Choice, Decision, and Measurement
A. A. J. Marley (Ed.),

ON SUBJECTIVE INTENSITY AND ITS MEASUREMENT

LOUIS NARENS

University of California, Irvine

ABSTRACT. Threshold functions and subjective estimations of ratios and differences are a few examples from a class of psychological functions that can be conceptualized as functions from a set of stimuli onto itself. This paper analyzes axiomatically the structural properties of a broad subset of such functions, with an emphasis on general properties of their psychological processing. Measurement-theoretic representation and uniqueness results are derived for Weber's Law, ratio magnitude estimation, and a version of Luce's Possible Psychophysical Laws. An explanation is also provided for the experimental findings of W. Torgerson and others that indicates a *qualitative* identity between subjective estimation of ratios and subjective estimation of differences.

1. INTRODUCTION

In many paradigms in psychophysics and other areas of psychology, a subject's behavior can often be idealized as sets of functions from a set of stimuli onto itself. Two examples from psychophysics are (i) threshold functions F_p on X, where $1 > p > .5$ and for each x in X, $F_p(x)$ is the stimulus such that if y has more of the physical attribute than $F_p(x)$, then the proportion of time y is judged subjectively more intense than x is $> p$; and (ii) ratio estimation functions G_r on X such that for each x and y in X, $y = G_r(x)$ if and only if the subject judges y as being r times as intense as x. Functions from X onto itself that result from subjects' responses to instructions are called *behavioral*, and this chapter presents an axiomatic treatment of behavioral functions that result from responses involving a subject's evaluation of subjective intensities of stimuli. One main result is a characterization of situations where for a nonempty set $\mathcal{B}$ of behavioral functions there exists a mapping of the stimuli into the positive reals such that each element of $\mathcal{B}$ is represented as a multiplication by a positive real. This result is used to characterize axiomatically a generalization of the psychophysical power law. Another result provides a new perspective for Luce's (1959b) seminal research on possible psychophysical laws.

As a motivation for the kind of modeling and theory developed in this chapter, consider the following important and puzzling empirical finding of Torgerson (1961).

Key words and phrases. Psychophysics, subjective intensity, ratio estimation, difference estimation, Weber's law, power law, psychophysical law.

Acknowledgments. Research for this chapter was supported by the NSF grant SBR-9520107 to the Department of Cognitive Sciences, University of California, Irvine.

Address for correspondence: Louis Narens, Department of Cognitive Sciences, University of California Irvine, Irvine, CA 92697. Email: lnarens@vmsa.oac.uci.edu

> The situation turns out to be much the same in the quantitative judgment domain. Again, we have both distance methods, where the subject is instructed to judge subjective differences between stimuli, and ratio methods, where the subject is instructed to judge subjective ratios. Equisection and equal appearing intervals are examples of distance methods. Fractionation and magnitude estimation are examples of ratio methods.
>
> In both classes of methods, the subject is supposed to tell us directly what the differences and ratios are. We thus have the possibility of settling things once and for all. Judgments of differences take care of the requirements of the addition commutative group. Judgments of ratios take care of the multiplication commutative group. All we need to show is that the two scales combine in the manner required by the number system. This amounts to showing that scales based on direct judgments of subjective differences are linearly related to those based on subjective ratios.
>
> Unfortunately, they are not. While both procedures are subject to internal consistency checks, and both often fit their own data, the two scales are not linearly related. But when we plot the logarithm of the ratio scale against the difference scale spaced off in arithmetic units, we usually do get something very close to a straight line. Thus, according to the subject's own judgments, stimuli separated by equal subjective intervals are also separated by approximately equal subjective ratios.
>
> This result suggests that the subject perceives or appreciates but a single quantitative relation between a pair of stimuli. This relation to begin with is neither a quantitative ratio or difference, of course – ratios and differences belong only to the formal number system. It appears that the subject simply interprets this single relation in whatever way the experimenter requires. When the experimenter tells him to equate differences or to rate on an equal interval scale, he interprets the relation as a distance. When he is told to assign numbers according to subjective ratios, he interprets the same relation as a ratio. Experiments on context and anchoring show that he is also able to compromise between the two. (*pp. 202–203*)

For the purposes of this chapter, Torgerson's conclusion may be restated as follows: There is a function Ψ that maps stimuli in X into sensations of a subject. When asked for kinds of difference judgments the subject responds in accordance with a subjective difference function on the set of sensations $\Psi(X)$; and when asked for kinds of ratio judgments the subject responds in accordance with a subjective ratio function on the same set of sensations $\Psi(X)$. Torgerson's empirical studies indicate that each difference function is a ratio function and visa versa. But, why should this be the case? That is, under what kinds of general conditions about subjective processing of stimuli should we expect a result like this?

By "general conditions" I mean conditions or principles that apply to many different kinds of phenomena, and not conditions specific to particular experimental paradigms or the data they generate. They should be construed as conditions or principles like the laws of physics. Einstein's Principle of Relativity is a particularly good example of what I call a "general condition."

A general condition that drives many of the mathematical results of this paper is that the sensations in $\Psi(X)$ are processed "homogeneously": that is, the sensations in $\Psi(X)$ are processed in a manner that does not distinguish *individual* elements of $\Psi(X)$. A precise statement of "homogeneity" is given later in the chapter. It is often a consequence of concepts used routinely in science. For example, whenever one is in a situation where there is a set of isomorphisms from an underlying qualitative structure into a numerical one that forms a ratio, interval, or ordinal scale, the underlying qualitative structure is homogeneous.

The axiomatizations presented in this chapter are designed to explain why we observe what we observe; they are not constructed to be slick descriptions of what is observed. Because of this, when there are axioms about both observables and mental phenomena, the axioms about observables are by design very weak. Great care is taken throughout the chapter to separate what is being assumed to be observed about the stimulus, what is being assumed to be observed about the subject's responses, and what is theoretically taking place in the mental processing of the subject in producing his or her responses to instructions and stimuli.

The intent of the chapter is to show how fairly simple and plausible assumptions about the production of psychophysical functions combine to produce powerful results about observable relationships. Discussions of these results in terms of the empirical literature are brief and generally limited to issues raised by the above quotation of Torgerson.

2. Preliminaries

The following definitions and conventions are observed throughout this chapter:

Definition 1. $\mathfrak{Y} = \langle Y, U_1, U_2, \ldots \rangle$ is said to be a *structure* if and only if Y is a nonempty set, called the *domain* of $\mathfrak{Y}$, and each U_i is either an element of Y, a set of elements of Y, a relation on Y, a set of relations on Y, a set of sets of relations on Y, etc. $Y, U_1, U_2, \ldots$ are called the *primitives* of $\mathfrak{Y}$.

$\mathbb{R}$ denotes the set of reals and $\mathbb{R}^+$ the set of positive reals.

A *scale* on a nonempty set Y is a nonempty set of functions $\mathcal{F}$ onto a subset R of $\mathbb{R}$.

$\mathcal{F}$ is said to be a *ratio scale* if and only if (1) $\mathcal{F}$ is a scale and the range of each element of $\mathcal{F}$ is a subset of $\mathbb{R}^+$, and (2) for any f in $\mathcal{F}$,

$$\mathcal{F} = \{rf | r \in \mathbb{R}^+\}.$$

$\mathcal{F}$ is said to be an *interval scale* if and only if (1) $\mathcal{F}$ is a scale, and (2) for any f in $\mathcal{F}$,

$$\mathcal{F} = \{rf + s | r \in \mathbb{R}^+ \text{ and } s \in \mathbb{R}\}.$$

$\mathcal{F}$ is said to be an *ordinal scale* if and only if (1) $\mathcal{F}$ is a scale and the range of each element of $\mathcal{F}$ is a subset of $\mathbb{R}^+$, and (2) for any f in $\mathcal{F}$,

$$\mathcal{F} = \{g * f | g \text{ is a strictly increasing function from } \mathbb{R}^+ \text{ onto } \mathbb{R}^+\},$$

where $*$ is the operation of function composition.

Note that the definitions of ratio and ordinal scales in Definition 1 are more restrictive than usually given in the literature; in particular, the ranges of the elements of ratio and ordinal scales are required to be subsets of $\mathbb{R}^+$.

Definition 2. A *theory of measurement* consists of a precise specification of how a scale $\mathcal{F}$ of functions is formed. The theory of measurement used throughout this chapter is a variant of the *representational theory of measurement.* This variant says that a scale $\mathcal{F}$ on a set Y results by providing a structure of primitives $\mathfrak{Y}$ with domain Y and a numerical structure $\mathfrak{N}$ with domain either $\mathbb{R}^+$ or $\mathbb{R}$ such that $\mathcal{F}$ is a set of isomorphisms of $\mathfrak{Y}$ onto $\mathfrak{N}$.

In the representational theory, elements of a scale $\mathcal{F}$ are often called *representations.*

Note that the variant of the "representational theory" given in Definition 2 is more restricted than the theory of measurement presented in Krantz, Luce, Suppes, & Tversky (1971) in that it requires (1) elements of $\mathcal{F}$ to be isomorphisms (instead of homomorphisms), and (2) the range of elements of $\mathcal{F}$ to be *onto* (instead of *into*) either $\mathbb{R}^+$ or $\mathbb{R}$.

Definition 3. A structure $\mathfrak{Y}$ is said to be *ratio* (respectively, *interval, ordinal*) *scalable* if and only if it has a ratio (respectively, interval, ordinal) scale of isomorphisms onto some numerical structure.

Definition 4. A structure $\langle X, \succsim \rangle$ is said to be a *continuum* if and only if it is isomorphic to $\langle \mathbb{R}^+, \geq \rangle$.

The definition of "continuum" given in Definition 4 is not qualitative. A famous qualitative characterization of "continuum" was given by Cantor (1895).(See pp. 31–35 of Narens, 1985).

Definition 5. $\langle X, \succsim, \oplus \rangle$ is said to be a *continuous extensive structure* if and only if $\langle X, \succsim, \oplus \rangle$ is isomorphic to $\langle \mathbb{R}^+, \geq, + \rangle$.

The definition of "continuous extensive structure" given in Definition 4 is not qualitative. Qualitative axiomatizations of it can be given (e.g., combining qualitative axiomatization of "continuum" with an axiomatization of "extensive structure," for example, the axiomatization of "closed extensive structure" given in Chapter 3 of Krantz et al., 1971).

Definition 6. Isomorphisms of a continuous extensive structure $\langle X, \succsim, \oplus \rangle$ onto $\langle \mathbb{R}^+, \geq, + \rangle$ are often called *additive representations.*

The following is a well-known theorem of representational measurement theory:

Theorem 1. *The set of additive representations of a continuous extensive structure forms a ratio scale.*

3. Basic axioms

The primitives for the basic axioms consist of *physical primitives* and *behavioral primitives.* The physical primitives capture some of the physical relationships inherent in the stimuli used in the experiment. The behavioral primitives capture, in terms of the stimuli, some of the important psychological structure inherent in the subject's responses to instructions and stimuli. Both the physical and behavioral primitives are assumed to be observable.

The physical primitives in this chapter will consist of a nonempty set of stimuli, X, a total ordering $\succsim$ on this set of stimuli, and occasionally a binary operation $\oplus$ on X called a *concatenation* operation. Most of the results of the chapter rely on the primitives X and $\succsim$. (Because many sets of stimuli considered in psychology have natural behaviorally induced total orderings on them, the results of this paper that do not use the concatenation operation $\oplus$ often extend to behavioral situations based on stimulus sets with such orderings.)

The *behavioral primitives* in this chapter consist of the set of stimuli, X, and one or more functions, $B_1, B_2, \ldots$, on X. (Cases of infinitely many B_i are allowed.)

Note that the set of stimuli X is considered both a physical and a behavioral entity.

The physical and behavioral primitives are assumed to be observable. They make up the *behavioral-physical structure,* which has the form

$$\langle X, \succsim, B_1, B_2, \ldots \rangle$$

or if the physical concatenation operation $\oplus$ is relevant, the form

$$\langle X, \succsim, \oplus, B_1, B_2, \ldots \rangle .$$

The following are two examples of behavioral functions:

(1) *Direct Ratio Estimation:* The subject is given stimuli from X and is asked for each such stimulus x to select a stimulus y from X such that "y is s times as intense as x." A function $y = R_s(x)$ on X results.

(2) *Direct Difference Estimation:* The subject is given stimuli from X and is asked for each such stimulus to select a stimulus y from X such that "the difference of y and x in intensity is s." A function $y = D_s(x)$ on X results.

The usual numerical representation of the behavioral function R_s is the function that is multiplication by the number s. I find this practice strange, unfounded, and non-rigorous. Unfortunately, it is a widely used practice in the behavioral and social sciences, and many important findings depend on it. In this chapter, a much weaker version of representing R_s by a multiplication (not necessarily multiplication by s) is pursued. (This issue is also discussed in Narens, 1996a.)

Definition 7. Let $\mathfrak{X}$ be the behavioral physical structure

$$\langle X, \succsim, B_1, B_2, \ldots \rangle .$$

Then a function β from $\mathfrak{X}$ onto $\mathbb{R}^+$ is said to be a *multiplicative representation* for $\mathfrak{X}$ if and only if the following two statements are true:

(a) For each x and y in X,

$$x \succsim y \text{ iff } \beta(x) \geq \beta(y) .$$

(b) For each B_i there exists a positive real r_i such that for all x and y in X,

$$y = B_i(x) \text{ iff } \beta(y) = r_i \cdot \beta(x) .$$

Note that in Definition 7, no mention is made about how the behavioral primitives were obtained; in particular, the real number r_i in Statement 2 does not depend on whether B_i was obtained through direct ratio estimation or direct difference estimation.

Definition 8. Let $\mathfrak{X}$ be the behavioral-physical structure

$$\langle X, \succsim, B_1, B_2, \ldots \rangle .$$

Then two multiplicative representations β and γ for $\mathfrak{X}$ are said to be *equivalent* if and only if for each B_i there exists a positive real r_i such that for all x and y in X,

$$y = B(x) \text{ iff } \beta(y) = r_i \cdot \beta(x) \text{ iff } \gamma(y) = r_i \cdot \gamma(x) .$$

In particular, note that by Definition 8, if β is a multiplicative representation of $\mathfrak{X}$, then $r\beta$ is an equivalent multiplicative representation of $\mathfrak{X}$ for each positive real r. As is indicated in the discussion following Theorem 2 below, there are examples of behavioral-structures that have equivalent multiplicative representations β and γ such that for all positive reals r, $\beta \neq r\gamma$.

Similar definitions hold for a *difference representation* for $\mathfrak{X}$ and *equivalent difference representations* for $\mathfrak{X}$.

The *Basic Axioms,* which are about the behavioral-physical structure

$$\langle X, \succsim, B_1, B_2, \ldots \rangle ,$$

consist of the following three axioms:

Axiom 1 (Physical Axiom). *$\langle X, \succsim \rangle$ is a continuum (Definition 4).*

Axiom 2 (Behavioral Axiom). *Each B_i is a function from X onto X.*

Axiom 3 (Behavioral-Physical Axiom). *For each B_i and each x and y in X,*

$$x \succsim y \text{ iff } B_i(x) \succsim B_i(y) .$$

Axioms 1 to 3 are indeed very basic, saying very little of mathematical or psychological substance.

4. Weber's Law and the Generalized Power Law

The Basic Axioms are not sufficient for establishing the existence of a multiplicative representation for the behavioral-physical structure. This section considers a particularly simple, observable, behavioral-physical condition that together with the Basic Axioms implies the existence of multiplicative representations. The additional behavioral-physical condition uses the physical concatenation operation $\oplus$. (Narens, 1996a, provides observable behavioral conditions that implies the existence of multiplicative representations without using any physical structure beyond the ordering $\succsim$.)

Definition 9. Assume $\langle X, \succsim \rangle$ is a continuum (Definition 4). A function B from X onto X is said to be a *threshold function* on X if and only if (1) for all x and y in X,

$$x \succsim y \text{ iff } B(x) \succsim B(y)\,,$$

and (2) for all x in X,

$$B(x) \succ x\,.$$

The "threshold interpretation" of B in Definition 9 is that for each x in X, $B(x)$ is the element of X such that for all elements y of X, if $y \succ B(x)$, then the subject according to some behavioral criteria is able to discriminate y as being more intense than x, and for all elements z of X, if $B(x) \succ z$, then the subject is not able according to the behavioral criteria to discriminate z as being more intense than x.

Threshold functions are often represented by Weber representations:

Definition 10. Assume B is a threshold function on the continuum $\langle X, \succsim \rangle$ and c is a positive real number. Then φ is said to be a *Weber representation* for $\langle X, \succsim, B \rangle$ with *Weber constant* c if and only if φ is an isomorphism of $\langle X, \succsim \rangle$ onto $\langle \mathrm{I\!R}^+, \geq \rangle$ such that for all x and y in X,

$$y \succ B(x) \text{ iff } \frac{\varphi(y) - \varphi(x)}{\varphi(x)} > c\,.$$

Suppose φ is a Weber representation for $\langle X, \succsim, B \rangle$ with Weber constant c. Then it easily follows that $1 + c$ is a multiplicative representation for $\langle X, \succsim, B \rangle$. Conversely suppose $\langle X, \succsim, F \rangle$ has a multiplicative representation as a multiplication $k > 1$. Then it easily follows that $\langle X, \succsim, F \rangle$ has a Weber representation with Weber constant $k - 1$.

The following theorem shows that each threshold function has a Weber representation, and therefore by the above observation, each threshold function has a multiplicative representation.

Theorem 2. (Existence Theorem). *Suppose B is a threshold function on the continuum$\langle X, \succsim, \rangle$. Then for some $c > 0$, $\langle X, \succsim, B \rangle$ has a Weber representation with Weber constant c.*

Proof. Theorem 5.3 of Narens (1994). □

The corresponding uniqueness theorem for Theorem 2 is a consequence of Theorem 4.1 of Narens (1994). The latter also shows that $\langle X, \succsim, B \rangle$ has Weber representations γ and θ with the same Weber constant c such that for all $r \in \mathrm{I\!R}^+$, $\gamma \neq r\theta$. Thus $\langle X, \succsim, B \rangle$ also has multiplicative representations β and δ such that $\beta \neq r\delta$ for all $r \in \mathrm{I\!R}^+$.

Weber's Law consists of much more than having a Weber representation: Weber's Law results when the stimuli have been measured priory in terms of a standard physical representation φ, and then *with respect to* φ, a Weber representation results. Thus for Weber's Law to hold for a threshold function B, a particular kind of compatibility between B and the physical structure is needed for φ to also be a multiplicative representation of the behavioral-physical structure. Theorem 4 below is one method of formulating the needed compatibility in terms of observables.

Note that Theorem 4 also provides for simultaneous Weber Law representations for several threshold functions.

Theorem 3. (Generalized Power Law Theorem). *Assume Axioms 1 to 3. Suppose* $\oplus$ *is a physical operation and* φ *is an additive representation for* $\langle X, \succsim, \oplus \rangle$. *Suppose the following (observable) psychophysical axiom: For all* B_i *and all* x *and* y *in* X,

$$B_i(x \oplus y) = B_i(x) \oplus B_i(y) .$$

Then the following two statements are true:

(a) (Existence) φ *is a multiplicative representation for behavioral-physical structure*

$$\langle X, \succsim, B_1, B_2, \dots \rangle .$$

(b) (Uniqueness) *Let* β *be a multiplicative representation for*

$$\mathfrak{X} = \langle X, \succsim, B_1, B_2, \dots \rangle .$$

Then there exists a multiplicative representation γ *of* $\mathfrak{X}$ *that is equivalent to* β *and a positive real number* t *such that for each* x *in* X,

$$\gamma(x) = \varphi(x)^t . \tag{1}$$

Furthermore, if $\mathfrak{X}$ *is ratio scalable, then* γ *in Equation 1 is* $r\beta$ *for some positive real* r.

The following is an immediate consequence of Theorem 3 and the above discussion about the relationship of multiplicative representations and Weber representations:

Theorem 4. (Existence Theorem). *Assume the hypotheses of Theorem 3. Suppose that for each* B_i *and each* x *in* X, $B_i(x) \succ x$. *Then for each* B_i, *there exists a positive real* c_i *such that* φ *is a Weber representation for* B_i *with Weber constant* c_i.

5. Cognitive Axioms

The *cognitive primitives* consist of a subset S of sensations, a binary relation $\succsim'$ on S, and additional primitives, $T_1, T_2, \dots$, which may be first-order (e.g., subsets of S, relations on S) or higher-order (sets of relations on S, relations between relations on S and elements of S, etc.) The cognitive primitives are theoretical in nature and presumed to be unobservable to the experimenter. Also, it is not assumed that all these primitives are observable to the subject in the sense that he or she is capable of becoming aware of each of them. $\mathfrak{S} = \langle S, \succsim', T_1, T_2, \dots \rangle$ is called the *structure of cognitive primitives.* Axioms about $\mathfrak{S}$ will provide a theory that is used to relate the observable behavioral functions B_i to non-observable processing of instructions presented to the subject.

Definition 11. Throughout this paper Ψ will denote a primitive relation. Axiom 4 below will establish that Ψ is a function from X onto S. Since X is both physical and behavioral, Ψ is considered to be both a *cognitive-physical* and a *cognitive-behavioral* primitive.

Axiom 4 (Cognitive-Physical Axiom). *Ψ is a function from X onto S.*

Axiom 5 (Cognitive-Physical Axiom). *For all x and y in X,*

$$x \succsim y \text{ iff } \Psi(x) \succsim' \Psi(y).$$

It is an immediate consequence of Axioms 1, 4, and 5 that the function Ψ^{-1} exists.

$\succsim'$ is intended to be an ordering consistent with subjective intensity. It is not assumed that for all x and y in X with $x \succ y$, $\Psi(x) \succ' \Psi(y)$ is phenomenologically observable by the subject. Indeed, for different x and y sufficiently close in terms of the $\succsim$ ordering, one might want as a theoretical axiom that they are not distinguished phenomenologically in terms of subjective intensity.[1]

The following cognitive-behavioral axiom describes how a stimulus item is processed in terms of an instruction to the subject and the structure $\mathfrak{S}$. For purposes of exposition, the instruction is specialized to a form of a direct ratio or difference estimation. The axiom and the results that depend on it extend to a wide variety of instructions.

Axiom 6 (Cognitive-Behavioral Axiom). *Let $\boldsymbol{I}$ be an instruction given to the subject. It is assumed that $\boldsymbol{I}$ is of one of the following two forms: (1) the* ratio instruction, *"Find y in X such that y is p times as intense as the stimulus presented;" or (2) the* difference instruction, *"Find y in X such that the difference in intensity between y and the presented stimulus is p." Then there exists a cognitive function $F_{\boldsymbol{I}}$ that is produced by an algorithmic procedure* using only S and primitives of $\mathfrak{S}$ *such that for each stimulus x in X, if x is presented to the subject, then the subject responds by selecting y in X, where*

$$y = \Psi^{-1}[F_{\boldsymbol{I}}(\Psi(x))].$$

Furthermore, it is assumed that each primitive behavioral function B of behavioral-physical structure results from such a cognitive function, i.e., there exists an instruction $\boldsymbol{J}$ such that for all x and y in X,

$$y = B(x) \text{ iff } \Psi(y) = F_{\boldsymbol{J}}(\Psi(x)).$$

The intuition for Axiom 6 is as follows: When given instruction $\boldsymbol{I}$ and presented with stimulus x, the subject responds with y. The subject does this by implementing $\boldsymbol{I}$ as a function $F_{\boldsymbol{I}}$ on S, which he or she applies to $\Psi(x)$ to yield $F_{\boldsymbol{I}}(\Psi(x))$, which happens to be $\Psi(y)$. The implementation of $\boldsymbol{I}$ as $F_{\boldsymbol{I}}$ is carried out by an algorithmic procedure that involves some of the primitives $\{S, \succsim', T_1, T_2, \dots\}$. Different instructions $\boldsymbol{J}$ may give rise to different algorithmic procedures, which may involve different primitives of $\{S, \succsim', T_1, T_2, \dots\}$. It is explicitly assumed that each such implemented function is algorithmic in terms of primitives of $\mathfrak{S}$.

The notion of "algorithm" intended here is much more general than the ones ordinarily encountered in computer science – the latter being usually a form of Turing computability or some equivalent. (Turing computability is too restrictive,

[1]An example of this is to have one of the cognitive primitives, say T_1, a semiorder (Luce, 1956) that is phenomenologically observable by the subject, and have $\succsim'$ be the total ordering that is induced by T_1.

since it can only apply to situations that are encodable into arithmetic, and $\Psi(X)$ cannot be appropriately so encoded, because $\Psi(X)$ has greater cardinality than that of the set of natural numbers.)

It should be noted that the proofs of results employing Axiom 6 use much weaker conditions than those needed for this general concept of "algorithm"–namely, that the functions F_I in Axiom 6 have precise mathematical descriptions in terms of the primitives of $\mathfrak{S}$. Thus, in particular, the results of this chapter that depend on Axiom 6 are valid for any *formal* concept of "algorithm" appropriate to the situation described in Axiom 6.

Definition 12. Let $\mathfrak{Y} = \langle Y, W_1, \ldots, W_n \rangle$ be a structure. Isomorphisms of $\mathfrak{Y}$ onto itself are called *automorphisms.* $\mathfrak{Y}$ is said to be *homogeneous* if and only if for each x and y in Y there exists an automorphism α of $\mathfrak{Y}$ such that $\alpha(x) = y$.

Axiom 7. $\mathfrak{S}$ *is homogeneous.*

I admit that because of the abstract nature of the above definition of "homogeneity," Axiom 7 looks more like arcane mathematics than substantive psychology. However, homogeneity is a logical consequence of concepts and hypotheses used routinely throughout psychophysics, and more generally throughout science. For example, many important cases in science involve ratio, interval, or ordinal scales. When such scales can be justified through the representational theory of measurement, homogeneity is a consequence:

Suppose $\mathfrak{Y}$ is a qualitative structure with domain Y, $\mathfrak{N}$ is a numerical structure with domain N, and $\mathcal{M}$ is the scale of isomorphisms of $\mathfrak{Y}$ onto $\mathfrak{N}$ (Definition 2). The following is a necessary condition for $\mathcal{M}$ to be a ratio, interval, or ordinal scale:

$$\text{For each } x \text{ in } Y \text{ and } r \text{ in } N \text{ there exists } \beta \text{ in } \mathcal{M} \text{ such that } \beta(x) = r\,. \tag{2}$$

Assume Equation 2. It immediately follows from the definition of "automorphism" that for all γ and δ in $\mathcal{M}$, $\delta^{-1} * \gamma$ is an automorphism of $\mathfrak{Y}$. Let x and y be arbitrary elements of Y, and let γ be an element of $\mathcal{M}$. By Equation 2, let δ in $\mathcal{M}$ be such that

$$\delta(y) = \gamma(x)\,.$$

Then $y = \delta^{-1} * \gamma(x)$, where $\delta^{-1} * \gamma$ is an automorphism of $\mathfrak{Y}$. Since x and y are arbitrary elements of Y, it has been shown that $\mathfrak{Y}$ is homogeneous.

By arguments similar to the above, it is easy to establish that $\mathfrak{Y}$ is homogeneous if and only if Equation 2 holds. In most scientific applications, the primitives of $\mathfrak{Y}$ would correspond to observable relations and Equation 2 would be a consequence of generalizations and idealizations of observed facts about the primitives. However, because of principled lack of knowledge about the cognitive relations $T_1, T_2 \ldots$, a corresponding strategy cannot be adopted for the cognitive structure $\mathfrak{S}$. Instead, general assumptions about $\mathfrak{S}$ are needed. In psychology this is often done by making scale type assumptions about numerical interpretations of subjects' responses without direct reference to the structure $\mathfrak{S}$. For example, a subject's ratings of intensities of items are often assumed to be a portion of a function from a ratio scale (or interval scale) without giving any indication of what is being assumed about the psychological system and how that is related to the numerical interpretations of responses that justifies this feat of measurement.

Intuitively, the condition of homogeneity is saying that from the point of view of the primitives of a structure, *individually* each element of the domain looks like each other element. This does not mean that for a pair of elements (or triple, etc.) that one element of the *pair* (triple, etc.) must look like the other element of the *pair,* (triple, etc.); for example, for the pair $\{\Psi(x), \Psi(y)\}$ of $\mathfrak{S}$, $\Psi(x)$ may be $\succ'$ $\Psi(y)$, but if this is the case, then certainly $\Psi(y)$ is not $\succ'$ $\Psi(x)$. In Chapter 4 of Narens (1996b), the following result is shown: If $\mathfrak{Y}$ is homogeneous, then for each predicate $P(x)$ that is defined in terms of the primitives of $\mathfrak{Y}$ and pure mathematics, if $P(a)$ holds for some element a in the domain of $\mathfrak{Y}$, then $P(b)$ holds for all elements b in the domain of $\mathfrak{Y}$. This result shows that clearly in terms of "predicates defined in terms of the primitives of $\mathfrak{Y}$ and pure mathematics" that each element of the domain looks like each other element. Narens (1996b) also shows that when the primitives of $\mathfrak{Y}$ are finite in number, this condition of all elements of the domain looking like each other for predicates defined in terms of the primitives of $\mathfrak{Y}$ and pure mathematics is logically equivalent to $\mathfrak{Y}$ being homogeneous.

Axioms 1 to 7 yield the following existence theorem:

Theorem 5. *Assume Axioms 1 to 7. Then there exists a multiplicative representation for* $\langle X, \succsim, B_1, B_2, \dots \rangle$.

6. EMPIRICAL CONSIDERATIONS

Assume Axioms 1 to 7. Suppose $\{B_1, B_2, \dots\} = \{R_1, R_2, \dots\} \cup \{D_1, D_2, \dots\}$, where $R_1, R_2, \dots$ are direct ratio judgments and $D_1, D_2, \dots$ are direct difference judgments. Then by Theorem 5, a multiplicative representation ρ for

$$\langle X, \succsim, R_1, R_2, \dots \rangle$$

exists that is also a multiplicative representation for

$$\langle X, \succsim, D_1, D_2, \dots \rangle .$$

This is consistent with the empirical findings of Torgerson (1961) discussed earlier.

Recall that Torgerson (1961) made the following observation about his findings:

> This result suggests that the subject perceives or appreciates but a single quantitative relation between a pair of stimuli. ... It appears that the subject simply interprets this single relation in whatever way the experimenter requires. (*p. 203*)

It appears to me that this observation is little more than a restatement of the empirical findings in cognitive terms, and therefore it should not be taken as an "explanation," because it lacks reason as to why "the subject simply interprets this single relation in whatever way the experimenter requires." In contrast Axioms 1 to 7 supply a reason: The subject uses a single homogeneous structure for forming his or her responses to instruction and stimulus inputs. The singleness of the structure is always achievable, e.g., if the subject employed $\langle \Psi(X), \succsim', U_1, U_2, \dots \rangle$ for direct ratio estimations and $\langle \Psi(X), \succsim', V_1, V_2, \dots \rangle$ for direct difference estimations, then he or she could employ the single structure

$$\langle \Psi(X), \succsim', U_1, U_2, \ldots, V_1, V_2, \ldots \rangle$$

for both. Thus it is the homogeneity of the (resulting) single structure that is the important consideration.

It is worthwhile to note that Torgerson's findings and his "observation" are also consistent with Axioms 1 to 6 and the assumption that the subject is employing a non-homogeneous structure for forming his or her responses to instruction and stimulus inputs. Because of these considerations, I take Axioms 1 to 7 to be substantively different from his "observation." Also, Axioms 1 to 7 are consistent with a wider range of direct estimation results than are obtainable by the kinds of analysis employed by Torgerson: Torgerson's method of representing direct estimation functions rely on representing them numerically in terms of the numbers and the kinds of estimation referred to in the instructions; e.g., the behavioral function that results from the instruction, "Estimate twice the stimulus presented," as the numerical function that is multiplication by 2. Theorem 5 does not require a strict relationship between numerical representations and the instructions that generated them; e.g., the above behavioral function that is multiplication by 2 may equally well be represented by the numerical function that is multiplication by 3.

Torgerson's empirically based conclusion that "The subject perceives or appreciates but a single quantitative relation between a pair of stimuli," is consistent with a number of empirical studies. In a review of the topic, Birnbaum (1982) writes,

> In summary, for a number of social and psychophysical continua, judgments of "ratios" and "differences" can be represented by the same comparison operation. If it is assumed that this operation is subtraction, the J_R function (for magnitude estimation of "ratios") can be approximated by the exponential, and the J_D function (for ratings of "differences") is approximately linear. ... In other words, judgments of "ratios" and "differences" are consistent with the proposition that the *same* operation underlies both tasks, but they do not permit specification of what that operation might be. (*p. 413*)

An important consideration in Axiom 6 is that the algorithms can be mathematically specified entirely in terms of primitives of $\mathfrak{S}$ and instructions. Thus if the mathematical specification of an algorithm depends in an essential way on individual elements of $\Psi(X)$, then, by the way Axiom 6 is formulated, these elements must be primitives of $\mathfrak{S}$. Axiom 7 keeps this from happening, for if an element a of $\Psi(X)$ is a primitive of $\mathfrak{S}$, then for each automorphism γ of $\mathfrak{S}$, $\gamma(a) = a$, and therefore $\mathfrak{S}$ cannot be homogeneous. Thus keeping the above relationship between Axioms 6 and 7 intact, it could happen that for paradigms that produce behavioral functions (of one input variable), Axioms 1 to 7 are valid, but for more complicated paradigms producing behavioral functions of several input variables, Axioms 1 to 5 and the appropriate modification of Axiom 6 to functions of several variables may be valid, but Axiom 7 fails because some of the algorithms in the modified version of Axiom 6 use inputed stimuli in essential ways in their mathematical specifications, i.e., the cognitive system uses "context" (the imputed stimuli) as well as the structure $\mathfrak{S}$ to produce algorithms for the more complicated behavioral functions of several variables.

Although Axioms 4 to 7 cannot be tested directly, the axiom system consisting of Axioms 1 to 7 is potentially falsifiable through tests of its consequence, the conclusion of Theorem 5. Because one would ordinarily be involved in situations where one believed Axioms 1 to 3 to be reasonable generalizations and idealizations, the empirical failure of the conclusion of Theorem 5 could be taken as a refutation of the conjunction of Axioms 4 to 7. The conclusion of Theorem 5 is testable by both quantitative and qualitative means: quantitatively by testing whether there is a multiplicative representation for $\mathfrak{X}$, and qualitatively by testing one of the following two qualitative consequences of it:

(1) For all primitive behavioral functions B_i and B_j of $\mathfrak{X}$ and all stimuli x in X,

$$B_i * B_j(x) = B_j * B_i(x),$$

where $*$ denotes function composition.

(2) For all primitive behavioral functions B_i and B_j of $\mathfrak{X}$ if $B_i(x) \succ B_j(x)$ for some x in X, then $B_i(y) \succ B_j(y)$ for all y in X.

7. THE POSSIBLE PSYCHOPHYSICAL POWER LAW

Luce (1959b) presented a theory that related hypotheses involving the scale types of the independent and dependent variables of a quantitative psychophysical function with its the mathematical form. The following is an application of one of his results:

Theorem 6. *Assume Axioms 1 to 6. Suppose the physical structure $\langle X, \succsim, \oplus \rangle$ has a ratio scale $\mathcal{U}$ of isomorphisms onto $\langle \mathbb{R}^+, \geq, + \rangle$ and the cognitive structure $\mathfrak{S}$ has a ratio scale of isomorphisms $\mathcal{V}$ onto a numerical structure with domain $\mathbb{R}^+$. Suppose for each φ in $\mathcal{U}$ there exists θ in $\mathcal{V}$ such that for all x in X,*

$$\Psi(\varphi(x)) = \theta(\Psi(x)). \tag{3}$$

Then there exists $r \in \mathbb{R}^+$ such that for all $\varphi \in \mathcal{U}$ and $\theta \in \mathcal{V}$ there exists $s \in \mathbb{R}^+$ such that for all x in X,

$$\theta(\Psi(x)) = s\varphi(x)^r. \tag{4}$$

The conclusion of Theorem 6 describes a power relation between an observable representation φ of the physical dimension of stimuli and a non-observable representation θ of a psychological dimension of sensations. The representation θ and the scale $\mathcal{U}$ are theoretical in nature; they are not assumed to be cognitive constructs of the subject.

In performing a direct estimation task, say estimating ratios of subjective intensities, one might theorize that the subject is using some cognitively constructed numerical function ξ from a ratio scale on $\Psi(X)$ as a basis for his or her responses, e.g., the subject selects stimulus $\Psi(y)$ as twice as intense as $\Psi(x)$ if and only if $\xi(\Psi(y)) = 2\xi(\Psi(x))$. Foundationally, there are grave difficulties with this account, because ξ, a cognitively constructed function assigns entities of pure mathematics – numbers – to sensations, which means that the mind mentally represents parts of pure mathematics as *pure mathematics,* a view that is metaphysically at odds with most current psychological theorizing. This kind of difficulty is avoided by saying that ξ is a function from $\Psi(X)$ onto an algebraic system of mental entities that is isomorphic to a fragment of the real number system. But even with this modification the problem still persists about the nature of the construction of the function ξ.

Axiom 6 provides an alternative to the use of cognitively constructed numerical-like functions like ξ: In Axiom 6, the instruction $\boldsymbol{I}$ causes the subject to relate a response sensation $\Psi(y)$ to the sensation $\Psi(x)$ of each stimulus x by an algorithmic process describable in terms of primitives of $\mathfrak{S}$. This process is viewed as a function $C_{\boldsymbol{I}}$ on $\Psi(X)$. An important contrast between the functions ξ of the previous paragraph and $C_{\boldsymbol{I}}$ is that $C_{\boldsymbol{I}}$ is the description in terms of sensations all possible results of a *single* instruction $\boldsymbol{I}$, whereas ξ is used to describe in terms of sensations all possible results of *all* instructions. Axiom 6 does not assume that the *set* $\mathcal{C}$ of all cognitive functions resulting from all instructions given to the subject is cognitively accessible or cognitively organized in a manner such that it can be employed to mimic the uses of the function ξ in the previous paragraph. Because of these considerations, the process described in Axiom 6 appear to me to be a fundamentally weaker cognitive process than one that uses a numerical-like measurement function like ξ in carrying out instructions.

Axiom 8. *The physical structure* $\mathfrak{P} = \langle X, \succsim, \oplus \rangle$ *is a continuous extensive structure (Definition 5), and for each automorphism* α *of* $\mathfrak{P}$ *there exists an automorphism* γ *of* $\mathfrak{S}$ *such that for each* x *in* X,

$$\Psi(\alpha(x)) = \gamma(\Psi(x)) .$$

Axiom 8 states a form of harmony between physics and psychology. It is similar to the principle of the "invariance of the substantive theory" of Luce (1959b) for a psychophysical functions with independent and dependent variables from ratio scales, except that it is qualitative and makes no reference to the scale type of the dependent variable (i.e., the scale type of $\mathfrak{S}$). Luce (1990b) revised his 1959 theory of possible psychophysical functions. The revised theory is formulated qualitatively in terms of automorphisms of structures that measure the independent and dependent variables of a function of a single variable. Axiom 8 is very close in spirit to principles employed by Luce (1990b), but is technically different in that the dependent variable may assume scale types not covered in Luce (1990b).

Assume Axiom 8. Then by Theorem 1, $\mathfrak{P} = \langle X, \succsim, \oplus \rangle$ has a ratio scale of isomorphisms onto $\langle \mathbb{R}^+, \geq, + \rangle$. From this it is an easy consequence that $\mathfrak{P}$ is homogeneous. The following lemma is an easy consequence of the homogeneity of $\mathfrak{P}$:

Lemma 1. *Assume Axioms 4 and 5. Then Axiom 8 implies Axiom 7.*

Theorem 7. *Assume Axioms 1 to 6 and Axiom 8. Then the Generalized Power Law holds, i.e., the following three statements are true:*

(a) (Qualitative Formulation) *For all behavioral primitives* B_i *of* $\mathfrak{X}$ *and all* x *and* y *in* X,

$$B_i(x \oplus y) = B_i(x) \oplus B_i(y)\,.$$

(b) (Existence: Quantitative Formulation) *By Axiom 8 let* φ *be an additive representation of* $\langle X, \succsim, \oplus\rangle$. *Then* φ *is a multiplicative representation for the behavioral structure*

$$\mathfrak{X} = \langle X, \succsim, B_1, B_2, \dots\rangle\,.$$

(c) (Uniqueness: Quantitative Formulation) *Suppose* γ *is a multiplicative representation for the behavioral structure*

$$\mathfrak{X} = \langle X, \succsim, B_1, B_2, \dots\rangle\,.$$

By Axiom 8 let φ *be an additive representation of* $\langle X, \succsim, \oplus\rangle$. *Then there exist a multiplicative representation* β *for* $\mathfrak{X}$ *that is equivalent to* γ *and there exists a positive real number* t *such that for all* x *in* X,

$$\beta(x) = \varphi(x)^t\,.$$

Furthermore, if $\mathfrak{S}$ *is ratio scalable, then for some positive real* r, $\beta = r\gamma$.

8. Empirical considerations

Because of the non-observable nature of Axioms 4, 5, 6, and 8, they cannot be tested directly. However, the axiom system consisting of Axioms 1 to 6 and Axiom 8 is potentially falsifiable through tests of various of its consequences. Because one ordinarily would be involved in situations where one believed Axioms 1 to 3 to be reasonable generalizations and idealizations, the falsification of a conclusion of Axioms 1 to 6 and Axiom 8 could be taken as a refutation of the conjunction of Axioms 4, 5, 6, and 8. Thus, in particular, by Lemma 1 the conjunction of Axioms 4, 5, 6 and 8 is potentially falsifiable through the tests discussed earlier of the axiom system consisting of Axioms 1 to 7. The following lemma provides a basis for additional ways of potentially falsifying the conjunction of Axioms 4, 5, 6, and 8:

Lemma 2. *Assume Axioms 1 to 6 and Axiom 8. Then each automorphism of the physical structure* $\langle X, \succsim, \oplus\rangle$ *is an automorphism of the behavioral-physical structure*

$$\langle X, \succsim, \oplus, B_1, B_2, \dots\rangle\,.$$

Let φ be an additive representation of $\langle \mathfrak{X}, \succsim, \oplus\rangle$. It is assumed that the experimenter has access to a highly accurate empirical rendering of φ. Let B_i be one of the behavioral primitives $B_1, B_2, \dots$, and let a and b be distinct elements of X such that it is observed that $b = B_i(a)$. Let

$$r = \frac{\varphi(a)}{\varphi(b)}\,.$$

Then it is a an easy consequence of Lemma 2 that for each x and y in X,

$$\text{If } \frac{\varphi(x)}{\varphi(y)} = r, \text{ then } y = B_i(x). \tag{5}$$

Equation 5 is testable. By the discussion preceding Lemma 2, tests of Equation 5 are also tests of the conjunction of Axioms 4, 5, 6, and 8.

9. Conclusions

The observable Axioms 1 to 3 about behavioral functions are very weak mathematically, and from the perspectives of behavioral psychophysics, they can be considered as minimal. The non-observable cognitive-physical Axioms 4 and 5 linking the observable ordered structure of stimuli $\langle X, \succsim \rangle$ to an unobservable structure of sensations $\langle \Psi(X), \succsim' \rangle$ are also very weak mathematically. On the surface, Axiom 4, which says a stimulus x from X presented to the subject produces within him or her a sensation $\Psi(x)$, appears to be obvious and have minimal psychological content. However, implicit in the axiom is that *each time* x is presented the *same* sensation $\Psi(x)$ is produced within this subject. This is clearly an assumption that has more than minimal psychological content. For example, for the paradigms discussed in this chapter, it is implicit in Axiom 4 that the sensation produced in the subject by stimulus x when the subject is presented x and instruction $\boldsymbol{I}$ is the same sensation produced when the subject is presented x and instruction $\boldsymbol{J}$. Thus as an idealization, Axiom 5 is more than "minimal" in psychological content.

The cognitive Axioms 6 and 7 provide considerable mathematical content. Axiom 6 is a general theory about the cognitive processing of instructions, and Axiom 7 is a theoretical hypothesis about the non-observable cognitive structure of sensations $\mathfrak{S}$ that the subject uses in his or her responses to instructions. Axiom 7 may be viewed as a cognitive version of a consequence of many prominent quantitative psychophysical models.

The ideas behind the axiomatization, Axioms 1 to 7, as well as the ideas behind the proof of its consequence, Theorem 5, are flexible, and are applicable to a wide range of psychophysical phenomena. Because other applications may involve different primitives, appropriate changes in the physical, behavioral, and behavioral-physical axioms may have to be made. Also the cognitive Axiom 6 may have to be changed to reflect the new behavioral primitives. This could be accomplished by an appropriate instantiation of the following principle inherent in Axiom 6: "The cognitive correlates of the behavioral primitives can be viewed as relations that are algorithmic in terms of the cognitive structure of primitives." However, the cognitive-behavioral Axioms 4 and 5 and the cognitive Axiom 7 could remain the same.

The power of the theorems presented in this chapter is largely due to the behavioral primitives being limited to functions of a single variable. Such functions, if required to remain invariant under rich sets of transformations, necessarily have highly restricted mathematical forms. As discussed in the following section, Axiom 6 requires a cognitive correlate C of a primitive behavioral function B to be invariant under the automorphisms of $\mathfrak{S}$, and thus, because by Axiom 7 $\mathfrak{S}$ has a rich set of automorphisms, it follows (from Axioms 4 and 5) that B must also be invariant under a rich set of transformations. In settings with primitives that are functions of more than two variables, this line of argument is greatly weakened. For the case of functions of two variables, it still yields interesting results (see, for example, the discussion of homogeneous concatenation structures in Luce & Narens, 1985).

10. Methods of Proof

Details of proofs are not presented in this chapter. There are three main theorems: Theorems 3, 5 and 7. Theorem 3 follows by applying the remarks just after Theorem 5.5 of Narens (1994) to several threshold functions. The proof of Theorem 7 relies on Theorem 5 and the method of proof of Theorem 3. Thus what is both mathematically and conceptually the most important theorem, and from the point of view of the proof the most novel, is Theorem 5. The following are the key ideas of its proof:

Let $\boldsymbol{I}$ be an instruction given to the subject. By Axiom 6, $\boldsymbol{I}$ produces a cognitive function $C_{\boldsymbol{I}}$ from $\Psi(X)$ onto itself that is algorithmic in terms of primitives of $\mathfrak{S}$. Assume $C_{\boldsymbol{I}}$ is different from the identity function on $\Psi(X)$. By Axiom 7, $\mathfrak{S}$ is homogeneous. Because $C_{\boldsymbol{I}}$ is algorithmic in terms of the primitives of $\mathfrak{S}$, it follows from results of Chapter 4 of Narens (1996b) that $C_{\boldsymbol{I}}$ is invariant under the set $\mathcal{A}$ of automorphisms of $\mathfrak{S}$; that is, for all α in $\mathcal{A}$ and all u and v in $\Psi(X)$,

$$v = C_{\boldsymbol{I}}(u) \text{ iff } \alpha(v) = C_{\boldsymbol{I}}(\alpha(u)) . \tag{6}$$

That Equation 6 holds for all elements of $\mathcal{A}$ is used to derive additional algebraic conditions on $\mathcal{A}$. (These are described in Chapter 7 of Narens, 1996b). In terms of the "homogeneity-uniqueness classification" of Narens (1981a, 1981b), $\mathfrak{S}$ is homogeneous and either is 1-point unique or satisfies a special variety of ∞-point uniqueness. The algebraic conditions on $\mathcal{A}$ are then used to produce a scale of isomorphisms $\mathcal{S}$ of $\mathfrak{S}$, and Equation 6 is used to derive the numerical form of the representation of $C_{\boldsymbol{I}}$ for each element of $\mathcal{S}$. Because the instruction $\boldsymbol{I}$ also produces a behavioral function B (related to $C_{\boldsymbol{I}}$ by $y = B(x)$ iff $\Psi(y) = C_{\boldsymbol{I}}(\Psi(x))$), φ and the numerical characterization of $C_{\boldsymbol{I}}$ by elements of $\mathcal{S}$ can be used to represent and characterize B numerically. Theorem 5 results by repeating the process for each instruction, using the same scale of isomorphisms $\mathcal{S}$.