because the goal of the paper is not historical, but to examine how Stevens' method of measurement based on subjective estimations is related to classical forms of measurement. His method was—and still is—very controversial, and it was universally held to be a completely different form of measurement than those based on observable additive operations. I will argue that in principle it can be put on as a firm foundation as classical measurement, and when thus done, it bears remarkable formal similarities to classical measurement. Thus, *when properly characterized*, there is nothing inherently radical about the most radical method Stevens introduced into science.

2. PRELIMINARIES

The following definitions and conventions are observed throughout this article:

DEFINITION 1. $\mathbb{R}$ denotes the set of reals, $\mathbb{R}^+$ the set of positive reals, $\mathbb{I}$ the integers, $\mathbb{I}^+$ the positive integers, and $*$ the operation of function composition.

$\preccurlyeq$ is said to be a *total ordering* on X if and only if X is a nonempty set and $\preccurlyeq$ is a binary relation on X that is *transitive* (for all x, y, and z in X, if $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$), *connected* (either $x \preccurlyeq y$ or $y \preccurlyeq x$ for all x and y in X), and *antisymmetric* (for all x and y in X, if $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$).

In the measurement literature, the term *scale* is used to denote a specific way of measuring, as in "the centimeter scale," as well as a set of specific ways of measuring, as in "a ratio scale for length." Throughout this article, the latter use of scale will be retained and the former will be replaced by other terms.

DEFINITION 2. Let Y be a nonempty set. A *scale* on Y is a nonempty set of functions from Y onto a subset of $\mathbb{R}$. Let $\mathscr{F}$ be a scale on Y. Then the elements of $\mathscr{F}$ are called *measuring functions*.

The following three kinds of scales are routinely applied in the behavioral sciences.

DEFINITION 3. $\mathscr{F}$ is said to be a *ratio scale* if and only if (1) $\mathscr{F}$ is a scale, and (2) for any f in $\mathscr{F}$,

$$\mathscr{F} = \{rf \mid r \in \mathbb{R}^+\}.$$

$\mathscr{F}$ is said to be an *interval scale* if and only if (1) $\mathscr{F}$ is a scale, and (2) for any f in $\mathscr{F}$,

$$\mathscr{F} = \{rf + s \mid r \in \mathbb{R}^+ \text{ and } s \in \mathbb{R}\}.$$

$\mathscr{F}$ is said to be an *ordinal scale* if and only if (1) $\mathscr{F}$ is a scale, and (2) for any f in $\mathscr{F}$, s into $\mathbb{R}^+$ and

$$\mathscr{F} = \{g * f \mid g \text{ is a strictly increasing function from } \mathbb{R}^+ \text{ onto } \mathbb{R}^+\}.$$

DEFINITION 4. $\mathfrak{Y} = \langle Y, U_1, U_2, \ldots \rangle$ is said to be a *structure (of primitives)* if and only if Y is a nonempty set, called the *domain* of $\mathfrak{Y}$, and each U_i is either an element of Y, a set of elements of Y, or a finitary relation on Y. Y, U_1, U_2, ... are called the *primitives* of $\mathfrak{Y}$.

DEFINITION 5. A *theory of measurement* consists of a precise specification of how a scale $\mathscr{F}$ of functions is formed. The currently dominant approach to measurement in the literature is the *representational theory*. For the purposes of this article, the representational theory may be formulated as follows: Scales $\mathscr{F}$ on a set Y result by providing a structure of primitives $\mathfrak{Y}$ with domain Y and a numerical structure $\mathfrak{R}$ with domain a subset of $\mathbb{R}$ such that $\mathscr{F}$ is the set of isomorphisms of $\mathfrak{Y}$ onto $\mathfrak{R}$.

Let $\mathscr{F}$ be a scale that results by the representational theory. In the representational theory, elements of $\mathscr{F}$ are usually called *representations*. Throughout this article elements of $\mathscr{F}$ will usually be referred to as *measuring functions* or isomorphisms.

Totally ordered structures that are isomorphic to an open interval of real numbers are of considerable importance to mathematics and science. They are called *continua*, and the following qualitative axiomatization in terms ordering properties was given by Cantor (1895).

Definition 6. $\langle X, \preccurlyeq \rangle$ is said to be a *continuum* if and only if the following four statements are true:

1. *Total ordering*: $\preccurlyeq$ is a total ordering on X.
2. *Unboundedness*: $\langle X, \preccurlyeq \rangle$ has no greatest or least element.
3. *Denumerable density*. There exists a denumerable subset Y of X such that for each x and z in X, if $x \prec z$ then there exists y in Y such that $x \prec y$ and $y \prec z$.
4. *Dedekind completeness*: Each $\preccurlyeq$-bounded nonempty subset of X has a $\preccurlyeq$-least upper bound.

Cantor (1895) showed the following theorem:

Theorem 1. *$\langle X, \preccurlyeq \rangle$ is a continuum if and only if it is isomorphic to $\langle \mathbb{R}^+, \leqslant \rangle$.*

Proof. Cantor (1895). (A proof is also given in Theorem 2.2.2 of Narens, 1985.)

The following is an immediate consequence of Theorem 1:

Theorem 2. *Let $\mathfrak{X} = \langle X, \preccurlyeq \rangle$ be a continuum. Then the set of isomorphisms of $\mathfrak{X}$ onto $\langle \mathbb{R}^+, \leqslant \rangle$ is an ordinal scale.*

Definition 7. $\mathfrak{Y} = \langle Y, \preccurlyeq, U_1, U_2, \ldots \rangle$ is said to be a *continuous structure* if and only if $\mathfrak{Y}$ is a structure and $\langle Y, \preccurlyeq \rangle$ is a continuum.

For simplicity of exposition, this article will only consider continuous structures. However, the results and claims made in the article do not depend in an essential way on the relevant structures being based on continua, and thus these results and claims can be appropriately generalized.

3. CLASSICAL MEASUREMENT

The classical approaches to measurement are based on an observable, empirical addition operation that is used to produce a ratio scale of measuring functions. It was the form of measurement developed by Helmholtz (1887), Hölder (1901), and Campbell (1920). Closely related forms of measurement were put forward by Frege (1903) and Whitehead and Russell (1913). In Stevens' era, Campbell was by far the most influential of these theorists. However, because Hölder provided the most rigorous, complete, and systematic presentation, modern measurement theory has tended to proceed much more in Hölder's tradition.

The following axiomatization and theorem, which are essentially the theory of Hölder (1901) applied to continuous domains, capture the concept of a qualitative addition operation.

Definition 8. $\mathfrak{X} = \langle X, \preccurlyeq, \oplus \rangle$ is said to be a *continuous extensive structure* if and only if $\oplus$ is a binary operation on X and the following seven axioms are true:

1. *Total Ordering:* $\preccurlyeq$ is a total ordering on X (Definition 1).
2. *Density:* For all x and z in X, if $x \prec z$ then for some y in X, $x \prec y \prec z$.
3. *Associativity:* $\oplus$ is a binary operation that is *associative*; that is,

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$

for all x, y, and z in X.

4. *Monotonicity:* For all x, y, and z in X,

$$x \preccurlyeq y \text{ iff } x \oplus z \preccurlyeq y \oplus z \text{ iff } z \oplus x \preccurlyeq z \oplus y.$$

5. *Solvability:* For all x and y in X, if $x \prec y$, then for some z in X, $y = x \oplus z$.
6. *Positivity:* $x \prec x \oplus y$ and $y \prec x \oplus y$ for all x and y in X.
7. *Dedekind Completeness:* Each nonempty, $\preccurlyeq$-bounded above subset of X has a $\preccurlyeq$-least upper bound.

THEOREM 3. *Suppose $\mathfrak{X} = \langle X, \preccurlyeq \oplus \rangle$ is a continuous extensive structure. Then the set $\mathscr{S}$ of isomorphisms of $\mathfrak{X}$ onto $\langle \mathbb{R}^+, \leqslant, + \rangle$ is a ratio scale.*

Theorems 2 and 3 are special cases of the representational theory. During Campbell's era they were essentially the only applications of the representational theory for continuous structures. Because ordinal scales are only of very limited use in science, Campbell believed that all strong forms of measurement were ultimately based on the classical approach. Today we know that Campbell's view is far too narrow, because there is a myriad of structures for which the representational theory yields strong forms of measurement (e.g., see Krantz, Luce, Suppes, & Tversky, 1971; Luce, Krantz, Suppes, & Tversky, 1990; Narens, 1985). Stevens also thought Campbell's view was too narrow, but for different reasons.

The classical method of showing Theorem 3 was by a construction employing standard sequences:

DEFINITION 9. Let $\langle X, \preccurlyeq, \oplus \rangle$ be a continuous extensive structure. For each x in X and each i in $\mathbb{I}^+$, let

$$x_1 = x \qquad \text{and} \qquad x_{i+1} = x_i \oplus x,$$

and let τ_x be the sequence $x_1, \ldots, x_i, \ldots$. Then τ_x is called the *standard sequence of $\mathfrak{X}$ based on x.*

Let x be an arbitrary element of X and $x_1, \ldots, x_i, \ldots$ be the standard sequence of X based on x. The construction of a measuring function φ_x on X proceeds as follows: For each i in $\mathbb{I}^+$ let $\varphi_x(x_i) = i$. Let y be an arbitrary element of X such that $x \prec y$. Then using the properties of a continuous extensive structure, it can be shown that n in $\mathbb{I}^+$ can be found such that

$$x_n \preccurlyeq y \prec x_{n+1}.$$

Then intuitively, $\varphi_x(y)$ should be such that

$$n = \varphi(x_n) \leqslant \varphi_x(y) < \varphi(x_{n+1}) = n+1.$$

Thus intuitively, the value of $\varphi_x(y)$ is known within an error of 1. If $y_2 = y \oplus y$ is doubling of y in size and m is such that

$$x_m \preccurlyeq y_2 \prec x_{m+1},$$

then intuitively $\varphi_x(y)$ should be such that

$$\frac{m}{2} \leqslant \varphi_x(y) < \frac{m+1}{2} = \frac{m}{2} + \frac{1}{2};$$

i.e., the value of $\varphi_x(y)$ is known within error of $\frac{1}{2}$. Similar considerations of the elements $y_3, y_4, \ldots$ of the standard sequence of $\mathfrak{X}$ based on y yield approximations of $\varphi_x(y)$ within errors of $\frac{1}{3}$, $\frac{1}{4}$, etc. $\varphi_x(y)$ is then defined to be the limit of these approximations. Properties of $\mathfrak{X}$ are used throughout this construction to guarantee that the limit exists and is well defined. For $y \prec x$, one finds k in $\mathbb{I}^+$ such that $x \prec ky$, and then define

$$\varphi_x(y) = \frac{\varphi_x(ky)}{k}.$$

Properties of $\mathfrak{X}$ are also used to show that for u and v in X,

$$\varphi_u = \varphi_u(v) \cdot \varphi_v.$$

The above results with other properties of $\mathfrak{X}$ establish that for all x in X,

1. φ_x is a measuring function from X onto $\mathbb{R}^+$;
2. for all y and z in X, $y \precsim z$ iff $\varphi_x(y) \leqslant \varphi_x(z)$; and
3. $\{\varphi_u \mid u \in X\}$ is a ratio scale.

Because of these results, the above construction involving standard sequences may be viewed as a *method of measurement* that yields a ratio scale of measuring functions on X. According to this view, it is just a bonus that $\oplus$ is represented by numerical addition, i.e., that for all x, u, and v in X,

$$\varphi_x(u \oplus v) = \varphi_x(u) + \varphi_x(v);$$

that is, it is the (constructive) measuring of X by a ratio scale that is of primary importance for measurement and not necessarily the representing of $\oplus$ by $+$.

4. STEVENS' THEORY OF MEASUREMENT

Stevens considered classical measurement and its construction of a scale through standard sequences to be a very special form of measurement. He proposed,

> measurement, in the broadest sense, is defined as the assignment of [numbers] to objects or events according to rules. The fact that [numbers] can be assigned under different rules leads to different kinds of scales and different kinds of measurement. The problem then becomes that of making explicit (a) the various rules for the assignment of [numbers], (b) the mathematical properties (or group structure) of the resulting scales, and (c) the statistical operations applicable to measurements made with each type of scale. (Stevens, 1946, p. 667.)

This definition, which is widely cited in the social and behavioral science literature, is scientifically inadequate, because it provides no general definition or theory of "rules."

It is clear that Stevens wanted his definition to generalize classical measurement. However, he failed to understand that the mathematical complexities in rigorously establishing scale types will in general be much more complex than in the simple case covered by classical measurement. In the new methods of measurement he introduced, he and his legions of followers never provided adequate, theoretical justification for the scale types they assigned to their measurements. However, Stevens did correctly realize that the classical approach was too narrow and that it could be extended by providing a general characterization of the measurement process. For discussions of the notion of measurement process see the critical comments of Adams (1966) and the elaborate and subtle formal development of Niederée (1992a,b).

Stevens considered the following method to be proper for establishing part of a measuring function from a ratio scale: In an appropriate psychophysical setting, the experimenter presents lights to a subject and asks him or her to estimate numerically their subjective brightnesses. For each presented light x the experimenter assigns to x the number named by the subject for the brightness of x.

This form of measurement appears to be a radical and dramatic departure from classical measurement: There is no apparent addition operation, and except for the trivial step of the assignment by the experimenter of numbers named by the subject to stimuli, whatever calculations are involved are entirely buried in the unobservable, mental processing of the subject. Claiming such procedures "scientific" obviously invited controversy, and much ensued. However, it is argued in the following sections that procedures such as this one can be placed on a philosophically sound foundation, and when so done, they closely parallel the construction by standard sequences of classical measurement.

5. RATIO MAGNITUDE ESTIMATION

In this section an axiomatic treatment of ratio magnitude estimation is given. It is argued that the axioms capture principles inherent in Stevens' methods of ratio magnitude and production estimation. Then a theorem of Narens (1996) is presented that shows if the axioms are true, then measurement through the representational theory produces the same scale of measuring functions as Stevens' methods.

There are various methods of eliciting subjective estimates from subjects, and Stevens distinguished them in terms of instructions to the subject. For the purposes of this article, subtle distinctions in terms of instructions to the subject will not matter, because they are not the driving force behind the arguments presented here; instead it is the *structure* of the data set of the subject's responses that is the important consideration. (For additional elaborations of this point see Narens, 1996, 1997.)

Throughout this section, X will denote a nonempty set and $\preccurlyeq$ will denote a binary relation on it. The intended interpretation is that elements of X are possible stimuli to be presented to a subject for judgment and $\preccurlyeq$ is an experimenter-determined intensity ordering on X. For psychophysical situations, X may be taken as a set of physical stimuli and $\preccurlyeq$ as the natural physical ordering on X.

Throughout this section, E will denote a coding of the ratio estimation behavior of a subject to stimuli from X. Elements of E are ordered triples of the form $(x, \mathbf{p}, t)$, where x and t are elements of X and p is a positive real number. (For technical reasons discussed below, p is placed in bold typeface when it is part of a triple in E.) Instructions to the subject can be formulated so that for each $\mathbf{p}$ of a triple in E, $p \in \mathbb{I}^+$. This latter form of instruction, which is described in Convention 1, is employed later in the article.

Two Principles of the Stevens' Theory of Ratio Estimation

Consider X to be a set of stimuli that is presented to a subject. Then *Stevens' method of ratio estimation* culminates in the experimenter producing a function φ_t from X into $\mathbb{R}^+$ with the following properties: An element t—that we will call the *modulus*—is selected from X. The subject is instructed to consider the number 1 as representing his or her subjective intensity of t, and keeping this consideration in mind to give his or her numerical estimate of his or her subjective intensity of stimuli x in X. The experimenter uses these verbal estimates of the subject to construct the function φ_t by assigning the number corresponding to the subject's numerical estimate of x as the value of the function $\varphi_t(x)$.

Stevens was vague about what underlaid and what was being accomplished by this and related methods of subjective estimations involving ratios. I believe the following two assumptions, which here will be referred to as *Stevens's assumptions*, are inherent in his published ideas concerning valid applications of the method of ratio estimation:

1. The function φ_t is an element of a ratio scale $\mathscr{S}$ that adequately measures the subject's subjective intensity of stimuli in X.

2. Each element x in X can be used as a modulus and the resulting representation φ_x is in the ratio scale $\mathscr{S}$; i.e., there exists r $(=r(x, t))$ in $\mathbb{R}^+$ such that $\varphi_x = r\varphi_t$.

Let $D = \{\varphi_t \mid t \in X\}$. Then D is the complete data set that is generated by conducting all possible magnitude estimations of stimuli in X with all possible moduli from $\mathfrak{X}$. For the purposes of this discussion, D will be recoded as

$$E = \{(x, \mathbf{p}, t) \mid \varphi_t(x) = p\}.$$

In triples $(x, \mathbf{p}, t)$ in E, $\mathbf{p}$ is put in bold typeface because it represents the subject's expression of an estimate, which is best conceptualized as a *(real) numeral* rather than a *(real) number*. Because numbers are highly abstract scientific objects, it would be unwise to assume subjects understood or used them in their calculations or responses, and similarly it would be unwise to assume that subjects had or used a philosophically sound correspondence between (scientific) numbers and numerals. Thus for the purposes of this article, the occurrence of the numeral $\mathbf{p}$ in the expression "$(x, \mathbf{p}, t)$" will be interpreted as an empirical behavioral item derived by the experimenter from the subject's responses.

Definition 10. Let X and E be as above. Then E is said to have the *multiplicative property* if and only if for all x and t in X and all p, q, and r in $\mathbb{R}^+$, if $(x, \mathbf{p}, t) \in E$, $(y, \mathbf{q}, x) \in E$, and $(y, \mathbf{r}, t) \in E$, then $r = q \cdot p$.

The multiplicative property obviously imposes powerful constraints on the subject's ratio estimation behavior.[1] This is reflected in the following theorem.

Theorem 4. *Let X and E be as above, and suppose Stevens' assumptions. Then E has the multiplicative property.*

Proof. Suppose $(x, \mathbf{p}, t) \in E$, $(y, \mathbf{q}, x) \in E$, and $(y, \mathbf{r}, t) \in E$. Then $\varphi_t(x) = p$, $\varphi_x(y) = q$, and $\varphi_t(y) = r$. By Stevens' assumptions, let u in $\mathbb{R}^+$ be such that

$$\varphi_x = u\varphi_t.$$

Then $1 = \varphi_x(x) = u\varphi_t(x) = u \cdot p$; i.e.,

$$u = \frac{1}{p}.$$

Therefore,

$$q = \varphi_x(y) = \frac{1}{p} \cdot \varphi_t(y) = \frac{1}{p} \cdot r;$$

i.e., $r = q \cdot p$.

Convention 1. For the purposes of this article it is preferable to have instructions so that the numerals appearing in triples of E name positive integers. As mentioned earlier, what drives this article's theory of ratio estimation is the structure of the data set E and not the instructions that produce E. Thus for concreteness it will be assumed throughout the remainder of the article that E arises from the following instruction: The experimenter presents a stimulus t to the subject and asks him or her to "find a stimulus x in X that appears to be p times greater in intensity than the stimulus t." This form of instruction guarantees that all numerals in triples of E will name positive integers. In the following axiomatization, the triples of E are employed indirectly through the notation $\alpha_{\mathbf{p}}$, where $\alpha_{\mathbf{p}}$ is defined by

$$(x, \mathbf{p}, t) \in E \qquad \text{iff} \quad \alpha_{\mathbf{p}}(t) = x.$$

Definition 11. $\mathbf{N}$ is the set of positive, integral numerals; i.e.,

$$\mathbf{N} = \{\mathbf{p} \mid p \in \mathbb{I}^+\}.$$

Axiom 1. *$\langle X, \preccurlyeq \rangle$ is a continuum (Definition 6).*

Axiom 2. *$\{\alpha_{\mathbf{p}} \mid \mathbf{p} \in \mathbf{N}\}$ is a set such that the following six statements hold for all* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbf{N}$*:*

[1] Narens (1996) suggests that these constraints are too powerful and are likely to fail empirically. He suggests that a weaker condition, called the "commutativity property," should be considered in its place. The commutativity property is implied by the multiplicative property and results in a ratio scale (Narens, 1996, 1997). Ellermeir and Faulhammer (2000) directly tested the multiplicative and commutativity properties in a standard psychophysical paradigm and found that the multiplicative property failed but the commutativity property held. Luce (2001) provides an axiomatization using the commutativity property in a more complicated setting that includes magnitude production.

1. $\alpha_{\mathbf{p}}$ *is a* $\preccurlyeq$*-strictly increasing function from* X *onto* X.
2. $x \preccurlyeq \alpha_{\mathbf{p}}(x)$.
3. α_1 *is the identity function* ι *on* X.
4. *For all* t *in* X, $\alpha_{\mathbf{p}}(t) \prec \alpha_{\mathbf{q}}(t)$ *iff* $p < q$.
5. *For all* x *and* t *in* X, *if* $t \prec x$, *then there exist* $\mathbf{m}$ *in* $\mathbf{N}$ *and* z *in* X *such that*

$$t \prec z \prec x \qquad \text{and} \qquad \alpha_{\mathbf{m}+1}(t) = \alpha_{\mathbf{m}}(z).$$

6. *(Multiplicative property)* If $r = q \cdot p$, then $\alpha_{\mathbf{r}} = \alpha_{\mathbf{q}} * \alpha_{\mathbf{p}}$.

Conditions 1 to 4 of Axiom 2 are straightforward and are very reasonable idealizations for ratio estimation. Condition 5 is an Archimedean condition that guarantees that no two elements are infinitesimally close with respect to estimation. As has been previously discussed, Condition 6—the multiplicative property—is a natural consequence of Stevens ideas about ratio estimation.

THEOREM 5. *Assume Axioms* 1 *and* 2. *For each* p *in* $\mathbb{I}^+$, *let* β_p *be the function from* $\mathbb{R}^+$ *onto* $\mathbb{R}^+$ *such that for all* u *in* $\mathbb{R}^+$,

$$\beta_p(u) = p \cdot u.$$

Then the set of isomorphisms of

$$\langle X, \preccurlyeq, \alpha_1, ..., \alpha_{\mathbf{p}}, ...\rangle \qquad \text{onto} \quad \langle \mathbb{R}^+, \leqslant, \beta_1, ..., \beta_p, ...\rangle$$

form a ratio scale. (Note by Definitions 2 *and* 3, *ratio scales are nonempty.)*

Proof. Theorem 12 of Narens (1996).

Theorem 5 justifies, through the representational theory of measurement, Axioms 1 and 2 as an adequate theory of ratio estimation. I believe it also provides a rigorous foundation for the theory behind the often confusing ideas that Stevens promulgated on the subject. Generalizations and empirical implications of Theorem 5 are discussed in Narens (1996, 1997).

6. CONSTANT DIFFERENCE SEQUENCES

As discussed earlier, the method of standard sequences has played a prominent role in the theory of measurement as a means for constructing measurement functions. Conceptually and mathematically the construction works because the elements of a standard sequence are equally spaced with the spacing having the size of the first element of the sequence.[2] In the literature, the equally spaced nature of a sequence is initially justified on intuitive considerations involving the construction of the sequence. Then additional justifications are provided by theorems that show

[2] Luce and Tukey (1964) introduced a form of standard sequence in which the indices of elements of the sequence range over integers instead of positive integers. Such sequences occur in certain interval scalable situations.

the existence of a function φ from the domain X into the positive reals such that for all equally spaced sequences $x_1, \ldots, x_i, \ldots$ and all positive integers n,

$$\varphi(x_{n+1}) - \varphi(x_n) = \varphi(x_1).$$

This section presents a qualitative theory of equally spaced sequences. Essentially, the theory states that systems of equally spaced sequences may be viewed formally as a form of ratio estimation.

DEFINITION 12. Let $\mathfrak{X} = \langle X, \preccurlyeq \rangle$ be a continuum. Then $\mathfrak{S}$ is said to be a *system of constant difference sequences* of $\mathfrak{X}$ if and only if the following three conditions hold:

1. Each element of $\mathfrak{S}$ is a sequence of elements of X.
2. For each element x of X there exists an element σ of $\mathfrak{S}$ such that x is the first element of σ.
3. There exists a function φ from X onto $\mathbb{R}^+$ such that for all y and z in X,

$$y \preccurlyeq z \quad \text{iff} \quad \varphi(y) \leqslant \varphi(z),$$

and for each sequence $x_1, \ldots, x_n, \ldots$ in $\mathfrak{S}$ and each n in $\mathbb{I}^+$,

$$\varphi(x_{n+1}) - \varphi(x_n) = \varphi(x_1).$$

Let $\mathfrak{S}$ be a system of constant difference sequences of the continuum $\mathfrak{X}$. Then, by definition, functions φ satisfying Condition 3 above are called $\mathfrak{S}$-*representing functions.*

The following theorem characterizes systems of constant difference sequences.

THEOREM 6. *Let $\mathfrak{X} = \langle X, \preccurlyeq \rangle$ be a continuum and $\mathfrak{S}$ be a nonempty set of sequences of elements of X. For each p in $\mathbb{I}^+$ and each t and y in X, let*

> $\alpha_{\mathrm{p}}(t) = y$ *if and only if there exists σ in $\mathfrak{S}$ such that t is the first element of σ and y is the p^{th} element of σ.*

Then the following two statements are equivalent:

1. *$\mathfrak{S}$ is a system of constant difference sequences for $\mathfrak{X}$.*
2. *Interpreting $\alpha_{\mathrm{p}}(t) = y$ as the subject estimates that y is p times as intense as t, Axioms* 1 *and* 2 *hold.*

Proof. Assume Statement 1. Let φ be a $\mathfrak{S}$-representing function (Definition 12). Let t be an arbitrary element of X. Then it easily follows that

$$\varphi[\alpha_1(t)] = 1 \cdot \varphi(t).$$

Suppose n is in $\mathbb{I}^+$ and $\varphi[\alpha_{\mathrm{n}}(t)] = n \cdot \varphi(t)$. Then by Definition 12,

$$\varphi[\alpha_{\mathrm{n+1}}(t)] = \varphi[\alpha_{\mathrm{n}}(t)] + \varphi(t) = n \cdot \varphi(t) + \varphi(t) = (n+1) \cdot \varphi(t).$$

Thus by induction, $\varphi[\alpha_{\mathrm{m}}(t)] = m \cdot \varphi(t)$, for each m in $\mathbb{I}^+$ and each t in X. From this and the hypotheses of the theorem, Statement 2 easily follows. (The details are left to the reader.)

Assume Statement 2. Then Statement 1 of Definition 12 is true by hypothesis, and Statement 2 of Definition 12 is true by Statement 2 of Axiom 2. By Theorem 5, let φ be an isomorphism of

$$\langle X, \precsim, \alpha_1, ..., \alpha_{\mathrm{p}}, ...\rangle \text{ onto } \langle \mathbb{R}^+, \leqslant, \beta_1, ..., \beta_p, ...\rangle,$$

where for each p in $\mathbb{I}^+$, β_p is the function from $\mathbb{R}^+$ onto $\mathbb{R}^+$ such that for all u in $\mathbb{R}^+$, $\beta_p(u) = p \cdot u$. It only needs to be shown that φ satisfies Statement 3 of Definition 12. Because φ is an isomorphism of $\langle X, \precsim\rangle$ onto $\langle \mathbb{R}^+, \leqslant\rangle$, the first equation in Statement 3 of Definition 12 is true. To show the second equation, it is sufficient to show that for each t in X and each n in $\mathbb{I}^+$,

$$\varphi[\alpha_{\mathrm{n}+1}(t)] = \varphi[\alpha_{\mathrm{n}}(t)] + \varphi(t).$$

By the choice of φ,

$$\varphi[\alpha_{\mathrm{n}+1}(t)] = (n+1) \cdot \varphi(t) = n \cdot \varphi(t) + \varphi(t) = \varphi[\alpha_{\mathrm{n}}(t)] + \varphi(t).$$

THEOREM 7. *Suppose $\mathfrak{X} = \langle X, \precsim\rangle$ is a continuum and $\mathfrak{S}$ is a system of constant difference sequences for $\mathfrak{X}$. Let $\mathscr{S}$ be the set of $\mathfrak{S}$-representing functions (Definition* 12*). Then $\mathscr{S}$ is a ratio scale.*

Proof. Theorems 5 and 6.

The following theorem shows that the standard sequences produced through the classical method for continuous extensive structures are constant difference sequences.

THEOREM 8. *Let $\mathfrak{X} = \langle X, \precsim, \oplus\rangle$ be a continuous extensive structure. Then the set of standard sequences of $\mathfrak{X}$ (Definition* 9*) is a system of constant difference sequences (Definition* 12*).*

Proof. Immediate from Definition 9, Theorem 3, and Definition 12.

7. QUALITATIVE MEASUREMENT

Measuring functions assign real numbers to empirical or qualitative entities. In science, "real numbers" are construed to be the same real numbers used by mathematicians, namely a variety of platonic entities whose existence do not depend on material objects or minds. In mathematics textbooks, the ontology of real numbers has its roots in set-theoretic constructions involving nonnegative integers, which in the more sophisticated texts are taken to be the finite ordinals. A more classical idea of "real number" is that of a qualitative relation corresponding to a notion of "ratio." The latter idea has it roots in Eudoxes' theory of proportions in Book V of Euclid and is closely connected with methods of classical measurement. This view of real number was held by Frege (1903) as well as by Whitehead and Russell (1913). (A modern, mathematical development is given in

Krull, 1960, 1962.) Of course, adoption of this approach to real numbers leaves the door open for other kinds of "numbers" associated with other kinds of measurement processes. Such a view of measurement numbers was advocated by Adams (1966) and was systematically and formally developed by Niederée (1992a,b). These measurement-based approaches to "number" reveal possible avenues for achieving metaphysical reduction in quantitative science.

This section presents a theory of qualitative measurement for ratio estimation in which the empirical or qualitative objects are measured by "numbers" corresponding to (qualitative) invariants of the measurement process. Given the importance of invariants in mathematics and science, this provides for an intriguing concept of "number."

Definition 13. By definition, an *automorphism* of a structure $\mathfrak{Y}$ is an isomorphism of $\mathfrak{Y}$ onto $\mathfrak{Y}$.

Definition 14. Let $\mathfrak{Y} = \langle Y, R_1, ..., R_i, ...\rangle$ be a structure and α be a binary relation on Y. Then α is said to be an *invariant* of $\mathfrak{Y}$ if and only if for all automorphisms γ of $\mathfrak{Y}$ and all x and y in X,

$$\alpha(x, y) \qquad \text{iff} \quad \alpha(\gamma(x), \gamma(y)).$$

Definition 15. Assume Axioms 1 and 2. Let

$$\mathfrak{X} = \langle X, \preccurlyeq, \alpha_1, ..., \alpha_{\mathrm{p}}, ...\rangle.$$

Then α is said to be a *magnitude number* of $\mathfrak{X}$ if and only if

(i) α is a function from X onto X;

(ii) α is an invariant binary relation of $\mathfrak{X}$; and

(iii) α is strictly $\preccurlyeq$-increasing, i.e., for all x and y in X, $x \preccurlyeq y$ iff $\alpha(x) \preccurlyeq \alpha(y)$.

Definition 16. By definition, let $\mathbb{N}$ be the set of magnitude numbers of $\mathfrak{X}$.

In the following, $\mathbb{N}$ will be taken as the set of "numbers," and X will be measured in terms of $\mathbb{N}$ by means of a scale that generalizes the idea behind Stevens' method of ratio estimation. The next definition provides a natural qualitative ordering on $\mathbb{N}$.

Definition 17. Define $\preccurlyeq'$ on $\mathbb{N}$ as follows: For all α and β in $\mathbb{N}$,

$$\alpha \preccurlyeq' \beta \text{ iff for all } x \text{ in } X, \alpha(x) \preccurlyeq \beta(x).$$

Theorem 9. *$\langle \mathbb{N}, \preccurlyeq' \rangle$ is isomorphic to $\langle \mathbb{R}^+, \leqslant \rangle$.*

Proof. For each p in $\mathbb{I}^+$, let β_p be the function from $\mathbb{R}^+$ onto $\mathbb{R}^+$ such that for all u in $\mathbb{R}^+$,

$$\beta_p(u) = p \cdot u,$$

and let $\mathscr{S}$ be the set of isomorphisms of

$$\mathfrak{X} = \langle X, \precsim, \alpha_1, \ldots, \alpha_p, \ldots \rangle$$

onto

$$\mathfrak{M} = \langle \mathbb{R}^+, \leqslant, \beta_1, \ldots, \beta_p, \ldots \rangle.$$

By Theorem 5, $\mathscr{S}$ is a ratio scale. Let ψ be an element of $\mathscr{S}$, α be an arbitrary element of $\mathbb{N}$, and f be the image of α under ψ. Then, by isomorphism, f is an invariant of $\mathfrak{M}$ and f is a strictly increasing function on $\mathbb{R}^+$. Because $\mathscr{S}$ is a ratio scale, the set of automorphisms of $\mathfrak{M}$ is the set of multiplications by positive reals. Thus, because f is an invariant of $\mathfrak{M}$, for all r and s in $\mathbb{R}^+$,

$$f(r \cdot s) = r \cdot f(s).$$

It is well known that the only solution to the above functional equation for strictly increasing f is multiplication by a positive real. Thus f is an automorphism of $\mathfrak{M}$. Therefore, by the isomorphism ψ^{-1}, α is an automorphism of $\mathfrak{X}$. Thus $\mathbb{N}$ is the set of automorphisms of $\mathfrak{X}$. Let $\leqslant'$ be the following ordering on the set H of automorphisms of $\mathfrak{M}$: For all h and k in H and all r in $\mathbb{R}^+$,

$$h \leqslant' k \qquad \text{iff} \quad h(r) \leqslant k(r).$$

Then, because each element of H is a multiplication by a positive real, it easily follows that $\langle H, \leqslant' \rangle$ is a continuum. By the isomorphism ψ^{-1}, $\langle \mathbb{N}, \precsim' \rangle$ is a continuum.

The following lemma is useful in proofs. It is also employed in extending the key idea behind Stevens' method for constructing measuring functions for subjective ratio estimations to the continuum $\langle \mathbb{N}, \precsim' \rangle$.

LEMMA 1. *The following three statements are true:*

1. *Each element of* $\mathbb{N}$ *is onto* X.
2. *For each* y *and* z *in* X, *there is exactly one* β *in* $\mathbb{N}$ *such that* $\beta(y) = z$.
3. *For all* α *and* β *in* $\mathbb{N}$,

$$\alpha * \beta = \beta * \alpha.$$

Proof. Assume the notation of Theorem 9 and its proof. In the proof of Theorem 9 it was shown that $\mathbb{N}$ was the set of automorphisms of $\mathfrak{X}$. Thus Statement 1 follows. Because (i) the set of automorphisms of $\mathfrak{M}$ is the set of multiplications by positive reals, and (ii) for each r and s in $\mathbb{R}^+$, there exists a multiplication by a positive real h such that $h(r) = s$, Statement 2 follows by the isomorphism ψ^{-1} between $\mathfrak{M}$ and $\mathfrak{X}$. By a similar argument, Statement 3 follows.

DEFINITION 18. For each t in X, let Φ_t be the function from X into $\mathbb{N}$ such that for each x in X,

$$\Phi_t(x) = \beta,$$

where, by Statement 2 of Lemma 1, β is the unique element of $\mathbb{N}$ such that $\beta(t) = x$.

Suppose Stevens' method is used to generate a ratio scale $\mathscr{T}$. Let t be an arbitrary element of X and θ_t be the element of $\mathscr{T}$ such that $\theta_t(t) = 1$. Suppose $p \in \mathbb{I}^+$ and $\alpha_{\mathbf{p}}(t) = x$. Then Stevens' method of assigning numbers yields $\theta_t(x) = p$. By Statement 2 of Lemma 1 and Definition 18, $\Phi_t(x) = \alpha_{\mathbf{p}}$. Because $\alpha_{\mathbf{p}}$ is a primitive of the structure

$$\langle X, \preccurlyeq, \alpha_1, ..., \alpha_{\mathbf{p}}, ...\rangle,$$

it is an invariant of $\mathfrak{X}$, and because by Statement 1 of Axiom 2 $\alpha_{\mathbf{p}}$ is also $\preccurlyeq$-strictly increasing, it follows from Definitions 15 and 16 that $\alpha_{\mathbf{p}}$ is in $\mathbb{N}$. In this way, Definition 18 may be viewed as a way of extending Stevens' method to the continuum $\langle \mathbb{N}, \preccurlyeq' \rangle$.

Scale types are defined in terms of quantitative relationships between elements of the scale. For qualitative scales onto qualitative number systems, the equivalent approach often encounters difficulty. Because of this, it is convenient to classify scales of qualitatively valued measuring functions in a more abstract manner than has been traditionally done for scales quantitatively valued measuring functions. For the scale $\{\Phi_t \mid t \in X\}$ this is accomplished by the following theorem.

THEOREM 10. *Let $\mathscr{S} = \{\Phi_t \mid t \in X\}$ and G be the set of automorphisms of*

$$\langle X, \preccurlyeq, \alpha_1, ..., \alpha_{\mathbf{p}}, ...\rangle.$$

Then the following five statements are true:

1. *Each Φ in $\mathscr{S}$ is onto $\mathbb{N}$.*
2. *$\mathscr{S}$ is ordered; i.e., for all x and y in X and all Φ in $\mathscr{S}$,*

$$x \preccurlyeq y \quad \text{iff} \quad \Phi(x) \preccurlyeq' \Phi(y).$$

3. *$\mathscr{S}$ is homogeneous; i.e., for all x in X and all β in $\mathbb{N}$, there exists Φ in $\mathscr{S}$ such that $\Phi(x) = \beta$.*
4. *$\mathscr{S}$ is generated by automorphisms; i.e., for each Φ in $\mathscr{S}$,*

$$\mathscr{S} = \{\Phi * \alpha \mid \alpha \in G\}.$$

5. *$\mathscr{S}$ is 1-point unique; i.e., for all Φ and Ψ in $\mathscr{S}$, if $\Phi(x) = \Psi(x)$ for some x in X, then $\Phi = \Psi$.*

Proof. 1. Let t be an arbitrary element of X and α be an arbitrary element of $\mathbb{N}$. Then by Definition 18, $\Phi_t(\alpha(t)) = \alpha$.

2. Let x and y be arbitrary elements of X and Φ be an arbitrary element of $\mathscr{S}$. Let t in X be such that $\Phi = \Phi_t$ and, by Statement 2 of Lemma 1, let β and γ be elements of $\mathbb{N}$ such that $\beta(t) = x$ and $\gamma(t) = y$. Then by Definition 18,

$$\Phi_t(x) = \beta \qquad \text{and} \qquad \Phi_t(y) = \gamma.$$

Using Lemma 1, it is not difficult to show that

$$\beta(t) \preccurlyeq \gamma(t) \qquad \text{iff} \quad \beta \preccurlyeq' \gamma.$$

Thus by Definition 17 and Theorem 9,

$$x \preccurlyeq y \text{ iff } \beta(t) \preccurlyeq \gamma(t) \text{ iff } \beta \preccurlyeq' \gamma \text{ iff } \Phi_t(x) \preccurlyeq' \Phi_t(y).$$

3. Let x be an arbitrary element of X and β be an arbitrary element of $\mathbb{N}$. Because by Statement 1 of Lemma 1 β is onto X, let t in X be such that $\beta(t) = x$. Then by Definition 18, $\Phi_t(x) = \beta$.

4. Only the following needs to be shown: For all x and y in X, there exists α in $\mathbb{N}$ such that

$$\Phi_x = \Phi_y * \alpha. \tag{1}$$

Let Φ_x and Φ_y be arbitrary elements of $\mathscr{S}$. By Statement 2 of Lemma 1, let α in $\mathbb{N}$ be such that

$$\alpha(x) = y.$$

Equation (1) will be shown by showing $\Phi_y^{-1} * \Phi_x = \alpha$. It follows from Statements 1 and 2 that $\Phi_y^{-1} * \Phi_x$ is a one-to-one function from X onto X. Let z be an arbitrary element of X. By Statement 2 of Lemma 1, let θ in $\mathbb{N}$ be such that

$$\theta(x) = z.$$

By Statement 3 of Lemma 1,

$$\theta * \alpha = \alpha * \theta.$$

Thus by Definition 18,

$$\Phi_y^{-1} * \Phi_x(z) = \Phi_y^{-1}(\theta) = \theta(y) = \theta * \alpha(x) = \alpha * \theta(x) = \alpha(z).$$

5. Suppose Φ and Ψ are in $\mathscr{S}$, $x \in X$, and $\Phi(x) = \Psi(x)$. By Statement 4, let α in $\mathbb{N}$ be such that $\Phi = \Psi * \alpha$. Then

$$\alpha(x) = \Psi^{-1} * \Phi(x) = x = \iota(x),$$

which by Statement 2 of Lemma 1 yields $\alpha = \iota$, and therefore, $\Phi = \Psi$.

Let $\mathscr{T}$ be a scale of isomorphisms of a continuous structure $\mathfrak{Y} = \langle Y, \preccurlyeq_Y, U_1, U_2, \ldots\rangle$ onto a numerical structure $\mathfrak{M} = \langle \mathbb{R}^+, \leqslant, V_1, V_2, \ldots\rangle$. In the obvious manner, the concepts of order, homogeneity, generation by automorphisms, and 1-point uniqueness of Theorem 10 can be modified so that they apply to $\mathscr{T}$. Because elements of $\mathscr{T}$ are isomorphisms, it easily follows that $\mathscr{T}$ is an ordered scale. Homogeneity and 1-point uniqueness are conditions characteristic of the ratio scalability of $\mathfrak{Y}$ in the following sense: If $\mathscr{T}$ is homogeneous and 1-point unique, then by using Theorem 4.3 of Chapter 2 of Narens (1985) it follows that there exists a numerical structure $\mathfrak{M}' = \langle \mathbb{R}^+, \leqslant, V_1', V_2', \ldots\rangle$ such that $\mathscr{T}$ is the set of isomorphisms $\mathfrak{Y}$ onto $\mathfrak{M}'$ and that $\mathscr{T}$ is a ratio scale. If $\mathscr{T}$ is a ratio scale, then it easily follows that it is generated by automorphisms. Thus the conditions of order, homogeneity, generation by automorphisms, and 1-point uniqueness may be viewed as algebraic properties that are characteristic of the ratio scalability of $\mathfrak{Y}$.

DEFINITION 19. $\mathfrak{F} = \langle F, \preccurlyeq', \oplus, \otimes, e \rangle$ is said to be a *positive continuum field* if and only if $\mathfrak{F}$ is isomorphic to $\langle \mathbb{R}^+, \leqslant, +, \cdot, 1 \rangle$.

Let $\mathfrak{F} = \langle F, \leqslant', \oplus, \otimes, e \rangle$ be a positive continuum field. Then $\leqslant'$ is called the *total ordering relation,* $\oplus$ the *addition operation,* $\otimes$ the *multiplication operation,* and e the *multiplicative identity*.

Positive continuum fields are the positive parts of what in algebra are known as Dedekind complete totally ordered fields. The following theorem follows from a well-known characterization of the Dedekind complete totally ordered fields.

THEOREM 11. *Let $\mathfrak{F}$ be a positive continuum field. Then there is exactly one isomorphism of $\mathfrak{F}$ onto $\langle \mathbb{R}^+, \leqslant, +, \cdot, 1 \rangle$.*

There are various ways of defining addition and multiplication operations $\mathbb{N}$ so that these defined operations together with $\precsim'$ provide a rich algebraic structure on $\mathbb{N}$. In the following, a way in which the operation of function composition $*$ is the multiplication operation is presented. (There are also natural constructions in which $*$ is the addition operation.)

It is easy to verify that the identity function ι on X is in $\mathbb{N}$ and that function composition $*$ is an operation on $\mathbb{N}$. $*$ will be taken as the multiplication operation on X. Because ι is an identity element of $*$, ι will be taken as the multiplicative identity of the qualitative number system.

With these choices made, the choice of an addition operation on $\mathbb{N}$ presents a problem, because there are many natural choices. Theorem 12 below describes one of these.

THEOREM 12. *There exists exactly one operation $\oplus'$ on $\mathbb{N}$ such that*

$$\langle \mathbb{N}, \precsim', \oplus', {}^*, \iota \rangle$$

is a positive continuum field and $\alpha_2 = \iota \oplus \iota$.

Proof. By the proof of Theorem 9 it easily follows that $\langle \mathbb{N}, \precsim', *, \iota \rangle$ and $\langle \mathbb{R}^+, \leqslant, \cdot, 1 \rangle$ are isomorphic. Thus let φ be an isomorphism of $\langle \mathbb{N}, \precsim', *, \iota \rangle$ onto $\langle \mathbb{R}^+, \leqslant, \cdot, 1 \rangle$. Let $c = \varphi(\alpha_2)$. Then by isomorphism, to show the theorem the following needs only to be shown: *There is exactly one operation $+'$ on $\mathbb{R}^+$ such that*

$$\langle \mathbb{R}^+, \leqslant, +', \cdot, 1 \rangle \textit{ is a continuum field and } 1 +' 1 = c.$$

Because $\iota \prec' \alpha_2$, it follows that $1 < c$. Let $+'$ be the operation on $\mathbb{R}^+$ such that for all u and v in $\mathbb{R}^+$,

$$u +' v = \left(u^{\frac{\log 2}{\log c}} + v^{\frac{\log 2}{\log c}}\right)^{\frac{\log c}{\log 2}}.$$

Then by direct verification, $\langle \mathbb{R}^+, \leqslant, +', \cdot, 1 \rangle$ is a positive continuum field and $1 +' 1 = c$. To show the uniqueness of $+'$, suppose $+''$ is another operation on $\mathbb{R}^+$ such that $\langle \mathbb{R}^+, \leqslant, +'', \cdot, 1 \rangle$ is a positive continuum field and $1 +'' 1 = c$. Because

$\langle \mathbb{R}^+, \leqslant, +, \cdot, 1\rangle$ and $\langle \mathbb{R}^+, \leqslant, +'', \cdot, 1\rangle$ are positive continuum fields, it easily follows that

$$\langle \mathbb{R}^+, \leqslant, +\rangle \qquad \text{and} \qquad \langle \mathbb{R}^+, \leqslant, +''\rangle$$

are continuous extensive structures. Then by Theorem 10.3 of Chapter 2 of Narens (1985), let r in $\mathbb{R}^+$ be such that for all u and v in $\mathbb{R}^+$,

$$u +'' v = (u^r + v^r)^{\frac{1}{r}}.$$

Then

$$c = 1 +'' 1 = (1^r + 1^r)^{\frac{1}{r}} = 2^{\frac{1}{r}},$$

and thus $r = \frac{\log 2}{\log c}$, and therefore, $+'' = +'$.

Definition 20. Let $\oplus'$ be the unique operation on $\mathbb{N}$ described in the statement of Theorem 12.

$\oplus'$ will be taken as the addition operation for $\mathbb{N}$. Let

$$\mathfrak{N} = \langle \mathbb{N}, \preccurlyeq', \oplus', *, \iota\rangle.$$

Then $\mathfrak{N}$ is defined through qualitative means. Because by Theorem 12 $\mathfrak{N}$ is a positive continuum field, $\mathfrak{N}$ is by Theorem 11 easily adequate as a system of "numbers" for performing calculations.

8. CONCLUSIONS

Stevens developed his theory of measurement to counter the view that all strong forms of measurement relied on qualitative addition operations. He developed methods of measuring based on subjective estimations as examples of measurement that were radical departures from the classical methods. It was universally accepted that his methods were radical departures, but it remained controversial whether they were valid methods of measurement. In Section 5, axioms in terms of observables were presented that described assumptions inherent in Stevens' theory of subjective estimation of ratios. Theorem 5 showed that these axioms provide, through the representational theory of measurement, a rigorous foundation for measuring phenomena through subjective estimations.

In classical measurement, a qualitative addition operation was used to produce standard sequences. Measurement was then accomplished through use of those sequences. Theorem 6 showed that the structure of standard sequences was the same as the structure that one obtained through ratio estimation satisfying Axioms 1 and 2. This result produces twin ironies: First, Stevens' radical method is not radical at all, except perhaps for data collection, since it is structurally the same as the key structure that classical measurement uses to measure. (And one can argue that the data collection is scarcely radical, because for centuries astronomers collected related kinds of data for star classification.) Second, Stevens' critics were wrong in holding that subjectively based estimations could not be used for establishing strong forms of measurement. Although these estimations are based on subjective experience, the estimations themselves are part of the observable world, and as such they can enter in methods of measurement in exactly the same manner as observable elements of standard sequences arising from a physical addition operation.

Stevens' method of measurement consisted of constructing real valued functions ϕ_t such that

$$p = \phi_t(x) \text{ iff the subject estimates that } x \text{ is } p\text{-times as intense as } t.$$

In Section 7 it was shown that instead of a real valued function a qualitative valued function Φ_t could be used, where for positive integral p,

$$\begin{aligned} \alpha_{\mathbf{p}} = \Phi_t(x) \quad & \text{iff} \quad \text{the subject estimates that } x \text{ is } p\text{-times as intense as } t \\ & \text{iff} \quad \alpha_{\mathbf{p}}(t) = x, \end{aligned}$$

and where the range of Φ_t consisted of invariants of the measuring process that were strictly increasing functions from the domain of stimuli onto itself. Because these invariants with naturally defined operations and relations on them form a structure isomorphic to $\langle \mathbb{R}^+, \leqslant, +, \cdot, 1 \rangle$, they can be taken as "numbers" for many scientific purposes. Accordingly, Φ_t may be viewed as a qualitative measuring function onto an algebraically rich number domain. Theorem 10 and the discussion following it establishes that

$$\{\Phi_t \mid t \text{ is in the domain of stimuli}\}$$

has the *algebraic* properties characteristic of ratio scales.

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