On the Relationship of the Psychological and the Physical in Psychophysics

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This article presents a theory of the relationship of the psychological and the physical and uses it to formulate a new kind of meaningfulness principle for psychophysical application. This new principle calls into question the psychological relevance of many kinds of quantitative psychophysical relationships. As an illustration, it is used to study comparisons of sensitivity involving Weber fractions, particularly comparisons across sensory modalities. The methods of the illustration extend easily to other psychophysical situations.

Fechner was the first to present a comprehensive psychophysical theory and methodology. He carefully formulated how the psychological and physical were to be treated differently, provided a theory of the psychophysical relationship based in part on this difference in treatments, and used the different treatments as part of a methodology to test his theory. Since his time, the differences in treatment of the psychological and the physical have become less sharp and psychophysical methodology much more eclectic and not particularly focused on unique properties of the psychophysical relationship.

In this article, a sharp theoretical distinction between the psychological and physical is reestablished. The distinction is used to formulate a new theoretical principle that restricts the kinds of concepts and analyses that can be applied to the psychological component of the psychophysical situation. The principle is based on concepts of modern logic and measurement theory and asserts that the manner in which the physical is formulated should not influence conclusions drawn about the psychological. Like Fechner, we use Weber's law as a prototypical psychophysical situation to illustrate our theory, and like Fechner, our methods extend easily beyond this highly constrained situation. Unlike Fechner our goal is to only clarify a specific theoretical relationship between the psychological and the physical and to understand its implications rather than to provide a general theory and foundation for psychophysics.

Because psychophysical theory relies heavily on mathematical formulations of psychological and psychophysical concepts, it is particularly important for psychophysics that the relationship between mathematical formulations and the qualitative situations they describe is made clear. The understanding of this relationship between the quantitative and the qualitative is a generic and important foundational problem in science.

One of the most obvious and powerful uses of mathematics in science lies in the description of subtle properties of substantive, empirical domains. However, with this power comes the potential for the generation of meaningless quantitative concepts—that is, for generating quantitative concepts that have neither empirical nor qualitative interpretations in the substantive domain. The following provides a simple illustration:

Suppose Monday's temperature were 10 °C, Tuesday's 20 °C, Wednesday's 15 °C, and Thursday's 30 °C. Then it would be an empirically true statement that the ratio of Monday's to Tuesday's temperature is the same as the ratio of Wednesday's to Thursday's temperature (all temperatures being measured in centigrade), the ratio being 1/2. It is also a logically (and empirically) true fact that the binary relation $\sim c$ defined on ordered pairs (x, y) of temperatures measured in centigrade by

$$(x, y) \sim_c (u, v) \text{ iff } \frac{x}{y} = \frac{u}{v}$$

is an equivalence relation. However, according to physical theory, $\sim c$ cannot correspond to a "physically relevant" equivalence relation on ordered pairs of qualitative, physical temperatures; that is, it cannot correspond to an equivalence relation within physical theory. The reason for this is that physical theory demands that all qualitative equivalence relations within it, "when appropriately measured," must be equivalence relations under measurement. Because clearly the ratio of Monday's to Tuesday's temperature is not in the same equivalence class as the ratio of Wednesday's to Thursday's temperature when measured in terms of Fahrenheit (i.e., that these ratios, when the temperatures are measured in Fahrenheit, are not equal), it follows that in any physical theory in which centigrade and Fahrenheit are appropriate ways of quantitatively representing temperature, $\sim c$ is not a physically relevant equivalence relation on ordered pairs of temperatures.

Ruling out from consideration relationships that are not invariant under appropriate changes of scale has been a much used practice in physics, and a subarea of physics known as *dimensional analysis* has evolved around its development and application. Stevens (1946, 1951) introduced similar ideas into psychology, which were refined by use of measurement-theoretic concepts by Suppes and Zinnes (1963). Luce (1959, 1964)

developed related ideas using concepts of dimensional analysis to provide for a theory of "possible psychophysical laws." The ideas of Suppes and Zinnes and Luce were extended by Falmagne and Narens (1983), and psychophysical applications of these extensions were given by Falmagne and Narens (1983) and Falmagne (1985). Roberts and Rosenbaum (1986) also integrated the approaches of Suppes and Zinnes and Luce, and substantive applications of this to behavioral sciences can be found in Roberts (1985). Recently, Luce (1990) has used the invariance under change of scale idea to update his 1959 theory of "possible psychophysical laws" to cross modality-matching situations.

(Some of Stevens's, 1946, original ideas, particularly invariance under change of scale as a criterion for "appropriate" statistics, have generated much controversy-e.g., see the recent series of *Psychological Bulletin* articles by Townsend & Ashby, 1984, Michell, 1986, and Stine, 1989. The statistical issues about invariance under change of scale are related to but are different from the psychophysical issues raised in this article.)

Stevens (1946, 1951) called relationships that were invariant under change of scale *meaningful* and those that were not *meaningless*. In this article this usage is adhered to. Essentially, meaningless quantitative relationships can be looked on as those that depend on particular conventions of measurement for their definition (e.g., the equivalence relation ~c noted earlier depends on the centigrade representation), whereas meaningful ones do not depend on such conventions.

In psychophysics there are both underlying qualitative physical and psychological structures to measure. One usually proceeds by measuring the physical variables in terms of some standard system of physical units and then represents the psychological relationships quantitatively in terms of the measured physical variables. Psychophysical laws usually have the characteristic that the form of the representing quantitative relation is invariant under change of physical units, and some (e.g., Luce, 1959; Falmagne & Narens, 1983; Roberts & Rosenbaum, 1986) have suggested this as a necessary condition for a psychophysical law. Thus, in psychophysics there has been a strong tendency to consider only quantitative psychological relationships that are *physically meaningful* in the sense that their mathematical forms are invariant under changes of physical units.

In this article, an additional invariance condition on quantitative psychological relationships is proposed. It is based on the facts that (a) *from the point of view of theoretical physics the stimulus can be characterized in many different but physically equivalent ways,* and (b) *the physical theory does not depend on which of these equivalent ways are used in the formulation.* Thus, from this point of view, *it is only a matter of convention which equivalent way is used to characterize the stimulus in psychophysics,* and *psychological conclusions in psychophysical settings should be invariant under equivalent physical formulations;* that is, *one should reach the same conclusion no matter which equivalent formulation of the physical stimulus is used.* Implications of this new invariance condition for the comparison of physically meaningful measures of sensitivity are investigated in this article, with particular emphasis on comparisons of Weber fractions. It is argued that this new invariance principle gives correct insights about the psychological relevance of certain kinds of quantitative psychophysical relationships.

Although the ideas in the article are highly mathematical, a deliberate effort has been made to keep mathematical concepts and notation to a minimum. For clarity, a mathematical formulation of some basic concepts and results discussed in the text is given in the Appendix. A formal presentation that incorporates the main ideas of the article can be found in Narens (1991).

Physical Measurement

Physical qualities are often classified into two types: *fundamental* and *derived*. Fundamental qualities (e.g., length, time, mass, charge, etc.) can be characterized qualitatively as structures of the form $\mathfrak{X} = \langle X, \geq_*, \oplus \rangle$, where $\geq_*$ is a total ordering on the physical quality X and $\oplus$ is an associative and monotonic operation on X that allows elements of X to be combined or concatenated to form other elements of X. In physics, fundamental qualities are also assumed to be *continuous* in the sense that $\langle X, \geq_* \rangle$ is a continuum, that is, isomorphic to the ordered structure of positive real numbers, $\langle \mathbb{R}^+, \geq \rangle$. In this article, structures like $\mathfrak{X}$ are called *continuous extensive structures,* and their exact qualitative description is given in the Appendix. All of these fundamental qualities—that is, all of these continuous extensive structures—are structurally identical in the sense that they are isomorphic. The fundamental quality $\mathfrak{X}$ is measured by a ratio scale of isomorphisms onto $\langle \mathbb{R}^+, \geq, + \rangle$. Derived physical qualities (e.g., density, energy, etc.) are measured in terms of certain products of powers of measurements of fundamental qualities. Exactly why this is the case is too complicated to present here, but the theory has been worked out in detail in Chapter 10 of Krantz, Luce, Suppes, and Tversky (1971) and Chapter 22 of Luce, Krantz, Suppes, and Tversky (1990).

A curious feature of physical measurement is that *a fundamental physical quality can be captured qualitatively by different continuous extensive physical qualities.* This odd fact is demonstrated in the following example involving the physical measurement of length:

A *measuring rod* is an infinitely thin, straight physical rod. Measuring rods x and y are said to be *equivalent* if and only if when placed side by side in the same direction with left endpoints corresponding, their right endpoints also correspond (see Figure 1a), and x is said to be *longer than* y if and only if when placed side by side in the same direction with left endpoints corresponding, the right end point of x goes beyond y (see Figure 1b).

In physics it is assumed that the aforementioned concept of equivalence is an equivalence relation (that is, is a reflexive, symmetric, and transitive relation). Let X be the set of equivalence classes of equivalent rods. Define the relation $\geq_*$ on X as follows: for all elements A and B of X,
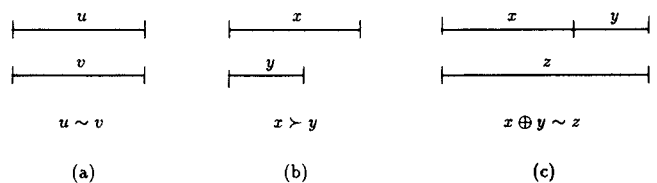

Figure 1. Extensive measurement of length.

$A \geq_* B$ if and only if there exist $a \in A$ and $b \in B$ such that a is longer than or equivalent to b.

Elements of X are called (*physical*) *length*.

Let X be the set of lengths. Classical physics assumes that $\langle X, \geq_* \rangle$ is a continuum (see Appendix for a precise definition of *continuum*). To appropriately measure length, more structure is needed on X. In the theory of physical measurement, this additional structure usually takes the form of adding the following concatenation operation to $\langle X, \geq_* \rangle$: Lengths A and B can be concatenated to form the length C, in symbols, $A \oplus B = C$, as

A $\oplus$ B = C if and only if for some x, y, and z in A, B, and C, respectively, z is equivalent to the rod that is formed by abutting x to y (see Figure 1c).

Classical physics assumes that $\langle X, \geq_*, \oplus \rangle$ forms a continuous extensive structure so that in particular $\oplus$ is an associative and monotonic operation. Physical length then can be measured by use of the following result, stated in modern terminology, by Helmholtz (1887): *Suppose $\mathfrak{X}_1 = \langle X_1, \geq_*, \oplus_1 \rangle$ is a continuous extensive structure and $\mathfrak{R} = \langle \mathbb{R}^+, \geq, + \rangle$. Then the set of isomorphisms of $\mathfrak{X}_1$ into $\mathfrak{R}$ forms a ratio scale.*

In the theory of measurement isomorphisms like those just mentioned are called *representations*, and this practice is used throughout the article. In the general literature they are usually called "scales," a term we prefer not to use because it is potentially confusable with concepts, such as "ratio scale," that refer to *sets of representations*.

In classical physics each fundamental quality (length, time, mass, etc.) is measured in terms of a ratio scale of representations of a continuous extensive structure for it. The measurements of other physical qualities (e.g., density, momentum, energy, and so on) are derived from the measurements of the fundamental ones, and their representations also form ratio scales. For example, the ratio scale customarily used to measure energy is the one that has representations of the form $k\varphi_M\varphi_D^2\varphi_T^{-2}$, where k is a positive real constant and φ_M, φ_D, and φ_T are representations for the fundamental qualities of mass, distance, and time, respectively. (The measurement–theoretic foundations for classical physical measurement have been systematically and comprehensively worked out in Chapter 10 of Krantz et al., 1971, and Chapter 22 of Luce et al., 1990; see these sources for a detailed presentation of many of the physical measurement concepts used in this article.)

It turns out that for classical physics, fundamental qualities can be appropriately measured in terms of different continuous extensive structures on the same attribute. Although this produces different ratio scales on the same physical attribute, it does not in any serious way affect the quantitative aspects of physics. Krantz, et al. (1971) wrote the following about this situation:

As Ellis (1966) pointed out, at least one other totally different interpretation of concatenation also satisfies the axioms [for a continuous extensive structure on lengths] and so leads to an additive representation; this measure of length is not linearly related to the usual one. Campbell (pp. 290–294 of 1957 edition) discussed other examples of a similar nature.

To present Ellis's interpretation we begin with a collection of rods. Let [the concatenation] $a*b$ be the hypotenuse of the right triangle whose sides are a and b. The comparison relation $\succsim$ is

determined by placing two rods side by side, with one end coinciding, and observing which one extends at the other end. Using properties of right triangles it is easy to verify that [the axioms of a continuous extensive structure] are satisfied. The only property that might present a slight difficulty is associativity. It is explained in Figure 2 where the lines are labeled by their lengths in the usual measure.

Since [the axioms of a continuous extensive structure are] satisfied, [Helmholtz' result obtains], hence there is a measure ψ that is order preserving and additive over this new concatenation. Since the usual measure φ is also order preserving, ψ and φ must be monotonically related, and by the properties of triangles it is easy to see that ψ is proportional to φ^2. To most people, the new interpretation seems much more artificial than the original one. In spite of this strong feeling, neither Ellis nor the authors know of any argument for favoring the first interpretation except familiarity, convention, and, perhaps, convenience. We are used to length being measured along straight lines, not along the hypotenuses of right triangles, but no empirical reasons appear to force that choice. Indeed, we could easily reconstruct the whole of physics in terms of ψ by replacing all occurrences of φ by $\psi^{\frac{1}{2}}$. This would make some equations appear slightly more complicated; others would be simpler. In fact, when φ^2 happens to be the more convenient measure, it is common to assign it a name and to treat it as the fundamental measure. Examples are the moment of inertia and the variance of a random variable. In the present case, if a and b are rods, the squares with side a and with side b can be concatenated by forming the square on the hypotenuse; φ^2 will be an additive (area) measure for such concatenation of squares (Krantz et al., 1971, pp. 87–88).

Let X be the set of lengths and $\oplus$ be the operation of concatenating lengths by abutting rods, and let $\oplus_2$ be the operation of concatenating lengths by the right triangle method described earlier. Let $\geq_*$ be the total ordering of lengths described earlier. Then by the aforementioned discussion, both $\mathfrak{X} = \langle X, \geq_*, \oplus \rangle$ and $\mathfrak{X}' = \langle X, \geq_*, \oplus_2 \rangle$ are continuous extensive structures. Let φ be a representation from $\mathfrak{X}$ onto $\langle \mathbb{R}^+, \geq, + \rangle$. Hence, it is easy to show that φ^2 is a representation from $\mathfrak{X}'$ onto $\langle \mathbb{R}^+, \geq, + \rangle$ and that for all x, y in X,

$$x \oplus_2 y = (\varphi^2)^{-1}[\varphi^2(x) + \varphi^2(y)].$$

From the point of view of qualitative concepts of meaningfulness (Luce, 1978; Luce et al., 1990; Narens, 1981, 1985, 1988, 1990, 1991), $\mathfrak{X}$ and $\mathfrak{X}'$ yield the same theories of meaningfulness because their automorphism groups (i.e., their groups of isomorphisms onto themselves) are identical; that is, they are *equally adequate for describing lawful physical phenomena*. The same can be said for the structures of the form $\mathfrak{X}_r = \langle X, \geq_*, \oplus_r \rangle$, where r is a positive real and $\oplus_r$ is defined by the following: for all x, y in X,
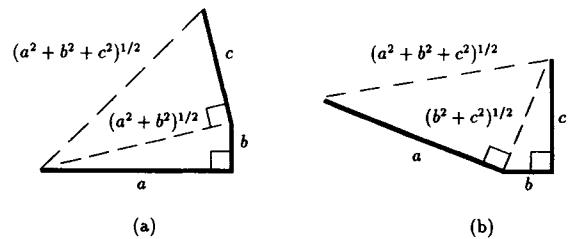

Figure 2. Orthogonal concatenation for length measurement illustrating left term (a) and right term (b) of the associative property.

$$x \oplus_r y = (\varphi^r)^{-1}[\varphi^r(x) + \varphi^r(y)].$$

In other words, for the theory of physics all the structures $\mathfrak{X}_r$ are equally adequate for the qualitative description of length. Furthermore, *theoretically* each of these structures is equally acceptable for the measurement of length through representations onto $\langle \mathbb{R}^+, \geq, + \rangle$. (Although from the *practical standpoint*, some of these, for example, $\mathfrak{X}_1$, may be preferable.) The same observation holds for other fundamental physical qualities. A similar observation obtains for derived physical variables (e.g., see the discussion of density in Krantz et al., 1971; Narens, 1991), but we do not go into it here.

This situation of having different but theoretically equivalent qualitative structures for measuring physical variables presents some difficulties for the interpretation of psychological concepts that are partially built out of physical measurements: Because from the point of view of theory, it is only by convention that a particular qualitative structure is used for measuring a physical variable, it should be demanded that substantive psychological conclusions on the basis of such measurements should remain the same when different but theoretically equivalent qualitative structures are used to measure the variable. In the following, this theoretical perspective is used to better understand the kinds of psychophysical sensitivity comparisons that can be made.

Psychophysical Structures and Weber Fractions

Psychophysical situations are usually characterized qualitatively in terms of qualitative primitive relationships that fall into two classes: (a) qualitative physical relationships and concepts, denoted by $P_1, \ldots, P_i, \ldots, P_m$, and (b) psychological relationships and concepts, denoted by $R_1, \ldots, R_j, \ldots, R_n$. In this article, we consider only those psychophysical situations that allow for such conceptual separations into physical and the psychological components.

The particular psychophysical situation that characterizes Weber's law allows for such a separation, and this case is later examined in some detail. Ideas and results developed for it extend easily to other psychophysical situations.

In the traditional psychophysical formulation of Weber's law, a physical continuum under consideration is *first* measured physically by some representation φ, and then *in terms of this physical measurement* a threshold for discriminability $\Delta_\varphi(x)$ is computed so that for all physical stimuli y, *the subject's behavior* shows y to be more intense psychologically than x (in symbols, $y \succ x$), if and only if $\varphi(y) > \varphi(x) + \Delta_\varphi(x)$, and finally, it is verified that there exists a positive real constant c such that for all x in the physical continuum under consideration,

$$\frac{\Delta_\varphi(x)}{\varphi(x)} = c. \tag{1}$$

Equation 1 is called *Weber's law*, its left-hand side the *Weber fraction*, and the real number c on its right-hand side the *Weber constant*. We make no assumptions about how the discrimination relation $\succ$ is obtained; in particular, for the purposes of this article it does not matter whether it is obtained through deterministic means or through probabilistic means. Weber's law stands at the very beginning of experimental psychology.

Weber's observation that all the senses, whose physical stimuli can be precisely measured on a one-dimensional physical scale, obeyed a uniform law, constituted the building-block on which Fechner grounded his well-known formula for the measurement of sensation. Weber's law, however, does *not refer to* (nonobservable) sensations, but only to (observable) sensitivities. Its character is therefore truly psychophysical; that is, it relates a purely physical entity and an observable psychological entity, namely, the threshold for discriminability.

In obtaining Weber's law, it is assumed that φ is a representation of a ratio scale of the physical continuous extensive structure $\langle X, \succ, \oplus \rangle$ onto $\langle \mathbb{R}^+, \geq, + \rangle$. (For the purposes of this article, there is no loss of generality in making this assumption, because (a) almost all the physical continua used in psychophysics—including derived ones such as energy—can be formulated as continuous extensive structures, and (b) arguments very similar to the ones below can also be made for general derived physical qualities.)

Narens (1980) characterized Equation 1 qualitatively in terms of a psychophysical situation. The following axiomatization closely resembles the one of Narens (1980; see also Suppes et al., 1990, Chap. 16) and is easily reducible to it.

The primitives consist of a nonempty set of objects X that is to be understood both as a set of physical objects and as a set of psychological stimuli (e.g., the elements of X as energy densities over the visible spectrum and as a set of lights to be presented to a subject). The other relations consists of a physical binary relation $\geq_*$ on X (that is used to physically order the stimuli), a physical concatenation operation $\oplus$ on X (that is used to physically add the stimuli), and a psychological binary relation $\succ$ on X (that is used to discriminate psychologically the stimuli in terms of intensity).

The first set of qualitative axioms are about the physical structure $\langle X, \geq_*, \oplus \rangle$ and says that $\langle X, \geq_*, \oplus \rangle$ is a continuous extensive structure (see Appendix). These axioms allow the physical stimuli to be measured by a ratio scale $\mathscr{S}$ in such a way that each representation in $\mathscr{S}$ interprets the qualitative physical relation $\geq_*$ as $\geq$ and the qualitative physical operation $\oplus$ as $+$.

The second set of qualitative axioms are about the psychological structure $\langle X, \succ \rangle$ and says $\succ$ is a *continuous semiorder* on X (see the Appendix for the formal definition of continuous semiorder; an informal definition is given shortly). Semiorders were introduced in the psychological literature by Luce (1956), and results about them have generally been confined to finite domains. In the present case, the semiorder of interest is on an infinite domain. (A far reaching analysis of semiorders on infinite domains can be found in Manders, 1981, and a systematic development of continuous semiorders can be found in Narens, 1990.) Intuitively, the expression "$x \succ y$" is to be interpreted as "the behavior of the subject discriminates the stimulus x as being (psychologically) more intense than the stimulus y." Empirically, $\succ$ may be implemented in a number of ways, for example, asking a subject if "x is noticeably brighter than y" or forcing the subject to choose the brighter of two stimuli and defining $x \succ y$ if and only if the probability of x as being chosen brighter than y is greater than r, where r is some specific number such that $0.5 < r < 1$, and so on.

One very important property of semiorders is that they are transitive (i.e., for all x, y, and z in Y, if $x \succ y$ and $y \succ z$, then

$x \succ z$). Letting "$x \not\succ y$" stand for "not $x \succ y$," one defines the indifference relation $\sim$ on Y as follows: For all x and y in X, $x \sim y$ if and only if $x \not\succ y$ and $y \not\succ x$. For semiorders $\sim$ is not in general a transitive relationship, that is, in general there will exist elements u, v, and w in X such that $u \sim v$ and $v \sim w$ but not $u \sim w$. Because $\sim$ is defined in terms of the psychological primitives X and $\succ$, it too can be considered a psychological relationship.

It can be shown (e.g., see Appendix) that the semiorder $\succ$ induces a weak ordering on the set X of psychological stimuli. That is, it follows from the semiorder axioms that the relationship $\geq_\Psi$ defined on X by the following: For all x and y in X,

$$x \geq_\Psi y \text{ iff } \forall z[(\text{if } y \succ z \text{ then } x \succ z) \text{ and } (\text{if } z \succ x \text{ then } z \succ y)]$$

(where "$\forall z$" is read as "for all z") is a transitive and connected relation. Because $\geq_\Psi$ is defined in terms of the psychological primitives, it is considered a psychological relationship.

A *continuous* semiorder $\langle X, \succ \rangle$ is a semiorder such that the induced ordered set $\langle X, \geq_\Psi \rangle$ is isomorphic to $\langle \mathbb{R}^+, \geq \rangle$. In particular, for continuous semiorders, $\geq_\Psi$ is a total ordering.

The next three axioms are about the *psychophysical* situation; that is, they involve both physical and psychological concepts and relationships. The first of these axioms is that the induced psychological ordering $\geq_\Psi$ is the physical ordering $\geq_*$, that is, that $\geq_\Psi = \geq_*$. The second is that for all x, y, u, and v in X,

$$\text{if } x \succ y \text{ and } u \succ v \text{ then } x \oplus u \succ y \oplus v.$$

The third psychophysical axiom is that for all x, y, u, and v in X,

$$\text{if } x \not\succ y \text{ and } u \not\succ v \text{ then } x \oplus u \not\succ y \oplus v.$$

Note that in the previous axiomatization, the last three axioms are about physical and psychological phenomena; that is, they are formulated in terms of both physical and psychological concepts. Because of this, they are called *psychophysical* axioms.

In the previous axiomatization, the primitive concept X, corresponding to the set of stimuli, can be given both psychological and physical interpretations. The primitive relation $\geq_*$ is to be interpreted as a physical relationship; however, because by a psychophysical axiom it is also the relation $\geq_\Psi$, it follows that $\geq_*$ is also interpretable as a psychological relationship, and correspondingly, $\geq_\Psi$ is interpretable as a physical relationship. However, Narens (1990) showed that (a) $\oplus$ is not qualitatively definable in terms of the psychological primitives X and $\succ$ even if extremely powerful logical languages are used for the defining, and (b) $\succ$ is definable in terms of the physical primitives X, $\geq_*$, and $\oplus$ in a logical language.

In the previous axiomatization, the physical primitives and axioms are used to measure the physical stimuli—that is, to represent X numerically; the psychological primitives and axioms are used to describe the psychological behavior, and the psychophysical axioms are used to describe how the physical and psychological primitives interact.

The Weber constant c in Equation 1 is invariant under physical changes of unit; that is, it is physically meaningful. By results of Narens (1988) this implies that it is logically equivalent to an expression formulated in terms of physical primitives, and

thus, it may be interpreted as a physical concept. Results of Narens (1990) show that c is not interpretable as a purely psychological concept; that is, there is no concept formulable purely in terms of the psychological primitives that correspond to the constant c. Other results of Narens (1990) show that the constant $1 + c$ is interpretable as a purely psychological concept. A detailed analysis of this strange result about Weber constants is beyond the scope of this article. However, the following gives some intuitive reasons why one might expect the result:

Let T be the function from X into X that is defined as follows: For all x and y in X,

$$y \succ x \text{ iff } y >_\Psi T(x),$$

where $\geq_\Psi$ is the ordering defined earlier. Because $\geq_\Psi$ is defined in terms of psychological primitives, the previous equation gives a definition of T in terms of psychological primitives. Because it easily follows from the definition of T and Equation 1 that

$$\varphi[T(x)] = (1 + c) \cdot \varphi(x),$$

T may be viewed as the interpretation of $1 + c$. Thus, $1 + c$ is psychologically relevant. We now argue that c is not psychologically relevant. We first note that T is related to the function Δ_φ by the following formula:

$$\Delta_\varphi(x) = \varphi[T(x)] - \varphi(x). \qquad (2)$$

The qualitative interpretation of $c \cdot \varphi(x)$ is a function $C(x)$ from X into X. (Because to this, we consider the function C to be the interpretation of c.) Then it follows by Equation 1 that $C(x)$ is the interpretation of $\Delta_\varphi(x)$. Therefore, because by Equation 2 $\Delta_\varphi(x) = T^*[\varphi(x)] - \varphi(x)$ where $T^*[\varphi(x)] = \varphi[T(x)]$, it follows that C is the interpretation of the *difference* of two functions of physically measured stimuli. Intuitively, the measurement theory of differences (e.g., see Krantz, et al., 1971, Chap. 4) should require much stronger properties than those inherent in continuous semiorders—or put differently, differences should not be meaningful for continuous semiorders. Thus, intuitively, one would not expect all differences of functions of $\varphi(u)$ to be meaningful for continuous semiorders. Technical results of Narens (1991) show that the *particular* difference $\varphi[T(x)] - \varphi(x)$ is not meaningful for continuous semiorders. By results discussed in the Appendix relating meaningfulness and definability, it then follows that the interpretation of the difference $T^*[\varphi(x)] - \varphi(x)$, that is, the interpretation of c, is not definable in terms of the psychological primitives.

In summary, the usual definition of the Weber constant uses both psychological and physical concepts, and there are other definitions of it in purely physical terms. However, there is no definition of it in purely psychological terms.

The Equivalence Principle

In the following definition of *physically equivalent* various concepts of "definability" may be used. For concreteness, we assume the one described by Narens (1988). There, qualitative objects, relations, relations of relations, and so forth may be defined in terms of finitely many other qualitative objects, relations, relations of relations, and so forth through formulae of a

powerful logical language (one of equivalent power to the language developed by Whitehead and Russell, 1925, in their *Principia Mathematica*) using finitely many parameters that are purely mathematical or purely logical objects, relations, and so forth. (The addition of the purely mathematical and logical parameters is tantamount to adding individual constant symbols for each purely mathematical and each purely logical object.) This definability concept corresponds to the kind of "definability" used in ordinary mathematical science and allows for the use of mathematical methods in defining a qualitative concept in terms of other qualitative concepts. For example, consider a continuous extensive structure $\mathfrak{X}$ measured in terms of a ratio scale $\mathscr{S}$ onto $\langle \mathbb{R}^+, \geq, + \rangle$, and let r be a positive real number. Then the qualitative binary relation R is properly defined in terms of the primitives of $\mathfrak{X}$ as follows: for each x and y in $\mathfrak{X}$ $R(x, y)$ if and only if for each φ in $\mathscr{S}$, $\varphi(x) = r\varphi(y)$. (Note that the purely mathematical parameters $\mathbb{R}^+, \geq, +,$ and r are used to define R.) It is beyond the scope of this article to give here a formal description of this "definability" concept. The interested reader should consult Narens (1988, 1991).

Two physical situations on the same domain, $\mathfrak{X} = \langle X, P_1, \ldots, P_i, \ldots, P_n \rangle$ and $\mathfrak{X}' = \langle X, P_1', \ldots, P_i', \ldots, P_n' \rangle$, are said to be *physically equivalent* if and only if

1. $\mathfrak{X}$ and $\mathfrak{X}'$ are isomorphic;
2. each P_i', $i = 1, \ldots, n$, is definable in terms of $P_1, \ldots, P_n$;
3. each P_i, $i = 1, \ldots, n$, is definable in terms of $P_1', \ldots, P_n'$.

Condition 1 of the previous definition says that $\mathfrak{X}$ and $\mathfrak{X}'$ are structurally identical; and Conditions 2 and 3 say that from the point of view of theory, the relationships $P_1, \ldots, P_n$ are just as physically relevant as the relationships $P_1', \ldots, P_n'$.

The two continuous extensive structures, $\langle X, \geq_*, \oplus \rangle$ and $\langle X, \geq_*, \oplus_2 \rangle$ discussed earlier for the measurement of length are physically equivalent. (See discussion at the end of the Appendix.) More generally, if $\mathfrak{Z} = \langle Z, \geq_*, \otimes \rangle$ and $\mathfrak{Z}' = \langle Z, \geq_*, \otimes' \rangle$ are physical continuous extensive structures and φ is a representation of $\mathfrak{Z} = \langle Z, \geq_*, \otimes \rangle$ onto $\langle \mathbb{R}, \geq, + \rangle$, then $\mathfrak{Z}$ and $\mathfrak{Z}'$ are physically equivalent if and only if there exists r in $\mathbb{R}^+$ such that for all x and y in Z,

$$\varphi(x \otimes' y) = [\varphi(x)^r + \varphi(y)^r]^{1/r}, \qquad [1]$$

which is in perfect agreement with the idea of "equally adequate for describing lawful physical phenomena" discussed in the Physical Measurement section. (For proofs of this result see Narens, 1991, and the discussion at the end of the Appendix.)

For the purposes of this article, the critical difference between *physically equivalent* and *equally adequate for describing lawful physical phenomena* is that the former is a logical concept, whereas the latter is a physical concept that is at the root of a particular physical theory (dimensional analysis). That these two concepts describe the same physical situation is a deep mathematical theorem. Because of its close connection with logic, *physically equivalent* is the superior concept for formulating and justifying the epistemological ideas that form the core of this article.[1]

The equivalence principle is a theory about the relationship of the physical and the psychological in psychophysics. It is interesting to note that this relationship is asymmetric—equivalent structures can be used for describing the underlying physical situation but not the underlying psychological situation.

The reason for this is that a primary goal of psychophysics is the characterization of (purely) psychological phenomena in terms of quantitative relationships among the measurements of physical variables. Formally, such psychological phenomena are viewed as *particular* qualitative relationships, which in our formulation are either primitives of the psychological structure or relationships that are definable in terms of the primitives. Because psychophysics is interested in statements about such *particular* psychological relationships, it makes no sense to demand that the truth values of these statements be invariant under substitution of psychologically equivalent relationships. *Particular* physical relationships are not of interest in psychophysics; instead they are used as vehicles for characterizing psychological relationships, and the equivalence principle is a description of their role as vehicles.

Suppose $\mathfrak{X} = \langle X, P_1, \ldots, P_n \rangle$ is a physical structure, $\mathfrak{Y} = \langle X, R_1, \ldots, R_m \rangle$ is a psychological structure, and $\mathfrak{X}$ and $\mathfrak{Y}$ describe a psychophysical situation. Then the equivalence principle states that a *necessary* condition for a qualitative or *quantitative* assertion to have psychological relevance is that its truth remain invariant under changes of physically equivalent structures; that is, if a structure $\mathfrak{X}' = \langle X, P_1', \ldots, P_m' \rangle$ that is physically equivalent to $\mathfrak{X}$ were substituted for $\mathfrak{X}$ in the formulation of the assertion, then the truth value of the assertion would not change.

The intuitive reasoning behind the equivalence principle is that psychological relevance should not depend on which of the physically equivalent structures the experimenter decides to use in describing the psychophysical situation. A more complete rationale is given in the Status of the Equivalence Principle section.

It should be emphasized that the concepts of *psychophysical*, (purely) *psychological*, and *psychologically relevant* used in this article are to be understood as technical concepts about statements about a psychophysical situation. In particular, we do not intend this terminology to imply that scientists engaged in psychophysical research are "not doing psychology," nor do we intend this terminology to imply that nonpsychologically relevant statements are not of value to the science of psychology.

Next, consequences and illustrations of the equivalence principle are presented for the psychophysical situations involving Weber's law. (Most of the observations made about this specialized situation easily extend to other psychophysical situations.)

Weber's Law in Single Modality Situations

Let $\mathfrak{X} = \langle X, \geq_*, \oplus \rangle$ and $\mathfrak{Y} = \langle X, \succ \rangle$ describe the previous Weber's law situation with the previous physical, psychological,

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[1] In addition, *equally adequate for describing lawful physical phenomena* is a somewhat flawed concept for psychophysical application: In dimensional analysis, it results from considerations about specific kinds of interactions between several physical quantities, whereas psychophysical situations generally involve either a single physical quantity or a few physical quantities that do not interact physically in the psychophysical situation. (Distinct physical quantities in a psychophysical cross modality matching experiment is an example of the latter.) For *equally adequate for describing lawful physical phenomena* to become an acceptable concept for psychophysical application, a theorem like the one above linking it to another concept like *physically equivalent* is needed.

and psychophysical axioms. Then, as is shortly discussed, the Weber constant being a particular size, for example, $c = 2$, is not psychologically relevant by the equivalence principle. However, Weber's law, formulated as,

"There exists a constant c such that $\dfrac{\Delta_\varphi(x)}{\varphi(x)} = c$,"

satisfies the equivalence principle and, as is discussed in Narens (1990), has a purely psychological interpretation. It is the identification of c with the number 2 that conflicts with the equivalence principle. To see why this is the case, the "2-ness" in the expression "$c = 2$" needs to be investigated.

Rewriting Weber's law as

$$\Delta_\varphi(x) = c\varphi(x),$$

it is seen that in this form of Weber's law, c can be viewed as a function of the values of the physical representation φ, namely, the function that is multiplication by a real constant, which by convention is also called c. Qualitatively, this function corresponds to a function γ from X onto X. For "$c = 2$," the "2-ness" of the qualitative function γ can be given a simple interpretation in terms of the *physical* structure, namely, $\gamma(x) = x \oplus x$ for all x in X (see Narens, 1990, for a detailed discussion). Thus, in terms of the physical structure, the "2-ness" consists of *doubling* the size of stimulus items with respect to the concatenation operation $\oplus$. (As mentioned before, $\oplus$ cannot be defined purely in terms of psychological primitives.) Because this concept of 2-ness depends on the particular physical operation $\oplus$, it conflicts with the equivalence principle. Furthermore, as is shown in Narens (1990, 1991), there are no operations definable in terms of the psychological primitives that could be used to give the "2-ness" in the expression "$c = 2$" a psychological interpretation.

(The same argument shows that the "3-ness" in the expression "$1 + c = 3$" has no psychological interpretation—even though, as mentioned earlier, $1 + c$, unlike c, has a purely psychological interpretation.)

Suppose $\succ^*$ is another psychological discrimination relation on X, for example, $\succ$ is one subject's discrimination on X and $\succ^*$ is a different subject's discrimination on X, where both subjects are performing the same discrimination task. Let c be the Weber constant associated with $\mathfrak{X}$ and $\succ$ and c^* the Weber constant associated with $\mathfrak{X}$ and $\succ^*$. Then $c > c^*$ satisfies the equivalence principle (see Appendix). It is also psychologically relevant, because (as is shown in the Appendix)

$c > c^*$ iff $\forall x \forall y (\text{if } x \in X \text{ and } y \in X,$

$\text{then } x \succ y \text{ implies that } x \succ^* y),$

is a true assertion, and the statement on its right-hand side of the iff is formulated purely in terms of psychological primitives and captures a theoretically correct comparison of psychological sensitivity.

Note that in the first example about an assertion specifying the magnitude of the Weber constant, the equivalence principle was used to decide that the assertion had no psychological relevance. This can be done because invariance under physically equivalent structures is assumed to be a *necessary* condition for psychological relevance. In the second example, which com-

pares the magnitudes of Weber constants for two discrimination relationships, the equivalence principle is satisfied. This alone is not enough to establish the psychological relevance of the comparison, because the equivalence principle is only a necessary condition. One still must understand something of the psychological situation to see the relevance—for example, in the second example understand that the comparison implicit in the statement given on the right-hand side of the iff is one possible way to measure psychological sensitivity.

Let $\succ$, $\succ^*$, c, and c^* be as in the previous equation and consider the following assertion: $1 + c^* = (1 + c)^2$. It can be shown (Narens, 1991) that this assertion satisfies the equivalence principle and that it is logically equivalent to an assertion formulated completely in terms of psychological primitives. However, unlike the first example where the "2-ness" in "$c = 2$" could *only be specified by reference to a function defined in physical* terms, the "2-ness" (of the exponent) in the assertion "$1 + c^* = (1 + c)^2$" can be specified purely in psychological terms (see Narens, 1991, for discussion and proof).

It is not difficult to show that the previous psychophysical axioms for Weber's law satisfies the equivalence principle in the following sense: If the physical situation $\mathfrak{X}' = \langle X, \geq_*, \oplus' \rangle$ that is physically equivalent to $\mathfrak{X}$ was used in place of X in the axioms, then the truth values of the physical and psychophysical axioms in terms of $\mathfrak{X}'$ would be the same as those of $\mathfrak{X}$. Because of this, Weber's law satisfies the equivalence principle, and this is reflected in the fact that the quantitative formulation of Weber's law,

$$\exists c \left[c \in \mathbb{R} \text{ and } \forall x \left(\text{if } x \in X, \text{ then } \frac{\Delta_\varphi(x)}{\varphi(x)} = c \right) \right],$$

(where $\exists$ stands for "there exists") satisfies the equivalence principle.

Weber's Law in Multiple Modality Situations

A particularly attractive feature of the Weber constant is that it is dimensionless, and thus makes no reference to the physical dimension from which it was derived. This suggests using ordinal comparisons of Weber constants for intermodal comparison of sensitivities. This idea can be traced back to at least Wundt (1908). Since then it can be found in the majority of textbooks on psychophysics, where sometimes it is suggested implicitly by presenting tables of Weber constants for different modalities according to magnitude, and other times spelled out explicitly. We quote three typical examples of the latter: Engen (1971, p. 19) stated that "the smaller the Weber fraction the keener the sense." Baird and Noma (1978, p. 43) argued that because the Weber ratio is dimensionless "one can compare sensitivities for different continua." In the same spirit, Coren and Ward (1989) said the following:

> Note that K (the Weber fraction) has no units (such as grams), so it does not depend on the physical units to measure I and ΔI. Thus, we can compare Weber fractions across different stimulus dimensions without having to worry about how the stimulus values were measured. (p. 36)

In contrast, it is argued here that such comparisons of Weber constants do not make psychological sense *unless additional psychological structure and assumptions are present*.

Consider the case of two Weber law structures $\mathfrak{W}_1$ and $\mathfrak{W}_2$ arising from a single subject on two separate modalities, each involving a different physical dimension. $\mathfrak{W}_1$ is composed of the physical structure $\mathfrak{X}_1 = \langle X_1, \geq_{*1}, \oplus_1 \rangle$ and the psychological structure $\langle X_1, \succ_1 \rangle$ and satisfies the qualitative axioms for Weber's law given earlier; and $\mathfrak{W}_2$ is composed of the physical structure $\mathfrak{X}_2 = \langle X_2, \geq_{*2}, \oplus_2 \rangle$ and the psychological structure $\langle X_2, \succ_2 \rangle$ and satisfies the qualitative axioms for Weber's law given earlier. Let $\geq_{\Psi_1}$ be the total ordering induced by $\succ_1$, and $\leq_{\Psi_2}$ be the total ordering induced by $\succ_2$. Also let φ_1 and φ_2, respectively, be representations from $\mathfrak{X}_1$ and $\mathfrak{X}_2$ onto $\langle \mathbb{R}^+, \geq, + \rangle$, and let c_1 be the Weber constant associated with $\mathfrak{W}_1$ and φ_1, and c_2 be the Weber constant associated with $\mathfrak{W}_2$ and φ_2. Then $c_1 > c_2$ is a physically meaningful assertion. However, unlike the single modality case considered earlier, it does not satisfy the equivalence principle, because the value of the Weber constant of $\mathfrak{W}_1$ changes with respect to structures that are physically equivalent to $\langle X_1, \geq_{*1}, \oplus_1 \rangle$, and all positive values are realizable by appropriate choices of physically equivalent structures.

The upshot of this is that for psychological purposes, one should not compare the order of Weber fractions across modalities on the basis of different physical dimensions *unless additional psychological primitives and axioms are assumed.* A little psychological theorizing leads to the same conclusion: One begins by asking what is needed to appropriately compare psychological sensitivity measures across modalities. In our view there are two obvious, closely interrelated answers:

The first assumes a model where (a) the subject has a single mental dimension D of subjective intensity, (b) stimuli from different physical dimensions have associated with them intensity values on D, and (c) the subject judges stimulus a to be subjectively less than or equally intense than stimulus b, in symbols $a \leq_. b$, if and only if his or her D-value for a is less than or equal to his or her D-value for b. In the present context this can be formulated as follows:

The physical structures $\mathfrak{X}_1$ and $\mathfrak{X}_2$ have been measured appropriately by the representations φ_1 and φ_2, psychophysical functions ψ_1 and ψ_2 exist that measure the psychological intensity of stimuli from X_1 and X_2 such that stimuli from $X_1 \cup X_2$ can be compared psychologically by comparing their values under the functions $\psi_1 * \varphi_1$ and $\psi_2 * \varphi_2$ (where $*$ is the operation of function composition); that is, there is a psychological ordering $\leq_.$ on $X_1 \cup X_2$ such that for all x and y in $X_1 \cup X_2$,

$$x \leq_. y \text{ iff}(x \leq_{*1} y \text{ or } x \leq_{*2} y \text{ or } \psi_1[\varphi_1(x)] \leq \psi_2[\varphi_2(y)]$$

$$\text{or } \psi_2[\psi_2(x)] \leq \psi_1[\psi_1(y)]).$$

(ψ_1 and ψ_2 are assumed to be strictly increasing functions into $\mathbb{R}^+$.) Then in terms of the psychophysical functions ψ_1 and ψ_2, the obvious psychological way of stating that $\succ_1$ is more sensitive than $\succ_2$ is that for each x in X_1 and y in X_2, if $\psi_1[\varphi_1(x)] = \psi_2[\varphi_2(y)]$ then $\psi_1[\Delta_{\varphi_1}(x)] < \psi_2[\Delta_{\varphi_2}(y)]$. In terms of the Weber constants c_1 and c_2 this becomes $\succ_1$ is more sensitive than $\succ_2$ if and only for each x in X_1 and y in X_2, if $\psi_1[\varphi_1(x)] = \psi_2[\varphi_2(y)]$ then $\psi_1[c_1\varphi_1(x)] < \psi_2[c_2\varphi_2(y)]$. Thus, in this case, where there is a definite understanding of psychological sensitivity, and it is clear that this understanding cannot in general be captured by ordinal comparisons of Weber fractions.

The second answer assumes that stimuli from X_1 and X_2 can

be compared by a cross-modality matching relation M. (Henceforth, the notation xMy is used for $M(x, y)$, and xMy is to be read as x matches y.) The proper interpretation of cross-modality matching is a subtle matter (e.g., see Krantz, 1972; Luce, 1990; Shepard, 1981, for theoretical discussions and Ward, 1990, for empirical results).

Assume that M has been added to the psychophysical situation as a psychological primitive relationship. Also assume the following three minimal psychological axioms: (a) for each x in X_1 there exists y in X_2 such that xMy, (b) for each y in X_2 there exist x in X_1 such that xMy, (c) for all x and x' in X_1 and all y and y' in X_2, if xMy and $x'My'$, then

$$x \geq_{\Psi_1} x' \text{ iff } y \geq_{\Psi_2} y'.$$

Then the following is the obvious *psychological* way of comparing sensitivity: $\succ_1$ is more sensitive than $\succ_2$ if and only if for all x and x' in X_1 and all y and y' in X_2, if xMy and $x'My'$ and $y \succ_2 y'$ then $x \succ_1 x'$. Luce (1990) described in detail compatibility axioms—which in the present case are easily convertible into psychophysical axioms—that imply the following result: There exist a positive real number r such that for all x in X_1 and y in X_2,

$$xMy \text{ iff } \varphi_1(x) = \varphi_2(y)^r. \tag{3}$$

Assuming these psychophysical axioms, the following is easily shown: $\succ_1$ is more sensitive than $\succ_2$ in the sense just described if and only if $1 + c_1 < (1 + c_2)^r$, where r is as given in Equation 3. From this it is clearly seen that in the previous situation the ordinal comparison of Weber fractions is not what is wanted, and a slightly more complicated comparison is needed.

Equation 3 satisfies the equivalence principle, and it can be shown (see Narens, 1990) that the assertion $1 + c_1 < (1 + c_2)^r$ is logically equivalent to a statement formulated purely in terms of psychological primitives.

Other Applications

The equivalence principle usually forces new interpretations to be given to familiar psychophysical results. Two examples of this are briefly described.

The first is what McGill and Goldberg (1968) call the near miss to Weber's law for discrimination of intensity of pure tones of a given frequency, which quantitatively is expressed as

$$\frac{\Delta_\varphi(x)}{\varphi(x)^{1-\delta}} = c, \tag{4}$$

where Δ_φ, φ, and c have the same meanings as in Weber's law, and δ is a small positive real number (hence the use of the modifier "near miss"), which is independent of frequency. (Note that c is a dimensional constant; that is, as the positive real r varies, c varies with $r\varphi$.) The near-miss version describes sensory discrimination data better than Weber's law. However, *given only such facts,* should any psychological significance be attached to Equation 4? By the equivalence principle the answer is no. Without going into details, here are the reasons: Let $\mathfrak{X} = \langle X, \geq_*, \oplus \rangle$ be the physical structure for which φ is an isomorphism onto $\mathfrak{R} = \langle \mathbb{R}^+, \geq, + \rangle$, and let $\mathfrak{Y} = \langle X, \geq', \oplus' \rangle$ be a structure that is physically equivalent to but different from $\mathfrak{X}$,

and μ be an isomorphism of $\mathfrak{Y}$ onto $\mathfrak{N}$. Then it can be shown that for all $0 < \gamma < 1$ and all positive d an element x in X exists such that

$$\frac{\Delta_\mu(x)}{\mu(x)^{1-\gamma}} \neq d,$$

and thus Equation 4 does not satisfy the equivalence principle. The authors take it as a necessary condition for the psychological relevance of Equation 4 that a statement $\ominus$, formulated in some sufficiently powerful logical language in terms of the psychological primitives X and $\succ$, can be added consistently to the psychological axioms (that say $\langle X, \succ \rangle$ is a continuous semiorder) such that the following assertion is implied by this expanded system of axioms:

Let $\mathfrak{W} = \{ \beta | \beta$ is physically equivalent to $\mathfrak{X} \}$. Then there exist β in $\mathfrak{W}$, an isomorphism τ of β onto $\mathfrak{N}$, $0 < \sigma < 1$, and $b > 0$ such that for all z in the domain of β,

$$\frac{\Delta_\tau(z)}{\tau(z)^{1-\sigma}} = b.$$

However, if the previous assertion were true, then it would follow from results of Narens (1988) that $\oplus$ is definable in terms of the psychological primitives. However, Narens (1990) showed that $\oplus$ is not definable in terms of $\langle X, \succ \rangle$.

The second example concerns a popular method for making intermodal comparisons of sensitivity on the basis of magnitude estimation data. Stevens and others have carried out hundreds of experiments on a wide variety of physical continua that have produced psychophysical power functions that relate standard physical measurements φ to subjects' magnitude estimations ψ of a continuum, that is, have produced functions ψ such that there exists a positive real number r, called the exponent, and a positive real number a such that for all x of the continuum,

$$\psi(x) = a\varphi(x)^r.$$

In analyzing such experiments (e.g., see Stevens, 1971, 1974), the following index of sensitivity is often used:

A subject is more sensitive to Continuum 1 than Continuum 2 if and only if the exponent of his or her psychophysical power function associated with Continuum 1 is less than the exponent of his or her psychophysical function associated with Continuum 2.

Because the choice of the physically equivalent structure used for measuring a continuum does not influence the subjects' magnitude estimations of that continuum but does influence the size of the exponent of the resulting psychophysical power function, it follows by the equivalence principle that the previous index of sensitivity does not have psychological relevance.

Both the method of magnitude estimation and the previous index of sensitivity have generated much criticism. To counter the criticism, Stevens and others used cross-modality matching techniques as an independent check of the validity of the exponents. This check is called "the method of transitivity of scales."

As before, assume M is a psychological matching relationship on two different physical continua $\langle X_1, \geq_1 \rangle$ and $\langle X_2, \geq_2 \rangle$, and assume psychophysical axioms so that for standard physical measurements φ and γ on X_1 and X_2, respectively, there exist

positive real numbers t and p such that for all x in X_1 and y in X_2,

$$xMy \text{ iff } \varphi(x) = p\gamma(y)^t.$$

Also assume that magnitude estimations have been made on both continua, and these result in psychophysical power functions associated with X_1 and X_2 that have, respectively, exponents r and s. Then transitivity of scales is said to hold if and only if

$$t = \frac{s}{r}.$$

It is not difficult to show that transitivity of scales is consistent with the equivalence principle. However, even if it holds, the ordinal comparison of exponents of psychophysical power functions associated with different continua is still by the equivalence principle not a valid psychological index of sensitivity.

(It should be noted that although most psychophysical researchers have ignored the potential problems involved in the choice of scale of the physical variable in determining psychophysical power functions, a few (e.g., Krueger, 1991; Myers, 1982; Weiss, 1981, 1989), have directly commented on this issue and noted that because of it, many kinds of psychophysical comparisons across modalities are not properly founded. Our methods for reaching similar conclusions are different, are more rigorously founded in theory, and are based exclusively on how the psychological and the physical are related using the equivalence principle.)

Status of the Equivalence Principle

The equivalence principle is a theory about the relationship of the psychological and the physical. In our opinion, it is not about empirical matters, and in particular, it cannot be refuted by experiment. Nevertheless, as is argued here, it is a very useful principle.

Let $\mathfrak{X} = \langle X, P_1, \ldots, P_n \rangle$ be a physical structure, $\mathfrak{Y} = \langle X, R_1, \ldots, R_m \rangle$ be a psychological structure, $\mathcal{A}_1$ the psychological axioms about $\mathfrak{Y}$, $\mathcal{A}_2$ the physical axioms about $\mathfrak{X}$, $\mathcal{A}_3$ the psychophysical axioms about

$$\langle X, P_1, \ldots, P_n, R_1, \ldots, R_m \rangle,$$

and $\mathfrak{X}' = \langle X, P_1', \ldots, P_n' \rangle$ be a structure that is physically equivalent to $\mathfrak{X}$. Suppose $\mathfrak{X}'$ is substituted for $\mathfrak{X}$. Then, because none of the psychological axioms $\mathcal{A}_1$ involve $P_1, \ldots, P_n$, they remain true under this substitution. The physical axioms remain true under this substitution because $\mathfrak{X}$ and $\mathfrak{X}'$ are isomorphic. The psychophysical axioms $\mathcal{A}_3$ need not remain true under this substitution. However, because $P_1, \ldots, P_n$ are definable in terms of $X, P_1', \ldots, P_n'$ through a logical language $\mathcal{L}$, psychophysical axioms $\mathcal{A}_3'$ can be formulated in $\mathcal{L}$ in terms of the primitives $X, P_1', \ldots, P_n'$ that impose exactly the same substantive restrictions as $\mathcal{A}_3$ by using the definitions of $P_1, \ldots, P_n$, and axioms $\mathcal{A}_3$. Thus, axioms $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_3$ make the same assertions about

$$\langle X, P_1, \ldots, P_n, R_1, \ldots, R_m \rangle$$

as (with appropriate substitutions) axioms $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}'_3$ do about

$$\langle X, P'_1, \ldots, P'_n, R_1, \ldots, R_m \rangle.$$

Thus, *every proper conclusion about the state of affairs described by*

$$\langle X, P_1, \ldots, P_n, R_1, \ldots, R_m \rangle$$

and axioms $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ *must be a proper conclusion about*

$$\langle X, P'_1, \ldots, P'_n, R_1, \ldots, R_m \rangle$$

and $\mathcal{A}_1$, $\mathcal{A}_2$ (*with appropriate substitutions*), *and* $\mathcal{A}'_3$, *and vice versa*. In particular, *the equivalence principle must be true*.

At first sight, the previous use of physically equivalent structures appears to be quite general and therefore imposes very little in the way of restrictions. However, this is not the case:

First, the earlier argument only applies to a special kind of psychophysical situation: one in which the psychology and physics have a common domain and can be divided into separate structures. This kind of limitation excludes from consideration psychophysical structures that have primitive relationships that are neither completely psychological nor completely physical.

Second, as previous examples have shown, the powerful uses of the equivalence principle concern quantitative psychophysical statements. For various reasons, psychophysics has found it desirable to form quantitative models by first representing the physical structure numerically, and then basing quantitative psychological and psychophysical concepts on these physical measurements. (Logically, one could have preceded by first representing the psychological structure numerically and basing quantitative physical and psychophysical concepts on psychological measurements.) Because of this practice, the equivalence principle has bite, because each pure number different from 1, that is, each ratio of physically measured stimuli different from 1, has a psychological meaning that depends on the physically equivalent structure used to measure the stimulus, and in general, this meaning changes with the physically equivalent structure. The implication of this for psychophysics is that one has to be very careful in using pure numbers in formulating theoretical concepts, particularly comparing pure numbers that arose from different physical dimensions (a practice that is neither needed nor engaged in classical physics).

Situations where the truth of the psychophysical axioms $\mathcal{A}_3$ remains invariant under substitution of physically equivalent structures deserve special note. (The psychophysical axioms of Weber's law as given earlier is an example of such an $\mathcal{A}_3$.) In such situations the psychophysical axioms may be said to describe a psychophysical law. This kind of "law" is more general than the kinds of laws described in the measurement literature (e.g., Falmagne & Narens, 1983; Luce, 1959, 1964, 1990; Roberts & Rosenbaum, 1986), but it is beyond the scope of this article to go deeply into this issue here. A discussion of it is given in Narens (1991).

Conclusion

Many psychophysical concepts and relationships can be formulated qualitatively in terms of psychophysical structures. The quantification of these concepts then proceeds by measuring the physical variables by usual methods and representing the psychological concepts and relationships in terms of the physical measurements. In such situations the sorting out of the quantitative formulas and statements that have psychological significance is often problematic. Although the measurement–theoretic concept of meaningfulness provides some help in accomplishing this, additional methods are needed. The equivalence principle, which is based on a theory of how the physical and psychological are related in psychophysics, is proposed as such a method. In cases involving comparing psychological sensitivities on the basis of Weber fractions, it appears to be a useful tool for the elimination of various meaningful concepts and relationships as having psychological relevance. Although the examples in this article focused on psychophysical situations involving Weber fractions, the basic concepts and methods used can be applied to a variety of other psychophysical situations (e.g., see Roskam, 1989, where ideas similar in spirit to the equivalence principle have been applied to various kinds of psychological phenomena.)

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(Appendix follows on next page)

Appendix

Definitions and Theorems

Definition 1: $\langle X, \geq' \rangle$ is said to be *totally ordered* if and only if X is a nonempty set and $\geq'$ is a transitive, connected, and antisymmetric relation on X.

Definition 2: Let $\langle X, \geq' \rangle$ be a totally ordered structure.

$\langle X, \geq' \rangle$ is said to be *dense* if and only if for each x and z in X, if $x >' z$, then for some y in X, $x >' y >' z$.

$\langle X, \geq' \rangle$ is said to satisfy *denumerable density* if and only if there is a denumerable subset Y of X such that for all x and z in X, if $x >' z$, then for some y in Y, $x \geq' y \geq' z$.

$\langle X, \geq' \rangle$ is said to be *Dedekind complete* if and only if each non-empty subset of X that is bounded above has a least upper bound in X.

Definition 3: $\langle X, \geq' \rangle$ is said to be a *continuum* if and only if the following six statements are true:

1. $X \neq \emptyset$.
2. $\geq'$ is a total ordering on X.
3. $\langle X, \geq' \rangle$ has neither greatest nor least element.
4. $\langle X, \geq' \rangle$ is dense.
5. $\langle X, \geq' \rangle$ is denumerably dense.
6. $\langle X, \geq' \rangle$ is Dedekind complete.

Definition 4: $\mathfrak{X} = \langle X, \geq_*, \oplus \rangle$ is said to be a *continuous extensive structure* if and only if the following five statements are true:

1. *Continuous ordering:* $\langle X, \geq_* \rangle$ is a continuum.
2. *Associativity:* $\oplus$ is a binary operation that is *associative;* that is,

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$

for all x, y, and z in X.

3. *Monotonicity:* For all x, y, and z in X,

$$x \geq_* y \text{ iff } x \oplus z \geq_* y \oplus z \text{ iff } z \oplus x \geq_* z \oplus y.$$

4. *Solvability:* For all x and y in X, if $x >_* y$, then for some z in X, $x = y \oplus z$.
5. *Positivity:* $x \oplus y >_* x$ and $x \oplus y >_* y$, for all x and y in X.

In essence Helmholtz (1887) showed the following theorem:

Theorem 1: Suppose $\mathfrak{X} = \langle X, \geq_*, \oplus \rangle$ is a continuous extensive structure. Then the following two statements are true:

1. The set $\mathscr{S}$ of isomorphisms of $\mathfrak{X}$ into $\mathfrak{N} = \langle \mathbb{R}^+, \geq, + \rangle$ is a *ratio scale*, that is, (a) $\mathscr{S} \neq \emptyset$, (b) for each φ in $\mathscr{S}$ and each r in $\mathbb{R}^+$, $r\varphi$ is in $\mathscr{S}$, and for each φ and ψ in $\mathscr{S}$, there exists s in $\mathbb{R}^+$ such that $\psi = s\varphi$.
2. Each element of $\mathscr{S}$ is onto $\mathfrak{N}$

Proofs of generalizations of Theorem 1 can be found in Krantz et al. (1971, Chap. 3) and in Narens (1985, Chap. 4, Sec. 9). (To obtain a proof of Theorem 1 from these generalizations, one merely uses solvability and continuous ordering to show that X satisfies the "Archimedean axiom," and continuous ordering to show that all elements of $\mathscr{S}$ are onto $\mathbb{R}^+$.)

Definition 5: $\succ$ is said to be a *semiorder* on X if and only if X is a nonempty set, $\succ$ is a binary relation on X, and the following three statements are true for all w, x, y, and z in X:

1. Not $x \succ x$.
2. If $w \succ x$ and $y \succ z$, then $w \succ z$ or $y \succ x$.
3. If $w \succ x$ and $x \succ y$, then $w \succ z$ or $z \succ y$.

Definition 6: Let $\succ$ be a semiorder on X. Define $\geq'$ on X as follows: For all x and y in X,

$x \geq' y$ iff $\forall z[(\text{if } y \succ z \text{ then } x \succ z) \text{ and } (\text{if } z \succ x \text{ then } z \succ y)]$.

$\geq'$ is called *the order induced by* $\succ$.

It easily follows from Definition 6 that $\geq'$ is a transitive and connected relation, that is, is a weak ordering on X.

Definition 7: Let $\langle X, \succ \rangle$ be a semiorder and $\geq'$ be the order induced by $\succ$. Then $\langle X, \succ \rangle$ is said to be *continuous* if and only if (a) $\langle X, \geq' \rangle$ is a continuum, and (b) for each x in X there exists y in X such that for all z in X,

$$z \succ x \text{ iff } z >' y.$$

Theorem 2: Let $\mathfrak{X} = \langle X, \geq_*, \oplus \rangle$ be a continuous extensive structure, $\mathscr{S}$ be the set of isomorphisms of $\mathfrak{X}$ onto $\langle \mathbb{R}^+, \geq, + \rangle$ (which exists by Theorem 1), $\langle X, \succ \rangle$ be a continuous semiorder, and $\geq'$ be the order induced by $\succ$. Suppose the following five statements are true for all x, y, z, and w in X:

1. $\geq_* = \geq'$.
2. If $x \geq_* y$ and $y \oplus z \not\succ y$, then $x \oplus z \not\succ x$.
3. If $x \succ z$ and $z \oplus w \not\succ z$, then $x \geq_* w$.
4. If $x \succ y$ and $z \succ w$, then $x \oplus z \succ y \oplus w$.
5. If $x \not\succ y$ and $z \not\succ w$, then $x \oplus z \not\succ y \oplus w$.

Then there exists a positive real number c such that for each φ in $\mathscr{S}$ and each u in X,

$$\frac{\Delta_\varphi(u)}{\varphi(u)} = c,$$

where $\Delta_\varphi(u) = \varphi(u^*) - \varphi(u)$ where u^* is the unique element of X such that $x \succ u$ for each x in X such that $x >' u^*$ and $y \not\succ u$ for each y in X such that $u^* >' y$.

Theorem 2 is a reformulation of Narens (1980) using concepts of this article. (For a proof of Theorem 2, see Suppes et al. (1990), Chap. 16, Theorem 26).

Lemma 1: Let $\mathfrak{X}$ be a physical structure that is a continuous extensive structure, $\mathfrak{X}'$ be physically equivalent to $\mathfrak{X}$, φ an isomorphism of $\mathfrak{X}$ onto $\langle \mathbb{R}^+, \geq, + \rangle$, and φ' an isomorphism of $\mathfrak{X}'$ onto $\langle \mathbb{R}^+, \geq, + \rangle$. Then for some r and s in $\mathbb{R}^+$,

$$\varphi' = s\varphi^r.$$

Proof: The theorem follows by results of Narens (1988) and Narens (1981, Theorem 2.7).

Theorem 3: Suppose the following five statements:

1. φ is a measurement of the physical continuous extensive structure $\mathfrak{X}$ onto $\langle \mathbb{R}^+, \geq, + \rangle$.
2. $\succ$ and $\succ^*$ are continuous semiorders on the domain of $\mathfrak{X}$.
3. $\mathfrak{X}$ and $\succ$ and $\mathfrak{X}$ and $\succ^*$, when measured by φ, give Weber representations for $\succ$ and $\succ^*$ with, respectively, Weber constants c and c^*.
4. $\mathfrak{X}_1$ is physically equivalent to $\mathfrak{X}$ and φ_1 is an isomorphism of $\mathfrak{X}_1$ onto $\langle \mathbb{R}^+, \geq, + \rangle$.
5. c_1 and c_1^*, respectively, are the Weber constants for $\succ$ and $\succ^*$ when $\mathfrak{X}_1$ is measured in terms of φ_1.

Then

$$c > c^* \text{ iff } c_1 > c_1^*.$$

Proof: For each x in X, let $T(x)$ be the element y of X such that for all z in X,

$$z \succ x \text{ iff } z >_{*} y,$$

where $\geq_{*}$ is the ordering induced by $\succ$. In a similar manner, $T'(x)$ is defined for $\succ^{*}$. Then it easily follows from Theorem 2 that

$$\varphi[T(x)] = (1 + c)\varphi(x) \text{ and } \varphi[T^{*}(x)] = (1 + c^{*})\varphi(x),$$

with similar equations holding for φ_1. By Lemma 1, let r and s be positive reals such that

$$\varphi_1 = s\varphi^r.$$

Then for each x in X,

$$c > c^* \text{ iff } 1 + c > 1 + c^*$$

$$\text{iff } (1 + c)\varphi(x) > (1 + c^*)\varphi(x)$$

$$\text{iff } \varphi[T(x)] > \varphi[T^*(x)]$$

$$\text{iff } s \cdot \varphi^r[T(x)] > s \cdot \varphi^r[T^*(x)]$$

$$\text{iff } \varphi_1[T(x)] > \varphi_1[T^*(x)]$$

$$\text{iff } (1 + c_1)\varphi_1(x) > (1 + c_1^*)\varphi_1(x)$$

$$\text{iff } 1 + c_1 > 1 + c_1^*$$

$$\text{iff } c_1 > c_1^*.$$

Discussion of Definability and Invariance

We now discuss the principle result of Narens (1991) used in this article, namely, the relationship of invariance under automorphisms and definability:

Let $\mathfrak{X} = \langle X, B_1, \ldots, B_n \rangle$ and $\mathfrak{Y} = \langle Y, C_1, \ldots, C_n \rangle$ be isomorphic structures, where the $B_1, \ldots, B_n$ are elements of X, relations on X, relationships of relationships on X, and so forth etc. X and $B_1, \ldots, B_n$ are called the *primitives* of $\mathfrak{X}$, and Y and $C_1, \ldots, C_n$ are called the *primitives of* $\mathfrak{Y}$. A widely used principle in mathematics is that isomorphisms preserve the truth of statements and preserve the definability of concepts formulated in terms of the primitives of $\mathfrak{X}$ and $\mathfrak{Y}$. In terms of automorphisms of $\mathfrak{X}$ —that is, isomorphisms of $\mathfrak{X}$ onto itself—this says that automorphisms preserve the truth of statements appropriately formulated in terms of the primitives of $\mathfrak{X}$ and preserve the definability of concepts appropriately formulated in terms of the primitives of $\mathfrak{X}$. Another way of saying the latter is that concepts definable in terms of the primitives of $\mathfrak{X}$ are invariant under the automorphisms of $\mathfrak{X}$, where *definable* means "definable in terms of any higher order logical language." From this it follows that *concepts that are not invariant under the automorphisms of $\mathfrak{X}$ can never be defined in terms of the primitives of $\mathfrak{X}$*. This result is the principal method used in the article for showing that certain concepts have no purely psychological interpretation.

Narens (1991) constructs a powerful logical language $\mathcal{L}$ for $\mathfrak{X}$ such that each concept that is invariant under the automorphisms of $\mathfrak{X}$ is definable in terms of the primitives of $\mathfrak{X}$ through $\mathcal{L}$. (For a formal statement of this language and result see Narens, 1988.) For the results of this article, only a much weaker result is needed: Namely, for the

case where $\mathfrak{X}$ is the continuous extensive structure $\langle X, \geq_*, \oplus \rangle$, each automorphism invariant operation $\oplus'$ such that $\langle X, \geq_*, \oplus' \rangle$ is isomorphic to $\mathfrak{X}$ is definable in terms of the primitives of $\mathfrak{X}$. We give the basic ideas of the proof of this for the continuous extensive structure $\mathfrak{R} = \langle \mathbb{R}^+, \geq, + \rangle$ that is isomorphic to $\mathfrak{X}$. (The general case will then follow by isomorphism.)

We first note that the automorphisms of $\mathfrak{R}$ are multiplications by positive reals. (This is a well-known result.) Results of Narens (1981) shows that the operations $\oplus'$ that are invariant under the automorphisms of $\mathfrak{R}$ and such that $\langle \mathbb{R}^+, \geq, \oplus' \rangle$ is isomorphic to $\mathfrak{R}$ are of the form $\oplus' = \oplus_r$, where

$$x \oplus_r y = [x^r + y^r]^{1/r}$$

and r is a positive real number. It is an easy verification that for each positive real r, $\oplus_r$ is invariant under the automorphisms of $\mathfrak{R}$ and that $\langle X, \geq, \oplus_r \rangle$ is isomorphic to $\mathfrak{R}$.

We next note that it is obvious that concept of automorphism of $\mathfrak{R}$ is definable in terms of the primitives of $\mathfrak{R}$, and thus, the set of automorphisms A of $\mathfrak{R}$ is definable in terms of the primitives of $\mathfrak{R}$. It is also obvious that the composition operation $*$ of elements of A is definable in terms of A and therefore of the primitives of $\mathfrak{R}$. From this it is obvious that the function $\ominus_2$ defined on A by

$$\ominus_2(\alpha) = \alpha * \alpha \text{ for each } \alpha \in A$$

is definable in terms of the primitives of $\mathfrak{R}$. Define the binary relation $\geq'$ and the binary operation $+'$ on A as follows: For all $\alpha, \beta,$ and γ in A, $\alpha \geq' \beta$ if and only if for all s in $\mathbb{R}^+$, $\alpha(s) \geq \beta(s)$, and $\alpha +' \beta = \gamma$ if and only if for all s in $\mathbb{R}^+$, $\alpha(s) + \beta(s) = \gamma(s)$. It is not difficult to show (e.g., see Cohen & Narens, 1979) that $\langle A, \geq', +' \rangle$ is a continuous extensive structure. Clearly, $\geq'$ and $+'$ are definable in terms of the primitives of $\mathfrak{R}$. Let T be the set of isomorphisms of $\mathfrak{R}$ onto $\langle A, \geq', +' \rangle$. Then T is definable in terms of the primitives of $\mathfrak{R}$. Then it is not difficult to show that the operation $\oplus_2$ has the following definition: For all $x, y,$ and z in $\mathbb{R}^+$, $x \oplus_2 y = z$ if and only if for all f in T,

$$\ominus_2^{-1}(\ominus_2[f(x)] +' \ominus_2[f(y)]) = f(z).$$

Thus, $\oplus_2$ is definable in terms of the primitives of $\mathfrak{R}$.

By using the same ideas, it is easy to formulate the concept of $\ominus_k$ (composing elements of A k times with themselves) and show that $\oplus_k$ is definable from the primitives of $\mathfrak{R}$ for each positive integer k. $\oplus_q$, where q is the ratio of positive integer k to the positive integer m, is similarly definable in terms of the primitives of $\mathfrak{R}$ by considering

$$\ominus_q = \ominus_k * \ominus_m^{-1}.$$

To obtain the definability of $\oplus_r$ from the primitives of $\mathfrak{R}$ for all positive real numbers r, the language $\mathcal{L}'$ for $\mathfrak{R}$ used in the "defining" must include at least a continuum of constant symbols for purely mathematical concepts. Constructing such a language for the definability of $\oplus_r$ from the primitives of $\mathfrak{R}$ is straightforward but is not given here.