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# Cores of dense exchange economies

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Dense exchange economies cardinal characteristic is that each trader has infinitely many traders close to him or her in terms of initial allocations of goods and preference orderings. For such economies, a qualitative concept of "core" is formulated, and it is shown that a natural, qualitatively defined subset of the core coincides with the quantitative concept of "competitive equilibria." The method of proof avoids the use of the Axiom of Choice. Dense exchange economies and other exchange economies are also discussed with respect to the appropriateness of the kinds of infinitistic assumptions used in their formulations and in the obtaining of characterizing theorems.

*Key words:* Core; exchange economies; competitive equilibrium.

## 1. Preface

This is a slightly modified version of a paper that appeared in 1975 in the (UC Irvine) *School of Social Sciences Working Paper Series.* At that time I was intrigued by the various concepts of 'core' presented by economic theorists. Upon deeper inspection, I found them all to be unrealistic idealizations of economic processes, and I decided to try to reformulate the basic ideas underlying them in a manner that would provide a much more realistic idealization. The result was the concept of 'core of a dense exchange economy' and a theorem relating it to the economic concept of 'competitive equilibrium'. At the time, I considered this to be a technical and conceptual improvement on the extant results. Recently Lewis (1990a,b) has shown that many of the traditional concepts of 'core' require the Axiom of Choice of set theory for them to characterize non-empty sets of competitive equilibria. Since this raises important foundational issues concerning the core concept, I decided to publish my 1975 result in the hope that it might help to clarify or perhaps even resolve some of the foundational issues raised by Lewis's results.

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I have made no attempt to include here any of the vast literature about the 'core' concept that have appeared after 1975 except for brief mention of some results of Lewis (1990a,b).

## 2. Introduction

At the beginning of his influential paper on exchange economies (Aumann, 1964), R. Aumann gives the following rationale for considering infinite economies:

> The notion of *perfect competition* is fundamental in the treatment of economic equilibrium. The essential idea of this notion is that the economy under consideration has a 'very large' number of participants, and that the influence of each individual participant is 'negligible'. Of course, in real life no competition is perfect; but, in economics, as in the physical sciences, the study of the ideal state has proved very fruitful, though in practice it is, at best, only approximately achieved.
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> Though writers on economic equilibrium have traditionally assumed perfect competition, they have, paradoxically, adopted a mathematical model that does not fit this assumption. Indeed, the influence of an individual participant on the economy cannot be mathematically negligible, as long as there are only finitely many participants. Thus a *mathematical model appropriate to the intuitive notion of perfect competition must contain infinitely many participants.*

Many formulations of 'perfect competition' have appeared in the literature, and as Aumann suggests, all found need to use infinitary concepts. The introduction of infinitary concepts for the idealized description of an inherently finitary concept raises philosophical concerns. One view about such matters – and the one adopted in this paper – holds that the metaphysical concepts involved with the introduction of infinity should be held to a minimum, particularly in the formulation of idealized concepts and results that are to be used in providing a substantive understanding of phenomena giving rise to the (infinite) idealization. In this paper, three well-known formulations of 'perfect competition' will be examined with respect to the metaphysics inherent in them, and it will be argued that concepts used in their formulations are highly flawed as idealizations of exchange economies with a finite number of participants. Then a new concept of 'perfect competition' will be presented that captures the essential intuition of the previous formulations but which is free of artificial restrictions and excess metaphysical baggage.

Throughout this paper, $Re$ will denote the reals, $Re^+$ the positive reals, $I$ the integers, and $I^+$ the positive integers. $E_n$ will denote Euclidean $n$-space, and $\varphi$ will denote the Euclidean metric for $E_n$. Points in $E_n$ will be denoted by $\bar{a}, \bar{b}, \bar{x}, \bar{y}$, etc. If $r \in Re$, then $\bar{r}$ will denote the point $(r, \ldots, r)$. If $\bar{u}$ is a point of $E_n$ and $i \in I^+$ such that $i \leq n$, then $u_i$ will denote the $i$th coordinate of $\bar{u}$. By definition

$\bar{u} \geq \bar{v}$ iff for all $i \leq n$, $u_i \geq v_i$,

$\bar{u} > \bar{v}$ iff $\bar{u} \geq \bar{v}$ and for some $i$, $u_i > v_i$,

$$\bar{u} \gg \bar{v} \quad \text{iff for all } i \leq n,\ u_i > v_i.$$

$\Omega_n$ will denote the positive orthant of $E_n$, i.e. $\Omega_n = \{\bar{u} \mid \bar{u} > \bar{0}\}$. In a natural way, $\Omega_n$ can be thought of as the set of commodity bundles of some fixed set of $n$ commodities. $B(\bar{u}, r)$ will denote the open ball of $E_n$ with center $\bar{u}$ and radius $r$.

Let $\succ'$ be a binary relation on $\Omega_n$. $\succ'$ is said to be *irreflexive* if and only if for each $\bar{x}$ in $\Omega_n$ it is not the case that $\bar{x} \succ' \bar{x}$. $\succ'$ is said to be *continuous* if and only if for each $\bar{x}, \bar{y}$ in $\Omega_n$, $\{\bar{z} \in \Omega_n \mid \bar{z} \succ' \bar{x}\}$ and $\{\bar{u} \in \Omega_n \mid \bar{y} \succ' \bar{u}\}$ are open subsets of $E_n$. $\succ'$ is said to be *monotonic* if and only if for each $\bar{x}, \bar{y}$ in $\Omega_n$, if $\bar{x} > \bar{y}$ then $\bar{x} \succ' \bar{y}$. $\succ'$ is said to be *convex* if and only if for each $\bar{x}, \bar{y}$ in $\Omega_n$ and each $0 < r < 1$, if $\bar{x} \succ' \bar{y}$ then $\bar{x} \succ' r\bar{x} + (1-r)\bar{y} \succ' \bar{y}$.

Let $\succsim$ be a binary relation on $\Omega_n$. $\succsim$ is said to be a *weak ordering* if and only if $\succsim$ is a transitive and connected relation. For each $\bar{x}, \bar{y}$ in $\Omega_n$, let $\bar{x} \sim \bar{y}$ if and only if $\bar{x} \succsim \bar{y}$ and $\bar{y} \succsim \bar{x}$. Then it is easy to show that $\sim$ is an equivalence relation on $\Omega_n$. Let $\bar{x} \succ \bar{y}$ hold if and only if $\bar{x} \succsim \bar{y}$ and not $\bar{x} \sim \bar{y}$. Then it is easy to show that $\succ$ is an irreflexive relation on $\Omega_n$. Naturally $x \precsim y$ will mean $y \succsim x$, and $x \prec y$ will mean $y \succ x$.

An exchange economy consists of a non-empty set $T$ of traders, a preference ordering $\succ_t$ over commodity bundles for each trader $t$ in $T$, and an initial allocation of commodities $\bar{a}_t$ for each trader $t$ in $T$. Formally, $\mathscr{E} = \langle \bar{a}_t, \succ_t \rangle_{t \in T}$ is said to be an *exchange economy* if and only if $T \neq \emptyset$, $\bar{a}_t \in \Omega_n$ for each $t \in T$, and $\succ_t$ is a binary relation for each $t \in T$. The key concept of an exchange economy is the redistribution of goods: $\{\bar{a}_t\}_{t \in T}$ is the initial distribution of goods; presumably trading takes place and the result is a new distribution of goods $\{\bar{x}_t\}_{t \in T}$, where each trader is better off with regards to his preference ordering. Redistribution of goods will be called *allocations.* $\{\bar{a}_t\}_{t \in T}$ is aptly called the *initial allocation* of $\mathscr{E}$. For finite economies, i.e. economies with a finite set of traders $T$, the characterization of allocation is simple: $\{\bar{x}_t\}_{t \in T}$ is an allocation of $\mathscr{E}$ if and only if (1) for each $t \in T$, $\bar{x}_t \in \Omega_n$ and (2) $\sum_{t \in T} \bar{x}_t = \sum_{t \in T} \bar{a}_t$. However, for infinite economies matters are much more complicated. Once allocations are defined, the concepts of *core* and *competitive* allocations can be defined. Intuitively, an allocation $\{\bar{x}_t\}_{t \in T}$ is in the core of $\mathscr{E}$ if and only if no coalition of traders $S$ can be found so that the members of $S$ can redistribute their initial allocation of goods $\{\bar{a}_t\}_{t \in S}$ among themselves so that they end up with a redistribution of goods $\{\bar{y}_t\}_{t \in S}$ such that each trader $t$ in $S$ prefers $\bar{y}_t$ to $\bar{x}_t$. Intuitively, $\{\bar{x}_t\}_{t \in T}$ is a competitive allocation if and only if a price structure $\bar{p} > \bar{0}$ can be found so that for each trader $t$ in $T$, $\bar{p} \cdot \bar{x}_t \leq \bar{p} \cdot \bar{a}_t$ and no element of $B_t = \{\bar{y} \in \Omega_n \mid \bar{p} \cdot \bar{y} \leq \bar{p} \cdot \bar{a}_t\}$ is preferred to $\bar{x}_t$. (Note that $B_t$ is the set of commodity bundles that $t$ can purchase at price $\bar{p} \cdot \bar{a}_t$.) Various authors have precisely formulated economies in which the core and competitive allocations coincide. We will now consider some of these formulations.

## 3. Some previous formulations

Scarf (1962) and later Debreu (1963) considered denumerable economies which are infinite replications of a finite economy. To be specific, let $\mathscr{E} = \langle \bar{a}_t, \succ_t \rangle_{t \in T}$ be an exchange economy where $T$ is a finite set, and for each $t \in T$, $\bar{a}_t \gg \bar{0}$, $\succsim_t$ is a weak ordering, and $\succ_t$ is monotonic, continuous, and convex. Let $T^\infty = \{t_j \mid t \in T$ and $j \in I^+\}$. For each $t_j$ in $T^\infty$, let $\bar{a}_{t_j} = \bar{a}_t$ and $\succ_{t_j} = \succ_t$. Then $\mathscr{E}^\infty = \langle \bar{a}_t, \succ_t \rangle_{t \in T^\infty}$ is a denumerable economy that is an infinite replication of $\mathscr{E}$. $\{\bar{x}_{t_j}\}_{t \in T, j \in I^+}$, is said to be an *allocation* of $\mathscr{E}^\infty$ if and only if

$$\lim_{n' \to \infty} \left( \sum_{j=1}^{n'} \sum_{t \in T} \bar{x}_{t_j} - n' \sum_{t \in T} \bar{a}_t \right) = \bar{0}.$$

Coalitions of traders of $\mathscr{E}^\infty$ are finite subsets of $T^\infty$, and an allocation $\{\bar{x}_t\}_{t \in T^\infty}$ of $\mathscr{E}^\infty$ is *blocked by an allocation* $\{\bar{y}_t\}_{t \in T^\infty}$ if and only if for some coalition of traders $S$ of $T^\infty$, $\sum_{t \in S} \bar{y}_t = \sum_{t \in S} \bar{a}_t$ and for each $t \in S$, $\bar{y}_t \succ_t \bar{x}_t$. The *core* of $\mathscr{E}^\infty$ consists of all of those allocations of $\mathscr{E}^\infty$ that are not blocked by any other allocation of $\mathscr{E}^\infty$. An allocation $\{\bar{x}_t\}_{t \in T^\infty}$ of $\mathscr{E}^\infty$ is said to be *competitive* if and only if there exists a price vector $\bar{p} > \bar{0}$ such that for all $t \in T^\infty$, $\bar{p} \cdot \bar{x}_t \leq \bar{p} \cdot \bar{a}_t$ and is not the case that there exists $\bar{z} \in \Omega_n$ such that $\bar{p} \cdot \bar{z} \leq \bar{p} \cdot \bar{a}_t$ and $\bar{z} \succ_t \bar{x}_t$. Scarf (1962) and then Debreu (1963) proved the following theorem:

**Theorem A.** *The core of $\mathscr{E}^\infty$ coincides with the competitive allocations of $\mathscr{E}^\infty$.*

I agree with the following observation of Aumann (1964) about this characterization of 'perfect competition':

> The notion of finitely many types might not at first sight seem objectionable. But it involves the further assumption that there are 'many' traders of each type; in fact the number of traders of each type must be very large compared to the number of types in order for their model to be applicable. This seems far from economic reality, where, in general, different traders cannot be expected to have the same initial bundles or the same preferences.

Another criticism of Scarf's characterization is that the concept of *allocation* is awkward and perhaps too restrictive.

Aumann (1964) considers the following economy with a continuum of traders. Let $T$ be the closed interval [0, 1], $a$ be a Lebesgue integrable function from $T$ into $\Omega_n$ such that $\int_T a > \bar{0}$, and for each $t \in T$, let $\succ_t$ be a continuous, monotonic, irreflexive relation on $\Omega_n$. Let $\mathscr{E} = \langle a(t), \succ_t \rangle_{t \in T}$. An *allocation* of $\mathscr{E}$ is a Lebesgue integrable function $x$ from $T$ into $\Omega_n$ such that $\int_T x = \int_T a$. *Coalitions of traders* of $\mathscr{E}$ are Lebesgue measurable subsets of $T$. A coalition of traders of $\mathscr{E}$ is said to be *non-null* if and only if its Lebesgue measure is non-zero. An allocation $x$ of $\mathscr{E}$ said to be *blocked by an allocation* $y$ if and only if for some non-null coalition of traders $S$, $\int_S y = \int_S a$ and for each $t \in T$, $y(t) \succ_t x(t)$. The *core* of $\mathscr{E}$ is the set of all those allocations of $\mathscr{E}$ that are not blocked by any allocation of $\mathscr{E}$. An allocation $x$ of $\mathscr{E}$ is said to be *competitive* if and only if there is a price vector $\bar{p} > \bar{0}$ such that for almost all $t$ in $T$, $\bar{p} \cdot x(t) \leq \bar{p} \cdot a(t)$ and for each $\bar{z}$ in $\{\bar{z} \in \Omega_n \mid \bar{p} \cdot \bar{z} \leq \bar{p} \cdot a(t)\}$ it is not the case that $\bar{z} \succ_t x(t)$. Aumann (1964) then shows the following theorem:

**Theorem B.** *The core of $\mathscr{E}$ coincides with the set of competitive allocations of $\mathscr{E}$.*

Aumann defends the introduction of a continuum of traders as follows:

> The idea of a continuum of traders may seem outlandish to the reader. Actually, it is no stranger than a continuum of prices or of strategies or a continuum of 'particles' in fluid mechanics. In all these cases, the continuum can be considered an approximation to the 'true' situation in which there is a large but finite number of particles (or traders or strategies or possible prices).

I have two things to say about Aumann's arguments. First, in physics and other places, the continuum is used as an idealization of various *ordered* phenomena. If the phenomena do not have a natural ordering, then infinity, and in particular denumerability, is the natural generalization of large finite. Second, by bringing in Lebesgue measurability, one also brings in Lebesgue non-measurability. What is the economic rationale behind the fact that some sets of traders (the Lebesgue measurable subsets of $T$) can form coalitions for the exchange of goods while others (the Lebesgue non-measurable subsets of $T$) cannot? Aumann's model of a perfect exchange economy explicitly assumes that there is a natural ordering of traders and that all trades (allocations) are somehow inherently tied to the natural ordering.

We will next consider the formulation of Brown and Robinson (1975a) which uses the concepts and methods of non-standard analysis. It is assumed that the reader is familiar with the basic concepts of non-standard analysis. $\mathscr{E} = \langle \bar{a}_t, \succ_t \rangle_{t \in T}$ is said to be a *non-standard exchange economy* if and only if the following seven conditions hold:

(i) $T$ is a non-standard finite set that has internal cardinality, $\omega$ where $\omega$ is an infinite positive integer;

(ii) $\{\bar{a}_t \mid t \in T\}$ is an internal subset of $^*E_n$ that is standardly bounded (i.e. there exists $\bar{r} \in E_n$ such that for all $t \in T$, $\bar{a}_t < \bar{r}$);

(iii) $(1/\omega) \sum_{t \in T} \bar{a}_t \gg \bar{0}$;

(iv) $(1/\omega) \sum_{t \in T} \bar{a}_t$ is not infinitesimal;

(v) $\{\succ_t \mid t \in T\}$ is an internal set;

(vi) for each $t \in T$, $\succ_t$ is irreflexive and monotonic;

(vii) for each $t \in T$ and each finite $\bar{x}, \bar{y}$ in $^*\Omega_n$, if $\bar{x} \gtrapprox \bar{0}$, then for each $\bar{\omega} \simeq \bar{x} + \bar{y}$ and each $\bar{z} \simeq \bar{y}$, $\bar{\omega} \succ_t \bar{z}$, where $\bar{x} \gtrapprox \bar{0}$ means that $\bar{x} > \bar{0}$ and some coordinate of $\bar{x}$ is non-infinitesimal, and $\bar{u} \simeq \bar{v}$ means that $\bar{u}$ and $\bar{v}$ are infinitesimally close.

An *allocation* $x$ of $\mathscr{E}$ is a standardly bounded, internal function from $T$ into $^*\Omega_n$ such that $(1/\omega) \sum_{t \in T} x(t) \simeq (1/\omega) \sum_{t \in T} \bar{a}_t$. *Coalitions of traders* of $\mathscr{E}$ are internal subsets of $T$. A coalition of traders $S$ is said to be *negligible* if and only if $|S|/\omega \simeq 0$, where $|S|$ is the number of traders in $S$. A coalition of traders $S$ is said to be *effective for an allocation* $y$ if and only if $(1/\omega)\sum_{t\in S} y(t) \simeq (1/\omega)\sum_{t\in S}\bar{a}_t$. By definition, $\bar{x} >>_t \bar{y}$ if and only if for all $\bar{w}$ in the monad of $\bar{x}$, $\bar{w} >_t \bar{y}$. An allocation $\bar{x}$ of $\mathscr{E}$ is said to be *blocked* by an allocation $\bar{y}$ via the coalition of traders $S$ if and only if $S$ is an effective coalition for $\bar{y}$ and for each $t\in S$, $\bar{y}(t) >>_t \bar{x}(t)$. The *core* of $\mathscr{E}$ is the set of allocations of $\mathscr{E}$ that are not blocked by any allocation via any non-negligible coalition of traders. A price vector is a finite vector $\bar{p} > \bar{0}$ such that each coordinate of $\bar{p}$ is non-infinitesimal. An allocation $\bar{x}$ of $\mathscr{E}$ is said to be *competitive* if and only if there exists a price vector $\bar{p}$ and an internal subset of traders $K$ of $T$ such that $|K|/\omega \simeq 1$, for each $t\in K$, $\bar{p}\cdot\bar{x}(t) \leqslant \bar{p}\cdot\bar{a}_t$, and there does not exist $\bar{y}\in\Omega_n$ and $t'\in K$ such that $\bar{y} >_{t'} x(t')$ and $\bar{p}\cdot\bar{y} \lesssim \bar{p}\cdot\bar{a}_{t'}$, where $\bar{u}\lesssim\bar{v}$ means that either $\bar{u}<\bar{v}$ or $\bar{v}$ exceeds $\bar{u}$ by at most an infinitesimal amount in each coordinate. Brown and Robinson then prove the following theorem:

**Theorem C.** *The core of $\mathscr{E}$ coincides with the competitive allocations of $\mathscr{E}$.*

Brown and Robinson's non-standard exchange economies lie somewhere between large finite economies and infinite economies; there are appropriate ways of viewing them as either of these. As infinite economies, they have certain flaws as models of 'perfect competition'. Principally, allocations are *internal* functions and coalitions are *internal* sets, and from an economic point of view there is no reason whatsoever to restrict these concepts in this way. Also even accepting this formulation of coalition and core, the economy is not quite perfect since some small set of traders (i.e. an internal subset $S$ of $T$ such that $|S|/\omega \simeq 0$) may not be treated fairly by the allocation or pricing.

Non-standard exchange economies can also be viewed as idealizations of arbitrarily large finite economies, which was probably the intention of Brown and Robinson. As such, the concept of non-standard exchange economies is translatable into limit theorems about sequences of finite economies. This is the approach of Brown and Robinson (1975b), and the interested reader should consult this work.

## 4. Dense economies

One of the basic difficulties in modeling perfect competition is that *allocation* is a difficult notion for infinite economies. To see this, let $\mathscr{E} = \langle \bar{a}_i, >_i\rangle_{i\in I^+}$ be the infinite exchange economy where for each $i\in I^+$, $\bar{a}_i = (1,1,\ldots,1)$ and $>_i$ is some monotonic relation. Now suppose the coalition of traders $I^+$ get together and work out the following 'trade': 1 keeps his allocation $\bar{a}_1$ and receives from 2, 2's allocation $\bar{a}_2$; in general, for each $i\in I^+$, if $i$ is of the form $2m$ or $2m-1$, $i$ will give his allocation $\bar{a}_i$ to trader $m$. The result of this 'trade' is that all traders end up with twice the amount of the allocation they started with. In order to avoid paradoxical results like these, some restriction must be placed on the notions of allocation and coalition of traders. But unlike the methods of Aumann (1964) and Brown and Robinson (1975a), the mode of restriction should have intuitive economic appeal. The following seems to me to be a reasonable collection of definitions for infinite economies.

Let $\mathscr{E}=\langle \bar{a}_t, \succ_t\rangle_{t\in T}$ be an exchange economy. *Coalitions of traders* of $\mathscr{E}$ are non-empty, finite subsets of $T$. $\{\bar{x}_t\}_{t\in T}$ is said to be an *allocation* of $\mathscr{E}$ if and only if there exists a partition $\mathscr{F}=\{A_j \mid j\in J\}$ of $T$ such that for each $j\in J$, $A_j$ is a coalition of traders of $\mathscr{E}$ and $\sum_{t\in A_j}\bar{x}_t=\sum_{t\in A_j}\bar{a}_t$. $\mathscr{F}$ is called an *allocation partition* for $\{\bar{x}_t\}_{t\in T}$. An allocation $\{\bar{x}_t\}_{t\in T}$ of $\mathscr{E}$ is said to be *blocked by an allocation* $\{\bar{y}_t\}_{t\in T}$ if and only if for some coalition of traders $S$, $\sum_{t\in S}\bar{y}_t=\sum_{t\in S}\bar{a}_t$ and for each $t\in S$, $\bar{y}_t\succ_t\bar{x}_t$. The *core* of $\mathscr{E}$ is the set of those allocations of $\mathscr{E}$ that are not blocked by any allocation of $\mathscr{E}$. An allocation $\{\bar{x}_t\}_{t\in T}$ is said to be *competitive* if and only if there exists a price vector $\bar{p}>\bar{0}$ such that for each $t\in T$, $\bar{p}\cdot\bar{x}_t\leq\bar{p}\cdot\bar{a}_t$, and there does not exist $u\in T$ and $\bar{y}\in\Omega_n$ such that $\bar{y}\succ_u\bar{x}_u$ and $\bar{p}\cdot\bar{y}\leq\bar{p}\cdot\bar{a}_u$.

Let $\mathscr{E}=\langle \bar{a}_t, \succ_t\rangle_{t\in T}$ be an exchange economy and $\{\bar{x}_t\}_{t\in T}$ be an allocation of $\mathscr{E}$. $\{\bar{x}_t\}_{t\in T}$ is said to be *lower dense* if and only if for each $\varepsilon>0$ and each $\bar{y}\in\Omega_n$ and each $t\in T$, there exist infinitely many $u\in T$ such that

(i) $\varphi(\bar{a}_t,\bar{a}_u)<\varepsilon$, where $\varphi$ is the Euclidean metric for $E_n$, and

(ii) if $\bar{y}+\bar{a}_t-\bar{\varepsilon}\succ_t\bar{x}_t+\bar{\varepsilon}$, then $\bar{y}+\bar{a}_u\succ_u\bar{x}_u$.

**Theorem 1.** *Let $\mathscr{E}=\langle \bar{a}_t, \succ_t\rangle_{t\in T}$ be an exchange economy where for each $t\in T$, $\bar{a}_t\gg\bar{0}$ and $\succ_t$ is irreflexive, monotonic, and continuous, and let $\{\bar{x}_t\}_{t\in T}$ be a lower dense allocation in the core of $\mathscr{E}$. Then $\{\bar{x}_t\}_{t\in T}$ is competitive.*

Throughout the proof of Theorem 1 (including Lemma 1), for each $i\in I^+$ and each $t\in T$, let

$$F_i(t)=\left\{\bar{z}\in\Omega_n \mid \text{ for all } \bar{u}\in B\left(\bar{z}+\bar{a}_t, \frac{1}{i}\right), \bar{u}\succ_t\bar{x}_t\right\},$$

$$F(t)=\bigcup_{i=1}^{\infty}F_i(t),$$

$$G_i(t)=F_i(t)-\bar{a}_t,$$

$$G(t)=\bigcup_{i=1}^{\infty}G_i(t),$$

and let $\Delta$ be the set of convex combinations of $\bigcup_{t\in T}G(t)$, i.e. let

$$\Delta=\left\{\alpha_1\bar{y}_1+\cdots+\alpha_k\bar{y}_k \mid k\in I^+,\ 0\leq\alpha_1,\ldots,\alpha_k\leq 1,\ \sum_{i=1}^{k}\alpha_i=1,\right.$$
$$\left.\text{and } \bar{y}_i\in\bigcup_{t\in T}G(t) \text{ for } i=1,\ldots,k\right\}.$$

Note that for each $t\in T$, $F(t)$, $G(t)$, and $\Delta$ are open.

**Lemma 1.** $\bar{0} \notin \Delta$.

**Proof.** Suppose $\bar{0} \in \Delta$. A contradiction will be shown. Since $\Delta$ is open, let $r \in Re^{+}$ be such that $B(\bar{0}, r) \subseteq \Delta$. Let $\bar{x} \in B(\bar{0}, r)$ be such that $\bar{x} \gg \bar{0}$. Then $-\bar{x} \in B(\bar{0}, r) \subseteq \Delta$. Then $-\bar{x}$ is a convex combination of $k$ points of $\bigcup_{t \in T} G(t)$, i.e.

$$-\bar{x} = \sum_{i=1}^{k} \beta_i \bar{y}_i, \text{ where } \sum_{i=1}^{k} \beta_i = 1 \text{ and for } i = 1, \dots, k,\ \beta_i > 0 \text{ and } \bar{y}_i \in G(t_i).$$

Since $G(t_i)$ is open, by selecting rational numbers $\gamma_i$ sufficiently close to $\beta_i$ and rational points (i.e. points of $E_n$ with rational coordinates) $\bar{s}_i \in G(t_i)$ sufficiently close to $\bar{y}_i$, we can find a rational point $\bar{s} \gg \bar{0}$ such that $-\bar{s} \in B(\bar{0}, r) \subseteq \Delta$, and

$$-\bar{s} = \sum_{i=1}^{k} \gamma_i \bar{s}_i.$$

Let $t_0$ be a trader of $T$ distinct from $t_1, \dots, t_k$. Since $\bar{s} \gg \bar{0}$, for sufficiently large rational $\gamma$,

$$\gamma \bar{s} + \bar{a}_{t_0} > \bar{x}_{t_0}.$$

Since $\succ_{t_0}$ is monotonic,

$$\gamma \bar{s} + \bar{a}_{t_0} \succ_{t_0} \bar{x}_{t_0}.$$

Let

$$\bar{s}_0 = \gamma \bar{s}, \qquad \alpha_0 = \frac{1}{\gamma + 1}, \qquad \alpha_i = \frac{\gamma \gamma_i}{\gamma + 1} \quad \textit{for } i = 1, \dots, k.$$

Then

$$\sum_{i=0}^{k} \alpha_i \bar{s}_i = \frac{\gamma \bar{s}}{\gamma + 1} + \frac{\gamma}{\gamma + 1} \sum_{i=1}^{k} \gamma_i \bar{s}_i = \frac{\gamma \bar{s}}{\gamma + 1} + \frac{\gamma}{\gamma + 1}(-\bar{s}) = \bar{0}. \tag{1}$$

Let $\delta$ be the common denominator of $\alpha_0, \dots, \alpha_k$. Let $\xi_i = \alpha_i \delta$. Then for $i = 0, \dots, k$, $\xi_i$ are positive integers and $\sum_{i=0}^{k} \xi_i \bar{s}_i = \bar{0}$. Since $\bar{s}_0 + \bar{a}_{t_0} \succ_{t_0} \bar{x}_{t_0}$ and for $i = 1, \dots, k$, $\bar{s}_i \in G(t_i)$, it follows that $\bar{s}_i + \bar{a}_{t_i} \succ_{t_i} \bar{x}_{t_i}$ for $i = 0, \dots, k$. Since $\succ_{t_i}$ is continuous, $\varepsilon > 0$ can be found such that $\bar{s}_i + \bar{a}_{t_i} - \bar{\varepsilon} \succ_{t_i} \bar{x}_{t_i} + \varepsilon$.

Since $\{\bar{x}_t\}_{t \in T}$ is lower dense, we can find for $i = 0, \dots, k$, $\xi_i$ distinct traders of $T$, $t_i^1, \dots, t_i^{\xi_i}$, so that for $j = 1, \dots, \xi_i$,

$$\bar{s}_i + \bar{a}_{t_i^j} \succ_{t_i^j} \bar{x}_{t_i^j}. \tag{2}$$

For $i = 0, \dots, k$, let $S_i = \{t_i^j \mid j = 1, \dots, \xi_i\}$. Let $S = \bigcup_{i=0}^{k} S_i$. For each $t \in T$, let

$$\bar{z}_t = \bar{a}_t \quad \text{if } t \in T - S$$

and

$$\bar{z}_t = \bar{s}_i + \bar{a}_{t_i^j} \quad \text{if for some } i, j,\ 0 \le i \le k,\ 1 \le j \le \xi_i,\ t \in S_i \text{ and } t = t_i^j.$$

Then

$$\sum_{t\in S} \bar{z}_t = \sum_{i=0}^{k} \sum_{j=1}^{\xi_i} (\bar{s}_i + \bar{a}_{t_i^j})$$

$$= \sum_{i=0}^{k} \left( \xi_i \bar{s}_i + \sum_{j=1}^{\xi_i} \bar{a}_{t_i^j} \right)$$

$$= \sum_{i=0}^{k} (\xi_i \bar{s}_i) + \sum_{i=0}^{k} \sum_{j=1}^{\xi_i} \bar{a}_{t_i^j}$$

(which from $\sum_{i=0}^{k} \xi_i \bar{s}_i = \bar{0}$ and the definition of $S$)

$$= \bar{0} + \sum_{t\in S} \bar{a}_t.$$

Thus since

$$\sum_{t\in S} \bar{z}_t = \sum_{t\in S} \bar{a}_t, \tag{3}$$

and $\bar{z}_t = \bar{a}_t$ for $t \in T - S$, $\{\bar{z}_t\}_{t\in T}$ is an allocation of $\mathscr{E}$. From equations (2) and (3) it follows that $\{\bar{x}_t\}_{t\in T}$ is blocked by $\{\bar{z}_t\}_{t\in T}$. Thus $\{\bar{x}_t\}_{t\in T}$ is not in the core of $\mathscr{E}$, which is a contradiction. □

**Proof of Theorem 1.** By Lemma 1, $\bar{0} \notin \Delta$. It is easy to show from the definition of $\Delta$ that $\Delta$ is convex. Thus by a well-known theorem of convex sets (see Berge and Ghouila-Houri, 1965, p. 54) let $\bar{p}$ be an element of $E_n$ such that $\bar{p} \neq \bar{0}$ and for each $\bar{u} \in \Delta$, $\bar{p} \cdot \bar{u} \geq 0$.

We will first show that for $t \in T$, $\bar{u} \in E_n$,

$$\text{if } \bar{u} > \bar{x}_t, \text{ then } \bar{u} \in F(t). \tag{4}$$

Suppose $t \in T$ and $\bar{u} > \bar{x}_t$. Let $r > 0$ be such that $\bar{a}_t > \bar{r}$. Choose $m \in I^+$ so that $1/m < r$. Then

$$\bar{u} + \bar{a}_t - \bar{r} > \bar{u} > \bar{x}_t.$$

Thus for all $\bar{v} \in B(\bar{u} + \bar{a}_t, 1/m)$,

$$\bar{v} > \bar{u} + \bar{a}_t - \bar{r} > \bar{x}_t,$$

which by monotonicity yields

$$\bar{v} \succ_t \bar{x}_t.$$

Thus $\bar{u} \in F_m(t)$. Therefore $\bar{u} \in F(t)$.

We will now show that

$$\bar{p} > \bar{0}. \tag{5}$$

Suppose that for some $i$, $p_i < 0$. A contradiction will be shown. Since $\Delta \supseteq G(t)$, it follows that for all $\bar{y} \in G(t)$, $\bar{p} \cdot \bar{y} \geq 0$. Thus for each $\bar{z} \in F(t)$, $\bar{p} \cdot \bar{z} \geq \bar{p} \cdot \bar{a}_t$. Let $\bar{w}$ be such that $w_j = 0$ if $j \neq i$ and $w_i$ is a large positive number. Then by expression (4),

$\bar{w}+\bar{x}_t \in F(t)$, and thus $\bar{p}\cdot(\bar{w}+x_t)\geq \bar{p}\cdot \bar{a}_t$. But

$$\bar{p}\cdot(\bar{w}+\bar{x}_t) = p_i \cdot w_i + \bar{p}\cdot \bar{x}_t < \bar{p}\cdot \bar{a}_t$$

since $p_i<0$ and $w_i$ is a large positive number. This is a contradiction. Thus $\bar{p}\geq \bar{0}$. Since $\bar{p}\neq \bar{0}$, $p>\bar{0}$.

We will now show that for each $t\in T$,

$$\bar{p}\cdot \bar{x}_t \geq \bar{p}\cdot \bar{a}_t. \tag{6}$$

Suppose for some $t\in T$, $\bar{p}\cdot \bar{x}_t<\bar{p}\cdot \bar{a}_t$. A contradiction will be shown. Let $\bar{z}\gg\bar{0}$ be an element of $\Omega_n$ such that $\bar{p}\cdot(\bar{x}_t+\bar{z})<\bar{p}\cdot \bar{a}_t$. Then because $\bar{x}_t+\bar{z}\succ_t \bar{x}_t$, it follows that $\bar{x}_t+\bar{z}\in F(t)$, and that $\bar{p}\cdot(\bar{x}_t+\bar{z})\geq \bar{p}\cdot \bar{a}_t$. This is a contradiction.

We will now show that for each $t\in T$,

$$\bar{p}\cdot \bar{x}_t = \bar{p}\cdot \bar{a}_t. \tag{7}$$

Suppose $\upsilon\in T$ is such that $\bar{p}\cdot \bar{x}_\upsilon \neq \bar{p}\cdot \bar{a}_\upsilon$. A contradiction will be shown. By expression (6), $\bar{p}\cdot \bar{x}_\upsilon>\bar{p}\cdot \bar{a}_\upsilon$. Since $\{\bar{x}_t\}_{t\in T}$ is an allocation of $\mathscr{E}$, let $\mathscr{F}$ be an allocation partition for $\{\bar{x}_t\}_{t\in T}$ and $S$ be the element of $\mathscr{F}$ such that $\upsilon\in S$. By expression (6), $\bar{p}\cdot \bar{x}_t\geq \bar{p}\cdot \bar{a}_t$ for each $t\in S$. Since $\bar{p}\cdot \bar{x}_\upsilon>\bar{p}\cdot \bar{a}_\upsilon$ and $\upsilon\in S$, it follows that

$$\sum_{t\in S}\bar{p}\cdot \bar{x}_t > \sum_{t\in S}\bar{p}\cdot \bar{a}_t.$$

But this contradicts that $S$ is in $\mathscr{F}$.

We will now show that for each $t\in T$ and each $\bar{z}\in\Omega_n$,

$$\text{if } \bar{z}\succ_t \bar{x}_t, \text{ then } \bar{p}\cdot z>\bar{p}\cdot \bar{a}_t. \tag{8}$$

Suppose $t\in T$, $\bar{z}\in\Omega_n$, and $\bar{z}\succ_t \bar{x}_t$. Since by expression (5) $\bar{p}>\bar{0}$, some coordinate of $\bar{p}$, say $p_1$, is positive. Since $U=\{\bar{u}\mid \bar{u}\succ_t \bar{x}_t\}$ is open and $\bar{z}\in U$, let $s$ be a sufficiently small positive number such that $\bar{y}=(z_1-s, z_2, \ldots, z_n)$ is in $U$. Then $\bar{y}\succ_t \bar{x}_t$. By expression (4), $\bar{y}\in F(t)$. Thus $\bar{p}\cdot\bar{z}>\bar{p}\cdot\bar{y}\geq\bar{p}\cdot \bar{a}_t$.

From expressions (5), (7), and (8) it follows that $\{\bar{x}_t\}_{t\in T}$ is competitive. □

**Lemma 2.** *Suppose $\mathscr{E}=\langle \bar{a}_t, \succ_t\rangle_{t\in T}$ is an exchange economy and $\{\bar{x}_t\}_{t\in T}$ is a competitive allocation of $\mathscr{E}$. Then $\{\bar{x}_t\}_{t\in T}$ is in the core of $\mathscr{E}$.*

**Proof.** Let $\bar{p}>\bar{0}$ be such that for each $t\in T$, $\bar{p}\cdot \bar{x}_t\leq \bar{p}\cdot \bar{a}_t$ and there does not exist $\bar{z}$ in $\{\bar{y}\in\Omega_n \mid \bar{p}\cdot\bar{y}\leq \bar{p}\cdot \bar{a}_t\}$ such that $\bar{z}\succ_t \bar{x}_t$.

Suppose $\{\bar{x}_t\}_{t\in T}$ is not in the core of $\mathscr{E}$. A contradiction will be shown. Let $\{\bar{y}_t\}_{t\in T}$ be an allocation of $\mathscr{E}$ and $S$ be a coalition of traders of $\mathscr{E}$ such that

$$\sum_{t\in S}\bar{y}_t = \sum_{t\in S}\bar{a}_t, \tag{9}$$

and for each $t\in S$,

$$\bar{y}_t \succ_t \bar{x}_t.$$

Since $S$ is a finite set, let $u$ be a fixed element of $S$ such that for each $t \in S$,

$$\bar{p} \cdot (\bar{y}_t - \bar{a}_t) \geq \bar{p} \cdot (\bar{y}_u - \bar{a}_u).$$

Since $\bar{y}_u \succ_u \bar{x}_u$,

$$\bar{p} \cdot \bar{y}_u > \bar{p} \cdot \bar{a}_u.$$

Therefore

$$\bar{p} \cdot (\bar{y}_u - \bar{a}_u) > 0,$$

and thus

$$\sum_{t \in S} \bar{p} \cdot (\bar{y}_t - \bar{a}_t) = \bar{p} \cdot \sum_{t \in S} (\bar{y}_t - \bar{a}_t) > 0.$$

However, from equation (9) it follows that

$$\bar{p} \cdot \sum_{t \in S} (\bar{y}_t - \bar{a}_t) = 0,$$

and this is a contradiction. □

$\mathscr{E} = \langle \bar{a}_t, \succ_t \rangle_{t \in T}$ is said to be a *dense* exchange economy if and only if the following four conditions hold for each $t$ in $T$:

(i) $\mathscr{E}$ is an exchange economy;

(ii) $\bar{a}_t \gg \bar{0}$;

(iii) $\succsim_t$ is a weak ordering and $\succ_t$ is monotonic and continuous;

(iv) *density*: for each $\varepsilon > 0$ there exist infinitely many $u$ in $T$ such that $\bar{a}_t \geq \bar{a}_u$, $\bar{a}_u + \bar{\varepsilon} > \bar{a}_t$, and for all $\bar{w}$, $\bar{z}$ in $\Omega_n$, if $\bar{w} \succ_t \bar{z}$, then $\bar{w} + \bar{\varepsilon} \succ_u \bar{z}$.

**Theorem 2.** *Let $\mathscr{E} = \langle \bar{a}_t, \succ_t \rangle_{t \in T}$ be a dense exchange economy and $\{\bar{x}_t\}_{t \in T}$ be an allocation of $\mathscr{E}$. Then $\{\bar{x}_t\}_{t \in T}$ is lower dense and is in the core of $\mathscr{E}$ if and only if $\{\bar{x}_t\}_{t \in T}$ is competitive.*

**Proof.** Suppose $\{\bar{x}_t\}_{t \in T}$ is lower dense and is in the core of $\mathscr{E}$. Then by Theorem 1, $\{\bar{x}_t\}_{t \in T}$ is competitive.

Suppose $\{\bar{x}_t\}_{t \in T}$ is competitive. Let $\bar{p} > \bar{0}$ be such that for each $t \in T$, $\bar{p} \cdot \bar{x}_t \leq \bar{p} \cdot \bar{a}_t$ and there does not exist $\bar{q}$ in $\{\bar{s} \in \Omega_n \mid \bar{p} \cdot \bar{s} \leq \bar{p} \cdot \bar{a}_t\}$ such that $\bar{q} \succ_t \bar{x}_t$.

By Lemma 2, $\{\bar{x}_t\}_{t \in T}$ is in the core of $\mathscr{E}$. Thus we need only to show that $\{\bar{x}_t\}_{t \in T}$ is lower dense. Suppose $\varepsilon$ is an arbitrary positive number, $\upsilon$ is an arbitrary element of $T$, and $\bar{y}$ is an arbitrary element of $\Omega_n$ such that

$$\bar{y} + \bar{a}_\upsilon - \bar{\varepsilon} \succ_\upsilon \bar{x}_\upsilon + \bar{\varepsilon}. \tag{10}$$

We need only show that for infinitely many $u$ in $T$, $\varphi(\bar{a}_\upsilon, \bar{a}_u) < \varepsilon$ and $\bar{y} + \bar{a}_u \succ_u \bar{x}_u$. By density, let $U$ be an infinite subset of $T$ such that for each $u$ in $U$,

$$\bar{a}_\upsilon \geq \bar{a}_u \quad \text{and} \quad \bar{a}_u + \frac{1}{4n} \bar{\varepsilon} > \bar{a}_\upsilon, \tag{11}$$

and

$$\text{for all } \bar{w}, \bar{z} \text{ in } \Omega_n, \text{ if } \bar{w} \succ_{\upsilon} \bar{z} \text{ then } \bar{w} + \frac{1}{4n}\bar{\varepsilon} \succ_u \bar{z}. \tag{12}$$

Let $u$ be an arbitrary element of $U$. Then by expression (11), $\bar{p}\cdot\bar{a}_u \le \bar{p}\cdot\bar{a}_{\upsilon}$. Since $\bar{p}\cdot\bar{x}_u \le \bar{p}\cdot\bar{a}_u$, $\bar{x}_u$ is in trader $\upsilon$'s budget set $\{\bar{s}\in\Omega_n \mid \bar{p}\cdot\bar{s} \le \bar{p}\cdot\bar{a}_{\upsilon}\}$. Thus, since $\{\bar{x}_t\}_{t\in T}$ is competitive, it is not the case that $\bar{x}_u \succ_{\upsilon} \bar{x}_{\upsilon}$. Since $\succsim_{\upsilon}$ is a weak ordering, it follows that

$$\bar{x}_{\upsilon} \succsim_{\upsilon} \bar{x}_u. \tag{13}$$

By expression (11),

$$(\bar{y} + \bar{a}_u - \bar{\varepsilon}) + \frac{1}{4n}\bar{\varepsilon} > \bar{y} + \bar{a}_{\upsilon} - \bar{\varepsilon}.$$

Thus by the monotonicity and transitivity of $\succ_{\upsilon}$, we can conclude from expression (10) that

$$\bar{y} + \bar{a}_u - \tfrac{3}{4}\bar{\varepsilon} \succ_{\upsilon} \bar{x}_{\upsilon} + \bar{\varepsilon}.$$

Thus by expression (13) and the transitivity and monotonicity of $\succ_{\upsilon}$,

$$\bar{y} + \bar{a}_u - \tfrac{3}{4}\bar{\varepsilon} \succ_{\upsilon} \bar{x}_u.$$

By expression (12) it then follows that

$$\bar{y} + \bar{a}_u - \tfrac{2}{4}\bar{\varepsilon} \succ_u \bar{x}_u,$$

which by the monotonicity and transitivity of $\succ_u$ yields

$$\bar{y} + \bar{a}_u \succ_u \bar{x}_u. \quad \square$$

Let $\mathscr{E} = \langle \bar{a}_t, \succ_t \rangle_{t\in T}$ be a dense economy and $C$ be the set of lower dense allocations of the core of $\mathscr{E}$. $C$ is called the *dense core* of $\mathscr{E}$. In view of Theorem 2, to show that $\mathscr{E}$ is 'perfectly competitive' we have to argue that the dense core of $\mathscr{E}$ rather than the core of $\mathscr{E}$ is the correct concept for final trading states under perfect competition. This argument is easy. Let $\{\bar{x}_t\}_{t\in T}$ be an allocation in the core of $\mathscr{E}$ and let $\mathscr{F}$ be an allocation partition for $\{\bar{x}_t\}_{t\in T}$. Let $\varepsilon > 0$, $\bar{y}\in\Omega_n$, and $\upsilon\in T$ be such that

$$\bar{y} + \bar{a}_{\upsilon} - \bar{\varepsilon} \succ_{\upsilon} \bar{x}_{\upsilon} + \bar{\varepsilon}. \tag{14}$$

We will argue that for infinitely many $u\in T$, $\varphi(\bar{a}_{\upsilon}, \bar{a}_u) < \varepsilon$ and $\bar{y} + \bar{a}_u \succ_u \bar{x}_u$. Let $U$ be the set of all $u$ in $T$ such that

$$\bar{a}_{\upsilon} \ge \bar{a}_u \quad \text{and} \quad \bar{a}_u + \frac{1}{4n}\bar{\varepsilon} > \bar{a}_{\upsilon} \tag{15}$$

and

$$\text{for each } \bar{w}, \bar{z} \text{ in } \Omega_n, \text{ if } \bar{w} \succ_{\upsilon} \bar{z} \text{ then } \bar{w} + \tfrac{1}{4}\bar{\varepsilon} \succ_u \bar{z}. \tag{16}$$

Then by density, $U$ is an infinite subset of $T$. From expressions (14) and (15) and the monotonicity and transitivity of $\succ_v$, we can conclude that

$$\bar{y}+\bar{a}_u-\tfrac{3}{4}\bar{\varepsilon}\succ_v\bar{x}_v+\bar{\varepsilon}. \tag{17}$$

We claim that under any semblance of fair competition that $v$ will prefer $\bar{x}_v+\frac{1}{4}\bar{\varepsilon}$ to $\bar{x}_u$. After all, $v$ starts out with at least as much of each commodity as $u$ (i.e. $\bar{a}_v\geq\bar{a}_u$) and therefore, if trading is fair, he should end up with at least as desirable commodity bundle in terms of his own preference ordering as the bundle which $u$ ends up with. In terms of our current set-up, let $S$ be the coalition of traders in the allocation partition $\mathscr{F}$ to which $v$ belongs, and $W$ be the coalition in $\mathscr{F}$ to which $u$ belongs. Let $Z=S\cup W$. Then $Z$ is a coalition of traders of $T$, and $u$ and $v$ are in $Z$. If a final trade among the members of $Z$ is about to be made where $v$ prefers what $u$ is ending up with to what he has, then, since $v$ had initially as much of each commodity as $u$, he could have struck almost as good of a deal with the traders of $Z$ as $u$ by offering them a little more than $u$ offered. Thus we can conclude that

$$\bar{x}_v+\tfrac{1}{4}\bar{\varepsilon}\succ_v\bar{x}_u. \tag{18}$$

By expressions (17) and (18) and the monotonicity and transitivity of $\succ_v$, we can conclude that

$$\bar{y}+\bar{a}_u-\tfrac{3}{4}\bar{\varepsilon}\succ_v\bar{x}_u.$$

Thus by expression (16),

$$\bar{y}+\bar{a}_u-\tfrac{2}{4}\bar{\varepsilon}\succ_u\bar{x}_u,$$

which by the monotonicity and transitivity of $\succ_u$ yields

$$\bar{y}+\bar{a}_u\succ_u\bar{x}_u.$$

Thus $\{\bar{x}_t\}_{t\in T}$ is lower dense.

## 5. Discussion

There are two important features to note about the proof of the equivalence of the qualitative notion of lower dense core allocation and the quantitative notion of competitive equilibrium: (i) the Axiom of Choice of set theory was not used in the proof, and (ii) the proof was by contradiction and thus no description was provided for obtaining competitive equilibria from the lower dense core.

Lewis (1990a,b) shows that the Axiom of Choice plays an important role in many of the models used in modern equilibrium theory and in particular that the nonemptiness of the core in many of these models is equivalent to the Hahn–Banach Extension Theorem. The Hahn–Banach Extension Theorem may be viewed as a weakened form of the Axiom of Choice, since it can be derived from set theory with the Axiom of Choice but cannot be derived from set theory without the Axiom of Choice. Brown and Robinson's (1975a,b) results necessarily depend upon weakened forms of the Axiom of Choice, since their formulation requires the existence of a non-standard model, which itself depends on a weakened form of the Axiom of Choice.

For the types of results discussed here, there are three places where the Axiom of Choice can enter: (a) in showing the existence of an appropriate exchange economy; (b) in showing the appropriate 'core' concept to have at least one allocation; and (c) in showing the equivalence of the appropriate 'core' concept with the appropriate 'competitive equilibrium' concept. I believe uses of the Axiom of Choice to be appropriate for economic modeling in (a) and (b), if one believes that one is idealizing situations in which 'Providence' or 'Accident' play a role in determining initial allocations and preference orderings. Whether or not the Axiom of Choice is appropriate in (c) depends, in my view, on how 'competitive equilibrium' is to be used in the analysis. If it is to be used as a quantitative description of a qualitative situation, then I see nothing wrong with its employment; if, however, it is to be viewed as a price structure *to be achieved by economic processes*, i.e. as some form of *economic computation*, then I see great difficulty in having it depend on the Axiom of Choice.

In the following example, which does not use the Axiom of Choice, a dense exchange economy with a non-empty lower dense core will be given. In this economy, there will be two commodities and denumerably many traders, and each trader will have a unique preference ordering over commodity bundles:

Let $T=\{r \mid 0<r<1 \text{ and } r \text{ is rational}\}-\{\frac{1}{2}\}$. For each $t\in T$, let $\bar{a}_t=(1,1)$ and $\succ_t$ be the following relation on $\Omega_2$: for all $(r,s)$ and $(u,v)$ in $\Omega_2$,

$$(r,s)\succ_t(u,v) \quad \text{iff } tr+(1-t)s>tu+(1-t)v.$$

Then it easily follows that $\mathscr{E}=\langle\bar{a}_t, \succ_t\rangle_{t\in T}$ is a dense exchange economy. Let

$$\mathscr{F}=\{\{t,1-t\} \mid t\in T\},$$

and let $\{\bar{x}_t\}_{t\in T}$ the allocation with allocation partition $\mathscr{F}$ such that for each $t\in T$,

$$\bar{x}_t=(0,2) \quad \text{if } t<\tfrac{1}{2} \quad \text{and} \quad \bar{x}_t=(2,0) \quad \text{if } t>\tfrac{1}{2}.$$

It will be first be shown that $\bar{x}_t$ is in the core of $\mathscr{E}$.

Suppose $\bar{x}_t$ is not in the core of $\mathscr{E}$. A contradiction will be shown. Let $\bar{y}_t$ be an allocation that blocks $\bar{x}_t$, and let $S$ be a coalition of traders such that for each $s\in S$, $\bar{y}_s\succ_s\bar{x}_s$ and such that

$$\sum_{s\in S}\bar{y}_s=\sum_{s\in S}\bar{a}_s.$$

Let $k$ be the number of traders in $S$. Then

$$\sum_{s\in S}\bar{a}_s=(k,k).$$

Thus

$$\sum_{s\in S}\bar{y}_s=(k,k).$$

Therefore, one of the following two cases must obtain. Case 1: for each $s \in S$, $\bar{y}_s = (c_s, d_s)$, where $c_s + d_s = 2$; or Case 2: there exists $p \in S$ such that $\bar{y}_p = (c_p, d_p)$, where $c_p + d_p < 2$. In Case 1, $\bar{x}_s \succsim_s \bar{y}_s$ for each $s \in S$, contrary to assumption, and in Case 2, $\bar{x}_p \succ_p \bar{y}_p$, contrary to assumption.

To show that $\{\bar{x}_t\}_{t \in T}$ is lower dense, let $t$ be an arbitrary element of $T$, $\varepsilon$ be an arbitrary positive real number, and $\bar{y}$ be an arbitrary element of $\Omega_2$. Without loss of generality, we may assume $t < \frac{1}{2}$, since the case $t > \frac{1}{2}$ will follow by a similar argument. Suppose that $\bar{y} = (c, d)$ and

$$\bar{y} + \bar{a}_t - \bar{\varepsilon} \succ_t \bar{x}_t + \bar{\varepsilon}. \tag{19}$$

Let $u$ be such that $0 < u < t$ and $u$ be arbitrarily close to $t$. (Note that there are infinitely many such $u$.) Then $\bar{x}_t = \bar{x}_u = (0, 2)$ and $\bar{a}_t = \bar{a}_u = (1, 1)$. From equation (19) and the definition of $\succ_t$,

$$t(c+1) + (1-t)(d+1) > t2\bar{\varepsilon} + (1-t)(2 + 2\bar{\varepsilon}).$$

Since $u$ is arbitrarily close to $t$,

$$u(c+1) + (1-u)(d+1) > u2\bar{\varepsilon} + (1-u)(2 + 2\bar{\varepsilon}),$$

which by the definition of $\succ_u$ yields,

$$\bar{y} + \bar{a}_u \succ_u \bar{x}_u + 2\bar{\varepsilon},$$

and thus

$$\bar{y} + \bar{a}_u \succ_u \bar{x}_u.$$

Also, since $\bar{a}_t = \bar{a}_u$, it follows that

$$\varphi(\bar{a}_t, \bar{a}_u) = 0 < \varepsilon,$$

where $\varphi$ is the Euclidean metric on $E_2$. Thus it has been shown that $\{\bar{x}_t\}_{t \in T}$ is lower dense.

Dense exchange economies describe a very general trading situation, and the existence of particular dense exchange economies or the existence of core allocations that are lower dense within particular economies may require the use of the Axiom of Choice. As the above example indicates, one should be able to describe without recourse to the Axiom of Choice an abundance of dense exchange economies with cores that have lower dense allocations. A characterization of such economies would be highly desirable.

Let $\mathscr{E} = \langle \bar{a}_t, \succ_t \rangle_{t \in T}$ be an exchange economy (not necessarily a dense one). Unless there is some compelling *economic* reason otherwise, I believe the following 'meaningfulness' criterion should be imposed on results about $\mathscr{E}$. If $\pi$ is an arbitrary permutation on $T$, then any result or conclusion that is reached about $\mathscr{E}$ should also be reached by permuting the traders in $T$ by $\pi$, e.g. if $\{\bar{x}_t\}_{t \in T}$ is an allocation of $\mathscr{E}$, then $\{\bar{x}_{\pi(t)}\}_{t \in T}$ should also be an allocation of $\mathscr{E}$. The economies of Aumann (1964) and Brown and Robinson (1975a,b) do not meet this criterion for reasons discussed in Section 2; dense exchange economies do meet it. This meaningfulness criterion is meant to capture the idea that the results and conclusions should depend only on traders' initial allocations and preference orderings and not on any scheme for organizing these. (If results do depend on particular schemes for organizing initial allocations and preference orderings, then I think the economic rationale for the particular schemes of organization should be presented along with the description of the exchange economy.) I believe it is very difficult for exchange economies to meet the above meaningfulness criterion, and I consider it be a great strength of dense exchange economies that they do meet it.

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