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The Measurement Theory of Dense Threshold Structures

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A dense threshold structure consists of a nonempty set X, a total ordering $\geqslant_*$ on X, and a function T from X onto itself such that (i) $\langle X, \geqslant_* \rangle$ is dense (i.e., for each x and y in X, if $x >_* y$, then for some z, $x >_* z >_* y$) and (ii) $T(w) >_* w$ for all w in X. The intended interpretation is that elements y of X are discriminated as being more intense than elements x if and only if $y >_* T(x)$. A measurement–theoretic foundation for dense threshold structures is given that includes representation and uniqueness results, a measurement–theoretic analysis of Weber's law, and meaningfulness considerations about Weber constants.

1. INTRODUCTION

Thresholds for discriminating stimuli play a prominent role in psychological theory and experimentation. In the literature, the qualitative theory underpinning such discriminations have been based on a binary relation $\succ$, where $x \succ y$ is interpreted as "x is discriminably more intense than y." Luce (1956) gave a qualitative axiomatization for such a binary relation which he called a "semiorder" and showed that each semiorder $\succ$ with a finite domain can be represented by a real valued function φ and a positive real number c such that for all stimuli x and y in the domain,

$$x \succ y \qquad \text{iff} \quad \varphi(x) > \varphi(y) + c. \tag{1}$$

Note that in Eq. (1), the "threshold for y," $\varphi(y) + c$, may not correspond to any stimulus in the domain, i.e., there may be no stimulus z such that $\varphi(z) = \varphi(y) + c$. In most of the important applications in psychology of discrimination structures, the stimuli form a sufficiently richly ordered set in which the "thresholds" for stimuli y correspond to other stimuli; i.e., there is a function T on the stimuli such that for all stimuli x and y,

$$x \text{ is discriminably more intense than } y \text{ iff } x >_* T(y),$$

where $>_*$ is the ordering on the stimuli. When such a threshold function T exists, the mathematical theory of discrimination becomes simpler.

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In this paper, infinite ordered structures with such threshold functions are examined. Representation and uniqueness theorems for them are given, and their theory is extended qualitatively to incorporate a very general version of Weber's law. The qualitative theory of Weber's law is used to argue through meaningfulness considerations that in psychophysical situations the Weber constant does not have a purely psychological interpretation but $1 +$ the Weber constant does.

Preliminaries

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers, $\mathbb{I}$ the set of integers, and $\mathbb{I}^+$ the set of positive integers. $*$ denotes the (partial) operation of composing functions.

By definition, $\succcurlyeq_*$ is said to be a *total ordering* on X, if and only if it is a transitive and reflexive relation on a nonempty set X, and, for all x and y in X, $x \succ_* y$, $y \succ_* x$, or $x = y$.

By definition, a set X is said to be *denumerable* if and only if there exists a one-to-one function from it onto $\mathbb{I}$.

By definition, $\langle X, R_j \rangle_{j \in J}$ is said to be a *relational structure* if and only if X is a nonempty set and for each j in J, R_j is a relation on X.

Let $\mathfrak{X} = \langle X, R_j \rangle_{j \in J}$ be a relational structure. Then X is called the *domain* of $\mathfrak{X}$, and X and R_j, for $j \in J$, are called the *primitives* or *primitive relations* of $\mathfrak{X}$. $\mathfrak{X}$ is said to be *denumerable* if and only if X is denumerable.

Let $\mathfrak{X} = \langle X, R_j \rangle_{j \in J}$ be a relational structure. By convention, substructures of $\mathfrak{X}$ are often described by using the relations R_j instead of their restrictions; e.g., for $\langle \mathbb{R}, \geqslant, + \rangle$ the substructure based on $\mathbb{I}$ is written as $\langle \mathbb{I}, \geqslant, + \rangle$ instead of using new symbols, e.g., $\geqslant'$ and $+'$, to denote the restrictions of $\geqslant$ and $+$ to $\mathbb{I}$.

2. Dense Threshold Structures

DEFINITION 2.1. $\langle X, \succcurlyeq_*, T \rangle$ is said to be a *dense threshold structure* if and only if the following three statements are true:

1. $\langle X, \succcurlyeq_* \rangle$ is a totally ordered set.

2. $\langle X, \succcurlyeq_* \rangle$ is *dense*; i.e., for each x and z in X, if $x \succ_* z$, then for some y in X, $x \succ_* y \succ_* z$.

3. T is a strictly increasing function from X onto X such that for each x in X, $T(x) \succ_* x$.

Let $\mathfrak{X} = \langle X, \succcurlyeq_* T \rangle$ be a dense threshold structure. Then T is called the *threshold function* of $\mathfrak{X}$.

Dense threshold structures are special cases of a more general class of structures known as semiorders.

DEFINITION 2.2. $\succ$ is said to be a *semiorder* on X if and only if X is a nonempty set, $\succ$ is a nonempty binary binary relation on X, and the following three statements are true for all w, x, y, and z in X:

1. Not $x \succ x$.
2. If $w \succ x$ and $y \succ z$, then $w \succ z$ or $y \succ x$.
3. If $w \succ x$ and $x \succ y$, then $w \succ z$ or $z \succ y$.

DEFINITION 2.3. Let $\succ$ be a semiorder on X. Define $\geqslant_*$ on X as follows: For all x and y in X,

$$x \geqslant_* y \quad \text{iff} \quad \forall z[(\text{if } y \succ z \text{ then } x \succ z) \text{ and } (\text{if } z \succ x \text{ then } z \succ y)].$$

$\geqslant_*$ is called the *order induced by* $\succ$.

THEOREM 2.1. *Suppose $\langle X, \geqslant_*, T\rangle$ is a dense threshold structure. Define $\succ$ on X as follows: for a ll x and y in X,*

$$x \succ y \quad \textit{iff} \quad x >_* T(y).$$

Then $\langle X, \succ\rangle$ is a semiorder and $\geqslant_*$ is the order induced by $\succ$.

Proof. Left to reader.

Let $\langle X, \succ\rangle$ be a semiorder. It easily follows from Definition 2.3 that $\geqslant_*$ is a transitive and connected relation, i.e., is a "weak ordering" on X.

For a semiorder to be a dense threshold structure, additional assumptions need to be made:

DEFINITION 2.4. Let $\mathfrak{X} = \langle X, \succ\rangle$ be a semiorder and $\geqslant_*$ be its induced ordering. Then $\mathfrak{X}$ is said to be a *dense T-structure* if and only if the following three conditions are satisfied:

1. $\geqslant_*$ is a total ordering on X.
2. $\langle X, \geqslant_*\rangle$ is dense.
3. For each x in X there exist a and b in X such that

$$b >_* x >_* a$$

and for all y and all z in X,

$$y \succ x \quad \text{iff} \quad y >_* b,$$

and

$$z \succ a \quad \text{iff} \quad z >_* x.$$

Infinite semiorders have been studied in Manders (1981). In terms of representation theorems presented there, it is apparent that *Condition* 3, which implies that $\succ$ can be defined in terms of $\geqslant_*$ and elements of X, puts a severe restriction on the general theory of infinite semiorders. Nevertheless, Condition 3 is a reasonable and natural assumption in most scientific contexts where infinite semiorders are applicable.

THEOREM 2.2. *Let $\mathfrak{X} = \langle X, \succ \rangle$ be a semiorder structure that is a dense T-structure and $\geqslant_*$ be the order induced by $\succ$. Define T on X as follows: for each x and y in X, $T(x) = y$ if and only if for all z in X,*

$$z \succ x \quad \text{iff} \quad z >_* y.$$

Then $\langle X, \geqslant_*, T \rangle$ is a dense threshold structure.

Proof. The proof follows from the above definitions and is left to the reader.

DEFINITION 2.5. Let $\mathfrak{X} = \langle X, \geqslant_*, T \rangle$ be a dense threshold structure. Then $\mathfrak{X}$ is said to be *Archimedean* if and only if for each x and y in X there exists a positive integer m such that

$$T^m(y) \geqslant_* x.$$

DEFINITION 2.6. Let $\mathfrak{X} = \langle X, \geqslant_*, T \rangle$ be a dense threshold structure. Then $\mathfrak{X}$ is said to be *Dedekind complete* if and only if each nonempty $\geqslant_*$-bounded above subset of X has a least upper $\geqslant_*$-bound in X.

THEOREM 2.3. *Suppose $\mathfrak{X} = \langle X, \geqslant_*, T \rangle$ is a Dedekind complete dense threshold structure. Then $\mathfrak{X}$ is Archimedean.*

Proof. Suppose $\mathfrak{X}$ were not Archimedean. A contradiction is shown. It easily follows from Definition 2.5 and the fact that T^k is defined for all integers k that x and y in X can be found such that for each positive integer m, $x \geqslant_* T^m(y)$. Let

$$A = \{z \mid x \geqslant_* T^m(z) \text{ for all positive integers } m\}.$$

Then A is nonempty (since $y \in A$), bounded above by x, and such that for each z in A, $T(z)$ is in A. Since $\mathfrak{X}$ is Dedekind complete, let the element a in X be the least upper bound of A. $a \notin A$, for if $a \in A$, then $T(a) \in A$, but then a could not be the least upper bound of A since $T(a) >_* a$. Since

$$a >_* T^{-1}(a)$$

and a is a least upper bound of A and $a \notin A$, let b in A be such that

$$a >_* b >_* T^{-1}(a).$$

Then

$$T(b) >_* T[T^{-1}(a)] = a,$$

which is impossible, since $T(b)$ is in A and a is the least upper bound of A.

3. Representation Theorems

Lemma 3.1. *Suppose $\langle X, \geqslant_*, T\rangle$ is an Archimedean, dense threshold structure and $a \in X$. Then for each x in X, there exists a unique integer k such that*

$$a \geqslant_* T^k(x) >_* T^{-1}(a).$$

Proof. The proof is left to the reader.

Theorem 3.1. *Suppose $\mathfrak{X} = \langle X, \geqslant_*, T\rangle$ and $\mathfrak{Y} = \langle Y, \geqslant', U\rangle$ are Archimedean dense threshold structures and that X and Y are denumerable. Then $\mathfrak{X}$ and $\mathfrak{Y}$ are isomorphic.*

Proof. Let a_0 be an element of X and b_0 an element of Y. Let A be the half-open interval $(T^{-1}(a_0), a_0]$, and B the half-open interval $(U^{-1}(b_0), b_0]$. Then it immediately follows from the hypotheses that A and B are denumerable and that $\langle A, \geqslant_*\rangle$ and $\langle B, \geqslant'\rangle$ are dense, totally ordered sets with greatest elements and without least elements. Therefore, by a classical theorem of Cantor (1895) characterizing order types, let f be an isomorphism of $\langle A, \geqslant_*\rangle$ onto $\langle B, \geqslant'\rangle$. Define F on X as follows: For each x in X, let k be the unique integer such that $T^k(x) \in A$, and let

$$F(x) = U^{-k}[f(T^k(x))].$$

Suppose x and y are arbitrary elements of X. By Lemma 3.1, let m and n be the unique integers such that $T^m(x) \in A$ and $T^n(y) \in A$.

Case 1. $m = n$. Then

$$\begin{aligned} x \geqslant_* y \quad & \text{iff} \quad T^m(x) \geqslant_* T^m(y) \\ & \text{iff} \quad f(T^m(x)) \geqslant' f(T^m(y)) \\ & \text{iff} \quad U^{-m}[f(T^m(x))] \geqslant' U^{-m}[f(T^m(y))] \\ & \text{iff} \quad F(x) \geqslant' F(y). \end{aligned}$$

Case 2. $m \neq n$. Then

$$\begin{aligned} x \geqslant_* y \quad & \text{iff} \quad n > m \\ & \text{iff} \quad -m > -n \\ & \text{iff} \quad U^{-m}[f(T^m(x))] \geqslant' U^{-n}[f(T^n(y))] \\ & \text{iff} \quad F(x) \geqslant' F(y). \end{aligned}$$

Cases 1 and 2 show that F is order preserving, and since $\geqslant_*$ is a total ordering, it then follows that F is a function into Y. To show that F is onto Y, let z be an arbitrary element of Y. Let k be the unique integer such that $U^k(z)$ is in B. Then, since f is an isomorphism of A onto B, let w in A be such that $f(w) = U^k(z)$. Then

$$F[T^{-k}(w)] = U^{-k}[f(T^k(T^{-k}(w)))] = U^{-k}[f(w)] = U^{-k}[U^k(z)] = z.$$

The following equation completes the proof that F is an isomorphism onto $\mathfrak{Y}$: Suppose x is an arbitrary element of X and m is the unique integer such that $T^m(x) \in A$. Then $T^{m-1}[T(x)] \in A$, and thus

$$\begin{aligned} F[T(x)] &= U^{1-m}[f(T^{m-1}[T(x)])] \\ &= U^{1-m}[f(T^m(x))] \\ &= U[U^{-m}(f(T^m(x)))] \\ &= U[F(x)]. \end{aligned}$$

Not all denumerable dense threshold structures are isomorphic. Let $\langle X, \geqslant', +', 1' \rangle$ and $\langle X'', \geqslant'', +'', 1'' \rangle$ be distinct isomorphic copies of $\langle R, \geqslant, +, 1 \rangle$, where R is the set of rational numbers. For each x in X, let $T(x) = x +' 1'$, and let

$$\mathfrak{X} = \langle X, \geqslant', T \rangle.$$

Let $Y = X \cup X''$. Define $\geqslant_1$ on Y as follows: For each x and y in Y, $x \geqslant_1 y$ if and only if (1) x and y are in X and $x \geqslant' y$, (2) x and y are in X'' and $x \geqslant'' y$, or (3) x is in X'' and y is in X. For each x and y in Y, let $W(x) = y$ if and only if either x is in X and $y = x +' 1'$ or x is in X'' and $y = x +'' 1''$. Let

$$\mathfrak{Y} = \langle Y, \geqslant_1, W \rangle.$$

Then $\mathfrak{X}$ and $\mathfrak{Y}$ are nonisomorphic dense threshold structures, and $\mathfrak{X}$ is Archimedean and $\mathfrak{Y}$ is non-Archimedean.

DEFINITION 3.1. Let $\langle X, \geqslant_* \rangle$ be a totally ordered structure. Then $\langle X, \geqslant_* \rangle$ is said to satisfy *denumerable density* if and only if there is a denumerable subset Y of X such that for all x and z in X, if $x >_* z$, then there exists y in Y such that $x \geqslant_* y \geqslant_* z$.

Denumerable density is a necessary condition for $\langle X, \geqslant_* \rangle$ to be isomorphic to $\langle \mathbb{R}, \geqslant \rangle$. Examples exist of Dedekind complete dense threshold structures that are not denumerably dense.

DEFINITION 3.2. A dense threshold structure is said to be *continuous* if and only if it is Dedekind complete and denumerably dense.

THEOREM 3.2. *Suppose* $\mathfrak{X} = \langle X, \geqslant_*, T\rangle$ *and* $\mathfrak{Y} = \langle Y, \geqslant', U\rangle$ *are continuous threshold structures. Then there exists an isomorphism of* $\mathfrak{X}$ *onto* $\mathfrak{Y}$.

Proof. By Theorem 2.3, $\mathfrak{X}$ is Archimedean. We first proceed as in Theorem 3.1: Let a_0 be an element of X and b_0 an element of Y. Let A be the half-open interval $(T^{-1}(a_0), a_0]$ and B be the half-open interval $(U^{-1}(b_0), b_0]$. We then use a different classical theorem of Cantor (1895) characterizing order types to find an isomorphism f of $\langle A, \geqslant_*\rangle$ onto $\langle B, \geqslant'\rangle$, and then continue as in Theorem 3.1.

DEFINITION 3.3. Let $\mathfrak{X} = \langle X, \geqslant_*, T\rangle$ be a dense threshold structure. Then $\mathfrak{X}' = \langle X', \geqslant'_*, T'\rangle$ is said to be a *Dedekind completion* of $\mathfrak{X}$ if and only if

(i) $X' \supseteq X$, $\geqslant'_* \supseteq \geqslant_*$, and $T' \supseteq T$,

(ii) $\mathfrak{X}'$ is a dense threshold structure, and

(iii) $\langle X', \geqslant'_*\rangle$ is Dedekind complete.

THEOREM 3.3. *Suppose* $\mathfrak{X} = \langle X, \geqslant_*, T\rangle$ *is an Archimedean dense threshold structure that is denumerably dense. Then the following three statements are true*:

1. $\mathfrak{X}$ *has a Dedekind completion.*
2. *All Dedekind completions of* $\mathfrak{X}$ *are isomorphic.*
3. *Each Dedekind completion of* $\mathfrak{X}$ *is a continuous threshold structure.*

Proof. It is very easy to show (e.g., by use of "Dedekind cuts") that X can be extended to a set X' and $\geqslant_*$ can be extended to a total ordering $\geqslant'_*$ of X' such that

(i) $\langle X', \geqslant'_*\rangle$ is Dedekind complete,

(ii) $\langle X', \geqslant'_*\rangle$ has neither a greatest nor least element, and

(iii) for all u and v in X', if $u >'_* v$, then for some x in X,

$$u >'_* x >'_* v.$$

For each x in X', let

$$T'(x) = \sup\{T(y) \mid x \geqslant'_* y \text{ and } y \in X\}, \tag{2}$$

where sup is taken with respect to the ordering $\geqslant'_*$. Then for each x in X, $T'(x) = T(x)$, i.e., $T \subseteq T'$.

It is now shown that T' is a strictly increasing function: Suppose u and v are arbitrary elements of X' and $u >'_* v$. By (ii) above, let z and w be elements of X such that

$$u >'_* z >'_* w >'_* v. \tag{3}$$

Then by Eq. (2),

$$T'(u) \geqslant'_* T'(z) \geqslant'_* T'(w) \geqslant'_* T'(v). \tag{4}$$

However, since $\geqslant_* \subseteq \geqslant'_*$ and $T \subseteq T'$ and z and w are in X, it follows from Eq. (3) that

$$T'(z) = T(z) >_* T(w) = T'(w),$$

and since $\geqslant_* \subseteq \geqslant'_*$, it then follows from Eq. (4) that

$$T'(u) >'_* T'(v).$$

Since u and v are arbitrary elements of X' it has been shown that T' is strictly increasing.

It is now shown that T' is onto X': Let b be an arbitrary element of X'. Let

$$A = \{x \mid x \in X \text{ and } b \geqslant'_* T(x)\}. \tag{5}$$

By (ii) above, let c in X' be such that $c >'_* b$. By (iii), let e in X be such that $c >'_* e >'_* b$. Then $T^{-1}(e) \notin A$, since $e = T[T^{-1}(e)] >'_* b$. Thus $T^{-1}(e)$ is an upper bound of A. By (i), let a be the least upper bound of A. Then by Eqs. (2) and (5),

$$T'(a) = \sup\{T(x) \mid x \in A\}.$$

Also by Eqs. 2 and 5, $b \geqslant'_* T'(a)$. However, $b \not>'_* T'(a)$, for if $b >'_* T'(a)$, then by (iii) above there would exist d in X such that

$$b >'_* d >'_* T'(a) \tag{6}$$

and thus

$$b >'_* T[T^{-1}(d)],$$

which implies that $T^{-1}(d)$ is in A, and thus $a \geqslant'_* T^{-1}(d)$, and therefore

$$T'(a) \geqslant'_* T[T^{-1}(d)] = d,$$

which contradicts Eq. (6). Therefore, $b = T'(a)$.

The above shows that $\langle X', \geqslant'_*, T' \rangle$ is a Dedekind completion of $\langle X, \geqslant_*, T \rangle$. Thus Statement 1 has been shown. Statements 2 and 3 follow by noting that by Definitions 3.3 and 3.2 all Dedekind completions of $\mathfrak{X}$ are continuous threshold structures and that by Theorem 3.2 all continuous threshold structures are isomorphic.

DEFINITION 3.4. A *canonical, numerical,* dense threshold structure is a dense threshold structure of the form $\mathfrak{N} = \langle N, \geqslant, S \rangle$, where $N \subseteq \mathbb{R}$, $\geqslant$ is the usual ordering on the reals, and S is the function on $\mathbb{R}$ defined by

$$S(x) = x + 1.$$

THEOREM 3.4. *Suppose $\mathfrak{X} = \langle X, \geqslant_*, T \rangle$ is a dense threshold structure. Then the following three statements are true:*

1. *If $\mathfrak{X}$ is Archimedean and denumerable, then $\mathfrak{X}$ is isomorphic to the canonical, numerical, dense threshold structure $\langle Ra, \geqslant, S \rangle$, where Ra is the set of rational numbers.*

2. *If $\mathfrak{X}$ is Archimedean and denumerably dense, then $\mathfrak{X}$ is isomorphic to a canonical, numerical dense threshold structure $\langle N, \geqslant, S \rangle$, where $N \subseteq \mathbb{R}$.*

3. *If $\mathfrak{X}$ is continuous, then $\mathfrak{X}$ is isomorphic to the canonical, numerical dense threshold structure $\langle \mathbb{R}, \geqslant, S \rangle$.*

Proof. It is easy to verify that $\langle Ra, \geqslant, S \rangle$ is a denumerable threshold structure and $\langle \mathbb{R}, \geqslant, S \rangle$ is a continuous threshold structure. Statements 1 and 3 then follow from Theorems 3.1 and 3.2. To show Statement 2, suppose $\mathfrak{X}$ is denumerably dense. Let $\mathfrak{X}'$ be a Dedekind completion of $\mathfrak{X}$. By Theorems 3.3 and 3.2, let F be an isomorphism of $\mathfrak{X}'$ onto $\langle \mathbb{R}, \geqslant, S \rangle$. Then the restriction of F to X is an isomorphism of $\mathfrak{X}$ into $\langle \mathbb{R}, \geqslant, S \rangle$.

4. UNIQUENESS RESULTS AND AUTOMORPHISM GROUP

CONVENTION 4.1. Throughout the rest of this paper, let

$$\mathfrak{R} = \langle \mathbb{R}, \geqslant, S \rangle$$

be the continuous threshold structure, where

$$S(x) = x + 1$$

for all x in $\mathbb{R}$.

We begin by describing the automorphism group of $\mathfrak{R}$.

DEFINITION 4.1. By definition, let $\mathscr{A}$ be the set of strictly increasing functions from the half-open interval $(0, 1]$ of the reals onto itself. Also by definition, for each α in $\mathscr{A}$ and each r in $\mathbb{R}$, let α_r be the function on $\mathbb{R}$ such that for all x in $\mathbb{R}$ and all m in $\mathbb{I}$,

$$\text{if } x \in (m, m+1], \text{ then } \alpha_r(x) = m + r + \alpha(x - m).$$

By definition, let

$$\mathscr{G} = \{\alpha_r \mid \alpha \in \mathscr{A} \text{ and } r \in \mathbb{R}\}.$$

LEMMA 4.1. *Each element of $\mathscr{G}$ is an automorphism of $\mathfrak{N}$.*

Proof. Suppose x and y are arbitrary elements of $\mathbb{R}$ and m and n are the unique integers such that $x \in (m, m+1]$ and $y \in (n, n+1]$. Also suppose that α is an arbitrary element of $\mathscr{A}$ and r is an arbitrary element of $\mathbb{R}$.

Since

$$\begin{aligned}\alpha_r[S(x)] &= \alpha_r[x+1]\\ &= (m+1)+r+\alpha[(x+1)-(m+1)]\\ &= m+r+\alpha[x-m]+1\\ &= \alpha_r(x)+1\\ &= S[\alpha_r(x)],\end{aligned}$$

it follows that α_r preserves the function S.

It is now shown that α_r preserves $\geqslant$. Without loss of generality, suppose $m \geqslant n$. There are two cases to consider:

Case 1. $m = n$. Then

$$\begin{aligned} x \geqslant y \quad &\text{iff} \quad x-m \geqslant y-m\\ &\text{iff} \quad \alpha(x-m) \geqslant \alpha(y-m)\\ &\text{iff} \quad m+r+\alpha(x-m) \geqslant m+r+\alpha(y-m)\\ &\text{iff} \quad \alpha_r(x) \geqslant \alpha_r(y).\end{aligned}$$

Case 2. $m > n$. Then, since $\alpha(y-n) \in (0, 1]$ and $\alpha(x-m) \in (0, 1]$,

$$\begin{aligned} x \geqslant y \quad &\text{iff} \quad m+r \geqslant n+r+\alpha(y-n)\\ &\text{iff} \quad m+r+\alpha(x-m) \geqslant n+r+\alpha(y-n)\\ &\text{iff} \quad \alpha_r(x) \geqslant \alpha_r(y).\end{aligned}$$

LEMMA 4.2. *Each automorphism of $\mathfrak{N}$ is in $\mathscr{G}$.*

Proof. Suppose β is an automorphism of $\mathfrak{N}$. Let $r = \beta(0)$. Let ι be the identity function on $(0, 1]$. Then by Definition 4.1, $\iota_r \in \mathscr{G}$, and therefore by Lemma 4.1 is an automorphism of $\mathfrak{N}$. Let α be the restriction of $\iota_r^{-1} * \beta$ to $(0, 1]$. It easily follows from the definition of ι_r that α is onto $(0, 1]$. Then, since $\iota_r^{-1} * \beta$ is an automorphism of $\mathfrak{N}$, it follows that α is a strictly increasing function. Thus $\alpha \in \mathscr{A}$. It is shown that $\beta = \alpha_r$.

Let x be an arbitrary element of $\mathbb{R}$, and m be the unique integer such that $x \in (m, m+1]$. Then

$$\begin{aligned}
\alpha_r(x) &= m + r + \alpha(x-m) \\
&= m + r + \iota_r^{-1} * \beta(x-m) \\
&= m + r + \beta(x-m) - r \\
&= m + \beta(x-m) \\
&= m + \beta[S^{-m}(x)] \\
&= m + S^{-m}[\beta(x)] \\
&= m + \beta(x) - m \\
&= \beta(x).
\end{aligned}$$

THEOREM 4.1. *$\mathscr{G}$ is the set of automorphisms of $\mathfrak{R}$.*

Proof. Lemmas 4.1 and 4.2.

LEMMA 4.3. *Suppose $\mathfrak{X} = \langle X, \geqslant_*, T \rangle$ is an Archimedean dense threshold structure that is denumerably dense, a is an element of X, and φ and ψ are isomorphisms of $\mathfrak{X}$ into $\mathfrak{R}$ such that $\psi(a) = \varphi(a) = 0$. Then for some δ in $\mathscr{G}$, $\psi = \delta * \varphi$.*

Proof. For each x in $(a, T(a)]$, let

$$\alpha[\varphi(x)] = \psi(x).$$

Since φ and ψ are isomorphisms of $\mathfrak{X}$ into $\mathfrak{R}$, for each x and y in X,

$$\varphi(x) \geqslant \varphi(y) \qquad \text{iff } \psi(x) \geqslant \psi(y) \qquad \text{iff } \alpha * \varphi(x) \geqslant \alpha * \varphi(y),$$

and thus α is a strictly increasing function from $\varphi((a, T(a)])$ onto $\psi((a, T(a)])$. Since both the domain and range of α are order dense in $(0, 1]$, it easily follows that α can be extended to a strictly increasing function $\bar{\alpha}$ from $(0, 1]$ onto $(0, 1]$. Then $\bar{\alpha}$ is in $\mathscr{A}$ and $\bar{\alpha}_0$ is in $\mathscr{G}$. The theorem is shown by showing that $\psi = \bar{\alpha}_0 * \varphi$.

Let y be an arbitrary element of X. Since $\mathfrak{X}$ is Archimedean, let m be the unique positive integer such that

$$y \in (T^m(a), T^{m+1}(a)].$$

Therefore, since φ is an isomorphism of $\mathfrak{X}$ into $\mathfrak{R}$,

$$\begin{aligned}
(\varphi(T^m(a)), \varphi(T^{m+1}(a))] &= (S^m(\varphi(a)), S^{m+1}(\varphi(a))] \\
&= (S^m(0), S^{m+1}(0)] \\
&= (m, m+1],
\end{aligned}$$

and thus $\varphi(y) \in (m, m+1]$. Similarly, $\psi(y) \in (m, m+1]$. Therefore,

$$\begin{aligned}\bar{\alpha}_0[\varphi(y)] &= 0 + m + \alpha(\varphi(y) - m) \\ &= m + \alpha(S^{-m}[\varphi(y)]) \\ &= m + \alpha(\varphi[T^{-m}(y)]) \\ &= m + \psi(T^{-m}(y)) \\ &= m + S^{-m}(\psi(y)) \\ &= m + \psi(y) - m \\ &= \psi(y).\end{aligned}$$

LEMMA 4.4. *Suppose $\mathfrak{X} = \langle X, \geqslant_*, T\rangle$ is an Archimedean dense threshold structure that is denumerably dense and φ and ψ are isomorphisms of $\mathfrak{X}$ into $\mathfrak{R}$. Then for some δ in $\mathscr{G}$, $\psi = \delta * \varphi$.*

Proof. Let a be an element of X. Let ι be the identity function on $(0, 1]$. Then ι is in $\mathscr{A}$. Let

$$\beta = \iota_{\varphi(a)}^{-1} * \varphi \qquad \text{and} \qquad \gamma = \iota_{\psi(a)}^{-1} * \psi.$$

Then, since $\iota_{\varphi(a)}^{-1}$ and $\iota_{\psi(a)}^{-1}$ are automorphisms of $\mathfrak{R}$, it is easy to verify that β and γ are isomorphisms of $\mathfrak{X}$ into $\mathfrak{R}$ and $\beta(a) = \gamma(a) = 0$. Thus by Lemma 4.3, let η be an element of $\mathscr{G}$ such that

$$\gamma = \eta * \beta.$$

Then

$$\begin{aligned}\psi &= \iota_{\psi(a)} * \gamma \\ &= \iota_{\psi(a)} * \eta * \beta \\ &= \iota_{\psi(a)} * \eta * \iota_{\varphi(a)}^{-1} * \varphi \\ &= [\iota_{\psi(a)} * \eta * \iota_{\varphi(a)}^{-1}] * \varphi,\end{aligned}$$

and $\delta = \iota_{\psi(a)} * \eta * \iota_{\varphi(a)}^{-1}$ is in $\mathscr{G}$.

THEOREM 4.2. *Suppose $\mathfrak{X} = \langle X, \geqslant_*, T\rangle$ is an Archimedean dense threshold structure that is denumerably dense and φ and ψ are isomorphisms of $\mathfrak{X}$ into $\mathfrak{R}$. Then the following two statements are true:*

1. *There exists δ in $\mathscr{G}$ such that $\psi = \delta * \varphi$.*
2. *For each η in $\mathscr{G}$, $\eta * \varphi$ is an isomorphism of $\mathfrak{X}$ into $\mathfrak{R}$.*

Proof. Statement 1 follows from Lemma 4.4. Statement 2 easily follows from the definitions of "isomorphism" and "automorphism."

The following theorem is useful for establishing automorphism groups of Archimedean dense threshold structures that are denumerably dense:

THEOREM 4.3. *Suppose* $\mathfrak{X} = \langle X, \succcurlyeq_*, T \rangle$ *is an Archimedean dense threshold structure that is denumerably dense and* $\mathfrak{X}' = \langle X', \succcurlyeq', T' \rangle$ *is a Dedekind completion of* $\mathfrak{X}$. *Then for each automorphism* α *of* $\mathfrak{X}$, *there is an automorphism* α' *of* $\mathfrak{X}'$ *such that* $\alpha \subseteq \alpha'$.

Proof. Let α be an automorphism of $\mathfrak{X}$. Let α' be the function on X' such that for all x in X',

$$\alpha'(x) = \sup\{\alpha(z) \mid z \in X \text{ and } x \succcurlyeq' z\}.$$

Then the following three statements easily follow:

1. For all x in X, $\alpha'(x) = \alpha(x)$, i.e., $\alpha \subseteq \alpha'$.
2. For all x and y in X', if $x >' y$ then $\alpha'(x) \succcurlyeq' \alpha'(y)$.
3. α' is onto X'.

It is now shown that α' preserves the ordering $\succcurlyeq'$: Suppose x and y are arbitrary elements of X'. Without loss of generality, suppose $x >' y$. It need only be shown that $\alpha'(x) >' \alpha'(y)$. Since $\mathfrak{X}'$ is a Dedekind completion of $\mathfrak{X}$, let u and v be elements of X such that

$$x >' u >' v >' y.$$

Then $\alpha'(u) = \alpha(u) >' \alpha(v) = \alpha'(v)$, and thus by Statement 2 above,

$$\alpha'(x) \succcurlyeq' \alpha'(u) >' \alpha'(v) \succcurlyeq \alpha'(y),$$

i.e., $\alpha'(x) >' \alpha'(y)$.

It is next shown that α' preserves T': Suppose that x is an arbitrary element of X'. $\alpha'[(T'(x)] = T'[\alpha'(x)]$ is shown by contradiction. Suppose $\alpha'[(T'(x)] \neq T'[\alpha'(x)]$. There are two cases to consider:

Case 1. $T'[\alpha'(x)] >' \alpha'[T'(x)]$. Let w in X be such that

$$T'[\alpha'(x)] >' w >' \alpha'[T'(x)]. \tag{7}$$

Since T and α are onto X, let u in X be such that

$$T[\alpha(u)] = w. \tag{8}$$

Then by Eqs. (7) and (8),

$$T'[\alpha'(x)] >' T[\alpha(u)] = T'[\alpha'(u)],$$

which, since T' and α' are strictly increasing, implies

$$x >' u. \tag{9}$$

Since α is an automorphism of $\mathfrak{X}$,

$$T[\alpha(u)] = \alpha[T(u)],$$

which by Eqs. (7) and (8) yield

$$\alpha'[T'(u)] = \alpha[T(u)] >' \alpha'[T'(x)],$$

i.e.,

$$u >' x,$$

which contradicts Eq. (9).

Case 2. $\alpha'[T'(x)] >' \alpha'[T'(x)]$. Similar to Case 1.

It is worthwhile noting that in Theorem 4.3 α' is the unique automorphism of $\mathfrak{X}'$ that extends α in the sense that if β is another automorphism of $\mathfrak{X}'$ that extends α then $\alpha' = \beta$. (This easily follows because for each a and b in X' such that $a >' b$ there exists c in X such that $a >' c >' b$.)

DEFINITION 4.2. Let $\imath$ be the identity function on $\mathbb{R}$, By definition, for each r in $\mathbb{R}$, let $\imath_r$ be the function on $\mathbb{R}$ such that for x in $\mathbb{R}$ and all integers m,

$$\text{if } x \in (m, m+1] \text{ then } \imath_r(x) = m + r + \imath(x - m).$$

Then it follows from Definition 4.1 that $\imath_r$ is an automorphism of $\mathfrak{R}$ for each r in $\mathbb{R}$. It also easily follows that for each x in $\mathfrak{R}$ and each r in $\mathbb{R}$,

$$\imath_r(x) = r + x.$$

β is said to be a *pure translation* of $\mathfrak{R}$ if and only if $\beta = \imath_r$ for some r in $\mathbb{R}$.

Let $\mathfrak{X} = \langle X, \succcurlyeq_*, T \rangle$ be a continuous threshold structure. Then a function β from X onto X is said to be a *φ-pure translation* of $\mathfrak{X}$ if and only if the image of β under an isomorphism φ of $\mathfrak{X}$ onto $\mathfrak{R}$ is a pure translation of $\mathfrak{R}$; that is, if and only if the function γ from $\mathbb{R}$ onto $\mathbb{R}$ is such that, for each x in X,

$$\gamma[\varphi(x)] = \varphi[\beta(x)]$$

is a pure translation of $\mathfrak{R}$. It easily follows from φ being isomorphism of $\mathfrak{X}$ onto $\mathfrak{R}$ that each pure translation of $\mathfrak{R}$ is an image under φ of a φ-pure translation of $\mathfrak{X}$.

THEOREM 4.4. *Let $\mathfrak{X}$ be a continuous threshold structure and φ be an isomorphism of $\mathfrak{X}$ onto $\mathfrak{R}$. Then the following two statements are true*:

1. *The set of φ-pure translations of $\mathfrak{X}$ form a group under function composition.*

2. *For each x and y in the domain of $\mathfrak{X}$, there exists a φ-pure translation β of $\mathfrak{X}$ such that $\beta(x) = y$.*

Proof. Statement 1 then follows by noting that by Definition 4.2 the pure translations of $\mathfrak{R}$ form a group, and therefore by isomorphism the φ-pure translations of $\mathfrak{X}$ form a group.

Let x and y be arbitrary elements of the domain of $\mathfrak{X}$. Statement 2 then follows by noting that, by Definition 4.2,

$$\iota_{\varphi(y)-\varphi(x)}(\varphi(x)) = \varphi(y),$$

and that by Definition 4.2, the automorphism β of $\mathfrak{X}$ whose image under φ is $\iota_{\varphi(y)-\varphi(x)}$ is a φ-pure translation of $\mathfrak{X}$.

5. Weber Representations

Definition 5.1. $\langle X, \geqslant_* \rangle$ is said to be a *continuum* if and only if the following six statements are true:

1. $X \neq \varnothing$.
2. $\geqslant_*$ is a total ordering on X.
3. $\langle X, \geqslant_* \rangle$ has neither greatest nor least element.
4. $\langle X, \geqslant_* \rangle$ is dense.
5. $\langle X, \geqslant_* \rangle$ is denumerably dense.
6. $\langle X, \geqslant_* \rangle$ is Dedekind complete.

Definition 5.2. A structure $\mathfrak{X} = \langle X, \geqslant_*, R_j \rangle_{j \in J}$ is said to be *continuous* if and only if $\langle X, \geqslant_* \rangle$ is a continuum.

The focus of the remainder of this paper is on continuous structures. (The results below, however, with appropriate modifications, apply to more general situations in which the toral order need not be Dedeking complete.)

Definition 5.3. Let $\mathfrak{X} = \langle X, \geqslant_*, R_j \rangle_{j \in J}$ be a continuous structure.

$\mathfrak{X}$ is said to be *homogeneous* if and only if for each x and y in X there exists an automorphism α of $\mathfrak{X}$ such that $\alpha(x) = y$.

$\mathfrak{X}$ is said to be *1-point unique* if and only if for all automorphisms α and β of $\mathfrak{X}$, if $\alpha(x) = \beta(x)$ for some x in X, $\alpha = \beta$.

Let $\mathfrak{X}$ be a continuous threshold structure. Then it follows from Statement 2 of Theorem 4.4 that $\mathfrak{X}$ is homogeneous. By considering the isomorphic structure $\mathfrak{R}$ and its automorphism group as characterized in Theorem 4.1, it is easy to see that there exist distinct automorphisms β and γ that agree on an infinite subset of the domain of $\mathfrak{X}$ and thus, in particular, $\mathfrak{X}$ is not 1-point unique.

Narens (1981) shows the following theorem:

THEOREM 5.1. *Suppose* $\mathfrak{X} = \langle X, \succsim_*, R_j \rangle_{j \in J}$ *is a continuous structure that is homogeneous and* 1*-point unique. Then there exists a structure of the form* $\mathfrak{D} = \langle \mathbb{R}^+, \geqslant, D_j \rangle_{j \in J}$ *such that the set* $\mathcal{S}$ *of isomorphisms of* $\mathfrak{X}$ *onto* $\mathfrak{D}$ *forms a* ratio scale, *i.e., is such that*

(1) $\mathcal{S} \neq \varnothing$;

(2) *if* $\varphi \in \mathcal{S}$ *and* $\psi \in \mathcal{S}$, *then for some* r *in* $\mathbb{R}$, $\psi = r \cdot \varphi$; *and*

(3) *if* $\varphi \in \mathcal{S}$ *and* $s \in \mathbb{R}$, *then* $s \cdot \varphi \in \mathcal{S}$.

(In modern measurement theory, it has become customary to consider the set of homomorphisms of continuous structure $\mathfrak{X}$ into a numerially based structure $\mathfrak{D}$ as the proper way to measure $\mathfrak{X}$. Since $\mathfrak{X}$ is totally ordered, all homomorphisms of $\mathfrak{X}$ into $\mathfrak{D}$ are isomorphisms into $\mathfrak{D}$. I believe this view of measurement to be in error (see Narens, 1981, 1994) and that measurement should consist of isomorphisms of $\mathfrak{X}$ *onto* $\mathfrak{D}$. However, for the important case of homogeneous, 1-point unique, continuous structures $\mathfrak{X}$, numerical structures $\mathfrak{D}$ are generally selected so that all homomorphisms into $\mathfrak{D}$ are in fact isomorphisms onto $\mathfrak{D}$, and thus for these kinds of structures there is nothing to argue about.)

DEFINITION 5.4. Let $\mathfrak{X} = \langle X, \succsim_*, T \rangle$ be a dense threshold structure. Then φ is said to be a *Weber representation* of $\mathfrak{X}$ if and only if for all x and y in X,

(i) $x \succsim_* y$ iff $\varphi(x) \geqslant \varphi(y)$, and

(ii) $\varphi[T(x)] = k \cdot \varphi(x)$, where $k > 1$.

The real number k in (ii) is called the *modified Weber constant*, and (ii) is called the *modified Weber formula*. These are related to the (usual) *Weber constant* c and *Weber formula* by the following:

$$k = 1 + c \qquad \text{and} \qquad \frac{\varphi[T(x)] - \varphi(x)}{\varphi(x)} = c.$$

DEFINITION 5.5. For each $k > 1$, let

$$\mathfrak{N}_k = \langle \mathbb{R}^+, \geqslant, S_k \rangle,$$

where S_k is defined as follows: For all x in $\mathbb{R}^+$,

$$S_k(x) = k \cdot x.$$

THEOREM 5.2. *For each* $k > 1$, *let* φ_k *be the function from* $\mathbb{R}$ *onto* $\mathbb{R}^+$ *that is defined by*

$$\varphi_k(x) = k^x.$$

Then the following two statements are true for each $k > 1$:

1. φ_k *is an isomorphism of* $\mathfrak{N}$ *onto* $\mathfrak{N}_k$.

2. *Let β be a pure translation of $\mathfrak{N}$, i.e., let r in $\mathbb{R}$ be such that $\beta(x) = x + r$. Then the image γ of β under the isomorphism φ_k is a φ_k-pure translation of $\mathfrak{N}_k$ and, for all y in $\mathbb{R}^+$, $\gamma(y) = k^r \cdot y$, and thus by isomorphism the set of φ_k-pure translations of $\mathfrak{N}_k$ consists of multiplications by positive constants.*

Proof. Immediate from the definitions of "$\mathfrak{N}_k$" and "φ_k-pure translation."

THEOREM 5.3. *Suppose $\mathfrak{X}$ is an Archimedean dense threshold structure that is denumerably dense and $k > 1$. Then $\mathfrak{X}$ has a Weber representation with modified Weber constant k.*

Proof. By Theorem 3.4, $\mathfrak{X}$ is isomorphically embeddable into $\mathfrak{N}$. Then by Statement 1 of Theorem 5.2, $\mathfrak{X}$ is isomorphically embeddable in $\mathfrak{N}_k$, and since the threshold function S of $\mathfrak{N}$ is a pure translation of $\mathfrak{N}$, $\mathfrak{X}$ has modified Weber constant k by Statement 2 of Theorem 5.2.

Theorem 5.3 is a modest generalization of a result of Householder and Young (1940).

THEOREM 5.4. *Let $\mathfrak{Y} = \langle X, \geqslant_*, R_j \rangle_{j \in J}$ be a continuous, homogeneous, 1-point unique structure, and let $\mathfrak{X} = \langle X, \geqslant_*, T \rangle$ be a continuous threshold structure. By Theorem 5.1, let $\mathscr{S}$ be a ratio scale of isomorphisms of $\mathfrak{Y}$ onto $\mathfrak{D} = \langle \mathbb{R}^+, \geqslant, D_j \rangle_{j \in J}$, and let $\varphi \in \mathscr{S}$. Then φ is a Weber representation for $\mathfrak{X}$ if and only if T is an automorphism of $\mathfrak{Y}$.*

Proof. Suppose φ is a Weber representation for $\mathfrak{X}$. Since $\mathscr{S}$ is a ratio scale onto $\mathfrak{D}$, it easily follows that the set of automorphisms of $\mathfrak{D}$ is the set of multiplications by positive constants. Let k be the modified Weber constant, and S_k be the function on $\mathbb{R}^+$ that is multiplication by k. Then φ is an isomorphism of $\mathfrak{X}$ onto $\mathfrak{N}_k = \langle \mathbb{R}^+, \geqslant, S_k \rangle$. Therefore S_k, the image of T under φ, is an automorphism of $\mathfrak{D}$, and therefore, by isomorphism, T is an automorphism of $\mathfrak{Y}$.

Suppose T is an automorphism of $\mathfrak{Y}$. Since $\mathscr{S}$ is a ratio scale of isomorphisms onto $\mathfrak{D}$, it easily follows that the set of automorphisms of $\mathfrak{D}$ is the set of multiplications by positive constants. Thus, since φ is an isomorphism of $\mathfrak{Y}$ onto $\mathfrak{D}$, it follows that the image of T under φ is multiplication by a positive constant k, and therefore for each x in X

$$\varphi[T(x)] = k \cdot \varphi(x).$$

In order to more easily apprehend the significance of Theorem 5.4, a special case is considered:

DEFINITION 5.6. $\mathfrak{E} = \langle X, \geqslant_*, \oplus \rangle$ is said to be a *continuous extensive structure* if and only if $\mathfrak{E}$ is a continuous structure, $\oplus$ is an associative and commutative operation on X that is strictly increasing in each variable, and for all x and y in X, (1) $x \oplus y >_* x$ and (2) if $x >_* y$, then for some z in X, $x >_* y \oplus z >_* y$.

Continuous extensive structures are prominent in physics and are used as a theoretical justification for the measurement of many physical qualities. The following characterization of them is due to Helmholtz (1887):

THEOREM 5.5. *Let $\mathfrak{E}$ be a continuous extensive structure. Then the set of isomorphisms of $\mathfrak{E}$ onto $\langle \mathbb{R}^+, \geqslant, + \rangle$ is a ratio scale.*

Let $\mathfrak{E} = \langle X, \geqslant_*, \oplus \rangle$ be a continuous extensive structure, $\mathscr{S}$ be a ratio scale of isomorphisms of $\mathfrak{E}$ onto $\langle \mathbb{R}^+, \geqslant, + \rangle$, and $\mathfrak{X} = \langle X, \geqslant_*, T \rangle$ be a continuous threshold structure. It easily follows from Theorem 5.5 that $\mathfrak{E}$ is homogeneous and 1-point unique. For concreteness, think of $\mathfrak{E}$ as some physical physical structure and T as some empirically determined psychological threshold function. Let $\varphi \in \mathscr{S}$. Then, by Theorem 5.4, the following two statements are logically equivalent:

1. For all x and y in X, $T(x \oplus y) = T(x) \oplus T(y)$.
2. There exists $k > 1$ such that for all x in X, $\varphi[T(x)] = k \cdot \varphi(x)$.

Note that Statement 1 above corresponds to possible experiments. Therefore, Statement 2 can be determined experimentally by the direct testing of Statement 1; that is, Statement 2 can be determined experimentally "without fitting curves to $\varphi[T(x)]$."

Theorem 5.3 implies that each continuous dense threshold structure has a modified Weber representation. Householder and Young (1940) mistakenly confused this conclusion with *Weber's Law*, a much cherished principle of psychophysics:

Let $\mathfrak{X} = \langle X, \geqslant_*, T \rangle$ be a continuous threshold structure. To obtain a Weber representation of $\mathfrak{X}$ one finds an isomorphism of $\mathfrak{X}$ onto $\mathfrak{N}_k$ for some positive real k. There is obviously nothing "lawful" about this. To obtain a *Weber's Law* representation of $\mathfrak{X}$, one *first* obtains a function φ of a ratio scale of isomorphisms of another structure $\mathfrak{Y}$ which does not have T as a primitive relation, and *then* verifies that φ is a Weber representation for $\mathfrak{X}$. What is "lawful" about the latter is that φ is simultaneously an isomorphism of a ratio scale for $\mathfrak{Y}$ and a Weber representation for $\mathfrak{X}$. Qualitatively, this law reduces to saying that T is an automorphism of $\mathfrak{Y}$—an experimentally testable condition if T and the primitive relations of $\mathfrak{Y}$ are empirically determinable.

A case between a Weber representation and a "Weber's Law" may arise when T appears among the primitives of $\mathfrak{Y}$. This is because the structure that results by removing T may not be ratio scalable. The following is an interesting case of this.

THEOREM 5.6. *Let $\mathfrak{X} = \langle X, \geqslant_*, T, U \rangle$, where $\langle X, \geqslant_*, T \rangle$ and $\langle X, \geqslant_*, U \rangle$ are continuous threshold structures, and suppose $\mathfrak{X}$ satisfies the following two conditions:*

(i) *For all x in X, $T[U(x)] = U[T(x)]$.*

(ii) *For all x, y, and z in $\mathfrak{X}$, if $x >_* y$ then there exist integers m and n such that*

$$x >_* T^m * U^n(z) >_* y.$$

Then the following four statements are true:

1. $\mathfrak{X}$ *is a continuous structure that is homogeneous and* 1*-point unique.*

2. *There exists a numerically based structure* $\mathfrak{C}$ *such that the set of isomorphisms of* $\mathfrak{X}$ *onto* $\mathfrak{C}$ *is a ratio scale.*

3. *Let* $\mathscr{S}$ *be a ratio scale of isomorphisms of* $\mathfrak{X}$ *onto* $\mathfrak{C}$*. Then each homomorphism of* $\mathfrak{X}$ *into* $\mathfrak{C}$ *is in* $\mathscr{S}$.

4. *Let* $\mathscr{S}$ *be a ratio scale of isomorphisms of* $\mathfrak{X}$ *onto* $\mathfrak{C}$*. Then each* φ *in* $\mathscr{S}$ *is a Weber representation for* $\langle X, \geqslant_*, T\rangle$ *and is a Weber representation for* $\langle X, \geqslant_*, U\rangle$.

Proof. Left to reader.

It is not difficult to establish the existence of a structure $\mathfrak{X}$ that satisfies the hypotheses of Theorem 5.6.

6. Meaningfulness

Definition 6.1. Let $\mathfrak{X} = \langle X, \geqslant_*, T\rangle$ be a continuous threshold structure. An automorphism β of $\mathfrak{X}$ is said to be $\mathfrak{X}$*-automorphism invariant* if and only if for all x in X and all automorphisms γ of $\mathfrak{X}$,

$$\gamma[\beta(x)] = \beta[\gamma(x)],$$

i.e., if and only if

$$\gamma * \beta = \beta * \gamma,$$

i.e., if and only if

$$\gamma * \beta * \gamma^{-1} = \beta.$$

Theorem 6.1. *Suppose* $0 < r < 1$ *and* β *is the pure translation of* $\mathfrak{R}$ *defined by*

$$\beta(x) = x + r.$$

Then β *is not* $\mathfrak{R}$*-automorphism invariant.*

Proof. Suppose β were $\mathfrak{R}$-automorphism invariant. A contradiction is shown.

By Definition 4.1 and Theorem 4.1 let γ be the following automorphism of $\mathfrak{R}$: For each x in X and each integer m,

$$\text{if} \quad x \in (m, m+1], \qquad \text{then } \gamma(x) = m + 0 + (x - m)^2.$$

Since β is $\mathfrak{R}$-automorphism invariant, for each x in $(0, 1]$,

$$\gamma[\beta(x)] - \beta[\gamma(x)] = 0,$$

i.e.,

$$(x+r)^2-(x^2+r)=0,$$

i.e.,

$$2xr+r^2-r=0 \tag{10}$$

But Eq. (10) is impossible for all x in $(0, 1]$ since it has at most one solution.

Let $\mathfrak{X}=\langle X, \succsim_*, T\rangle$ be a continuous threshold structure and $\mathcal{S}$ a scale of isomorphisms of $\mathfrak{X}$ into $\mathfrak{N}_k$. Then (by Theorem 5.3 and the proof of Theorem 5.2) for all φ and ψ in $\mathcal{S}$ and all x in X,

$$\varphi[T(x)]=k\cdot\varphi(x) \qquad \text{and} \qquad \psi[T(x)]=k\cdot\psi(x), \tag{11}$$

i.e., all isomorphisms in $\mathcal{S}$ yield the same modified Weber constant k. Some measurement theorists might want to use this result to say that "k is meaningful." I think this would be an error: This *by itself* is not enough to conclude that "k is meaningful"; it is only enough to conclude that the sentence

$$\varphi[T(x)]=k\cdot\varphi(x)$$

is a meaningful assertion. To properly conclude "k is meaningful," additional observations like the following are needed:

> Multiplication by the constant k is an automorphism of $\mathfrak{N}_k$, and it is $\mathfrak{N}_k$-automorphism invariant since it is the threshold function S_k. Through the isomorphisms of $\mathcal{S}$, it has an interpretation in $\mathfrak{X}$ as the threshold function T.

Observations similar to the above do not hold for the Weber constant c.

Consider the Weber formula

$$\varphi[T(x)]-\varphi(x)=c\cdot\varphi(x), \tag{12}$$

where $\varphi\in\mathcal{S}$ and $c=k-1$. By Theorem 6.1 and the isomorphism of $\mathfrak{N}$ and $\mathfrak{N}_k$, multiplication by c is not $\mathfrak{N}_k$-automorphism invariant. By using results of Narens (1988), this means that the automorphism that corresponds to multiplication by c via φ^{-1} is not definable in terms of the primitives X, $\succsim_*$, and T, no matter how powerful a logical language is used.

Note that the statement in Eq. (12) is meaningful in the sense that if ψ is any element of $\mathcal{S}$, then

$$\psi[T(x)]-\psi(x)=c\cdot\psi(x).$$

The fact that this statement is "meaningful" does not mean that every part of it—e.g., the constant c—has a proper interpretation in $\mathfrak{X}$.

Let $\mathfrak{E}=\langle X, \succsim_*, \oplus\rangle$ be a continuous extensive structure and $\mathfrak{X}=\langle X, \succsim_*, T\rangle$ a continuous threshold structure. For this discussion, X is considered to be a set of physical objects as well as a set of psychological stimuli, $\succcurlyeq_*$ is considered a physical relation on physical objects as well as a psychological relation on psychological stimuli, $\oplus$ is considered a physical operation on physical objects, and T is considered a psychological function on psychological stimuli. Thus $\mathfrak{E}$ characterizes a physical situation and $\mathfrak{X}$ characterizes a psychological situation. Let $\mathscr{S}$ be a ratio scale of isomorphisms of $\mathfrak{E}$ onto $\langle \mathbb{R}^+, \geqslant, + \rangle$. As discussed above, the modified Weber constant that results from measurement by $\mathscr{S}$ has an interpretation in the psychological structure $\mathfrak{X}$, whereas the Weber constant has no such interpretation. By using results of Narens (1988), both constants have interpretations in the physical structure $\mathfrak{E}$. (Here "interpretability" means definable from the primitives of $\mathfrak{E}$ in terms of some sufficiently powerful logical language.)

Meaningfulness and interpretability in psychophysical situations, like the above, in which the psychophysical situation can be described by two structures—one having only psychological primitives and the other only physical primitives—are discussed in some detail in Narens and Mausfeld (1990) and Narens (1994).

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