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Articles

Measurement Scales on the Continuum

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In a seminal article in 1946, S. S. Stevens noted that the numerical measures then in common use exhibited three admissible groups of transformations: similarity, affine, and monotonic. Until recently, it was unclear what other scale types are possible. For situations on the continuum that are homogeneous (that is, objects are not distinguishable by their properties), the possibilities are essentially these three plus another type lying between the first two. These types lead to clearly described classes of structures that can, in principle, be incorporated into the classical structure of physical units. Such results, along with characterizations of important special cases, are potentially useful in the behavioral and social sciences.

THE MAIN RESEARCH ACTIVITIES TODAY ON THE MATHEmatics underlying numerical representations of qualitative orderings of objects or events—theories of measurement—center not on the classical methods that evolved in physics, which are well understood, but on alternative methods that may prove useful in other sciences where measurement has proved elusive. There are several different thrusts, and this article concentrates on one that has been developed by Luce and Narens and others associated with them. It is a scheme of classifying structures according to the degrees of uniqueness of their numerical representations. The results all concern a very general situation in the sciences, namely, where a phenomenon of interest can be described in terms of monotonic, continuous variables as functions of other monotonic, continuous variables.

The term "measurement" has many meanings, the most common being that of assigning numbers to empirical objects according to some definite scheme. Empirical measurements based on such schemes almost always involve error, and the means for understanding and dealing with error is of fundamental importance in practice. However, in the theory of measurement consideration of error often is not treated explicitly. There are at least two good reasons for this. First, no general qualitative concept of measurement error has yet emerged, which makes it very difficult to incorporate error into developed theories of measurement. Second, for a large body of measurement issues, error considerations play little or no role. The latter is especially true of those issues, such as dimensional analysis in physics, that rely on an understanding of the interconnections of various numerical representations rather than on the practical production of accurate representations. This article is concerned exclusively with issues for which error is not a significant factor.

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Classical Measurement of Physical Units

A continuous monotonic variable is nothing more than a qualitatively ordered set that can be mapped in an order-preserving way onto an interval of the ordered real numbers. Such ordered sets are called continua in mathematics (*1*). In measurement theory, such order-preserving mappings are called "representations," or sometimes "measurements," since they "measure" the qualitative objects by assigning numbers in a consistent way to the objects.

For many scientific purposes, such representations of variables as monotonic and continuous are idealizations, but ones that are ubiquitous throughout all of science. Many philosophers of science object to the use of continua as accurate descriptions of empirical variables, which are often believed to assume only finitely many values or are at most potentially infinite. We consider this a valid issue, but one about which we cannot comment in any detail in this short article. Suffice it to say we believe that valid arguments can be presented to establish that continuous variables are the correct kind of idealization for many, if not most, of the ordered empirical situations encountered in science (*2, 3*).

A continuum has many different representations. For example, if a continuum has a representation ϕ onto the positive real numbers, which we denote Re^+, then $f \cdot \phi$ (where $\cdot$ denotes functional composition) is also a representation onto Re^+ for all strictly monotonic functions f from Re^+ onto Re^+, and it is easy to show that all such representations have this form. The set of representations of a continuum onto Re^+ is an example of what is called an ordinal scale (*4*). Although ordinal scales are abundant in the behavioral and social sciences—rating scales of all sorts are the most common examples—they are avoided in the physical sciences because they are correctly viewed as a very weak form of measurement. This weakness is overcome because physical variables are always constrained in additional ways that greatly narrow the possible representations.

For example, in a number of situations two objects exhibiting the attribute to be measured can be combined to form another object that also exhibits the attribute. Formally, such combinations generate a binary operation that is given the generic name "concatenation." In measurement theory of continuous variables, it is postulated as an empirical law that concatenation of qualitative objects is monotonic with respect to the qualitative ordering of the attribute. This means that if $\succsim$ denotes the ordering and $\circ$ the operation, then for any objects x, y, z in the domain X,

$$x \succsim y \leftrightarrow (x \circ z) \succsim (y \circ z) \leftrightarrow (z \circ x) \succsim (z \circ y)$$

Mass and length measurement are familiar examples. In addition, they also satisfy the properties $x \circ (y \circ z) \sim (x \circ y) \circ z$, called associativity, and $(x \circ y) \sim (y \circ x)$, called commutativity, where $\sim$ denotes equivalence in the sense that both $x \succsim y$ and $y \succsim x$ hold.

For such relational structures, $\langle X, \succsim, \circ \rangle$, measurement proceeds by concatenating copies of various elements in the domain. Let n be a positive integer and u and element of X; then nu denotes the concatenation of n copies of u. By associativity and commutativity, it does not matter in which order these concatenations are formed. Suppose x is an object qualitatively greater than u_1 ($x \succ u_1$). If u_1 is taken as a unit—assigned the value 1—then the number to be assigned to x can be estimated approximately by finding the positive integer n_1 such that both $x \succsim n_1 u_1$ and $(n_1 + 1)u_1 \succ x$ (see Fig. 1). Then x will be assigned a number in the interval $(n_1, n_1 + 1)$. The error is as much as 1. By repeating the process, using as unit an element u_2 with the property $u_2 \circ u_2 \sim u_1$, a measurement of x is produced within an error of one u_2 unit, which is 1/2 when translated into u_1 units. By continuing in this way, a precise measure of x is achieved as a limit. Structures $\langle X, \succsim, \circ \rangle$ admitting such measurement are called extensive (*5*). Note that the measurement is reduced to two mathematical processes: counting and taking a limit. Establishing the existence of the limit and the properties of the representation φ so generated depend upon the structure's satisfying certain axioms in addition to commutativity and associativity.

The major feature of the representation, in addition to its being order-preserving, is that the operation $\circ$ is interpreted numerically as addition: $\varphi(x \circ y) = \varphi(x) + \varphi(y)$. Changing the unit produces a different additive representation, and all additive representations can be achieved through just a change of unit. This pleasant state of affairs is described by saying that the set of additive representations forms a ratio scale (*4*). In a ratio scale, any two representations are related by a similarity transformation, that is, multiplication by a positive real number. Ratio scales measure objects in a stronger way than do ordinal scales, and in physics these stronger ways are ultimately reflected in the structure of physical units as well as the forms of physical laws.

Not all physical measures are extensive, but the remaining ones are expressed as products of powers of extensive ones. This will be examined more fully below.

Is Fundamental Nonextensive Measurement Fundamentally Impossible?

An almost total absence in the behavioral and social sciences of empirical concatenation operations that meet the conditions of extensive measurement was recognized early, especially by the physicist and philosopher of science N. R. Campbell, who placed great weight on this feature of physical measurement. Indeed, he treated all other physical measurement, such as the multiplicative structures among fundamental physical variables, as a distinctly secondary form of "derived measurement" (*6*).

This work led to the question: What sort of fundamental measurement, if any, is possible in the other sciences? Broadly speaking, the attempts to answer the question in the behavioral and social sciences have focused primarily on two research issues—the measurement of utility and the measurement of sensations. Although we cite some of the main measurement contributions by economists, we deal in greater detail with the issues that were raised vis-à-vis psychology because they are more germane to the research described here.

During the 1930s the British Association for the Advancement of Science appointed a distinguished committee to conduct an inquiry into the question of whether fundamental measurement was possible in psychology. Potentially at stake was whether psychology (and, more generally, social science) could ever be legitimately considered a mathematical science, since at the time it was primarily through measurement that mathematics entered into science. The inquiry had been stimulated by the fact that psychologists were attempting to measure various things, probably the most satisfactory, although not the most important socially, being levels of sensation.

The resulting report was a series of short essays and rebuttals in which the physicists, to a man, concluded that measurement meant

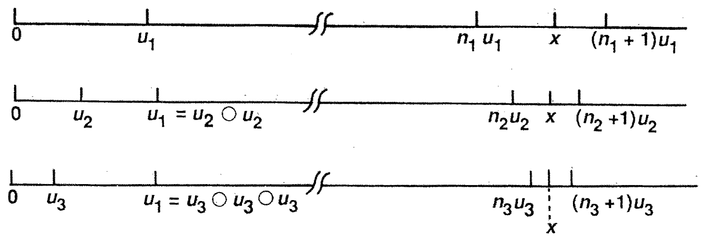

Fig. 1. A schematic rendering of the first three levels of approximation for the qualitative measurement of a length x in terms of a unit u_1.

an observable, extensive, concatenation operation, and, since no one contested the fact that psychology had few such operations, if any, they concluded that the strong forms of measurement found in physics were necessarily impossible in psychology. Perhaps the clearest statement of this position was that of Guild (*7*, p. 345):

> To insist on calling these other processes [he was referring to sensory procedures based on "just noticeable differences" and judgments of "equal intervals"] measurement adds nothing to their actual significance but merely debases the coinage of verbal intercourse. Measurement is not a term with some mysterious inherent meaning, part of which may have been overlooked by physicists and may be in course of discovery by psychologists. It is merely a word conventionally employed to denote certain ideas. To use it to denote other ideas does not broaden its meaning but destroys it: we cease to know what is to be understood by the term when we encounter it; our pockets have been picked of a useful coin.

This attitude is, of course, the antithesis of those having a mathematical or philosophical bent; the latter are likely to seek what is really essential in important situations and to investigate where else those same concepts may arise. The psychologist S. S. Stevens argued that the important thing was not the extensive nature of concatenations but rather the fact that continuous variables with such operations are blessed with a relatively unique (additive) representation, namely, they form a ratio scale (*4*). Indeed, Campbell concurred that this condition is very important, but remarked that (*8*, p. 340)

> Only one way of fulfilling this condition has ever been discovered: in it use is made of the primary function of numerals to represent number, a property of all groups. The rule is laid down that the numeral to be assigned to any thing X in respect of any property is that which represents the number of standard things or "units," all equal in respect of the property, that have to be combined together in order to produce a thing equal to X in respect of the property.

Stevens clearly believed this to be incorrect, although at the time he lacked any real examples to show otherwise. He did, however, cite the fact that some operations are represented not by addition but by weighted averages, and that such a representation lies somewhere in strength between ratio and ordinal scales; namely, affine transformations of the form $x \rightarrow rx + s$, $r > 0$, generate all representations from a single one. Such a measurement he described as forming an interval scale because ratios of intervals, not ratios themselves, are invariant under these transformations (*4*).

Actually, an example of such interval scale measurement did exist, but it was little known to measurement theorists at the time. This was a system of utility measurement due to the philosopher Ramsey (*9*) that coupled features of two distinct systems that were later explored separately and very thoroughly. The first was the axiomatization of expected utility, begun in 1947 by the mathematician von Neumann and the economist Morgenstern (*10*), and subsequently elaborated by, among others, Pfanzagl (*11*), Savage (*12*), and Suppes (*13*), respectively mathematician, statistician, and philosopher. A dozen years later, the economist Debreu (*14*) explored the second system by axiomatizing in a mixed topological-algebraic context additive utility over commodity bundles, and a few years later Luce and Tukey (*15*), psychologist and statistician, formulated a purely algebraic version. Their and related versions are called additive conjoint measurement, which is now a broadly familiar technique in many social sciences. In both expected utility and additive conjoint measurement, the representations form interval scales. No ratio scale axiomatization, other than extensive measurement, arose before 1976 (*16*).

Although Stevens' writings failed to raise the question of how scales not based on extensive operations might fit into the structure of physical units, it is fairly evident from his later work on magnitude estimation—in particular, the power relations he felt he had established among sensory attributes of intensity—that he believed some kind of close interlock to exist (*17*).

In contrast, he laid great emphasis on the question of which numerical assertions, especially statistical ones, are really meaningful in the sense of corresponding to something qualitative in the underlying observations rather than being purely mathematical statements about numbers with no empirical content. He discussed this almost entirely in terms of measurement-theoretic justifications for using or not using particular statistics, and his somewhat imprecise formulation of the issues generated a rather confused controversy which we do not enter into here. He was aware of some connections between these ideas and the importance of invariance in geometry, but he overlooked the much closer connections between them and the concept of dimensional invariance in physics (*18, 19*).

What Is Needed to Fulfill Stevens' Alternative to Campbell?

In a sense, much of our work over the past 12 years can be viewed as an attempt to work out fully the implications of Stevens' general position. In particular, we have undertaken to make precise the following five general questions and, to a degree, with the help of several who began as our students, have provided answers to them.

1) What is meant by the general concept of scale type, and can the types be classified in some useful way? In particular, why are ratio, interval, and ordinal scales so important, and are there others to be considered?

2) Given a particular scale type, what can be said about the numerical structures exhibiting that scale type? These are of interest since they become candidates for possible measurement representations. For example, was Campbell correct in believing that $\langle \mathrm{Re}, \geq, +\rangle$, which means the real numbers, Re, together with their natural order, $\geq$, and addition, $+$, is the sole candidate for ratio scaling of an empirical operation?

3) To what extent is it possible to couple one-dimensional measurement structures with conjoint (factorial) ones in such a way as to maintain the structure of units typical of classical physics? Clearly, it can be done when the one-dimensional structure has an operation that can be represented additively and the conjoint structure can be represented multiplicatively as in the case of physical measurement. The question is whether generalizations are possible that maintain the valuable pattern of physical units, namely, products of powers, that is often so much taken for granted.

4) Given answers to these questions, can we work out the empirical regularities that must be satisfied by phenomena in order for such a representation to come about? That is, can we axiomatize the qualitative systems corresponding to the possible representations?

5) And finally, given a concept of scale type, what then is meant by a meaningful statement within such a measurement system? In particular, what philosophically sound justifications can be given for the invariance conditions often invoked in meaningfulness arguments, as in dimensional analysis, and in discussions of the applicability of statistical methods to measurement?

Historically, we (and others in the field) did not work on the problems in the order given. For example, early on, the focus was mostly on questions 3 and 4, and only later did important results about questions 1, 2, and 5 arise. The first four questions are discussed in the order presented; the last must be omitted for lack of space.

Classification of Scale Types

For measurement on the continuum, the transformations discussed by Stevens that allow one to pass among equally good representations of a qualitative structure can be shown to correspond to internal symmetries of the qualitative structure. By a symmetry (the physicist's term) or an automorphism (the mathematician's term) is meant an isomorphism (structure-preserving) map of the structure onto itself. Thinking in these terms and reflecting on some specific examples that had arisen in our research, Narens (*20*) proposed a classification that allows one to understand the possible scale types that might be of scientific interest.

A recurring, key concept in science and mathematics is that of homogeneity. The intuition is that a domain is homogeneous if its elements are distinguishable not by their properties but only by their identity: in other words, if a property is true for one element, it is true for all. Homogeneity corresponds to much of the regularity observed in science and is essential for understanding what scientific laws might be. Quite often it is a consequence of observed properties of relations on the domain, as in the case of extensive operations on a continuum. It can also appear in other ways. Narens recognized that saying a qualitative domain based on a continuum was either ratio, interval, or ordinal scalable was tantamount to saying that the domain was homogeneous, because corresponding to each fixed real number a representation can be found that takes any particular object into that number. Since representations are intimately connected with automorphisms, this implies the following proposition: For each pair of objects, x and y, in the qualitative domain, there exists an automorphism of that domain that takes x into y. This proposition is the characterization of homogeneity used in mathematical logic, and it can be shown that for particularly powerful languages describing the domain it is equivalent to saying that the objects of the domain are indistinguishable from one another (*2*).

More formally, Narens classified measurement structures as follows. Consider a qualitative relational structure of the form $\mathscr{X} = \langle X, \succsim, S_j\rangle_{j\in J}$, where X is a set of entities, $\succsim$ is a total ordering of them (by the attribute being measured), $\langle X, \succsim\rangle$ is a continuum, and S_j are other relations of finite order on X where j lies in some index set J. In the extensive case discussed earlier, $J = \{1\}$ and S_1 is an operation, which as a relation is of order 3. Let $\mathscr{H}$ be a subset of the set $\mathscr{A}$ of all automorphisms, and let M and N be non-negative integers. Then $\mathscr{H}$ is said to be M-point homogeneous if and only if for each $x_i, y_i \in X$, $i = 1, ..., M$, such that $x_i \succ x_{i+1}$ and $y_i \succ y_{i+1}$, there is some α in $\mathscr{H}$ such that $\alpha(x_i) = y_i$. If $\mathscr{A}$ is M-point homogeneous, $\mathscr{X}$ is said to be M-point homogeneous. If $\mathscr{X}$ is M-point homogeneous for each M, it is said to be ∞-point homogeneous. Homogeneity as discussed earlier is just 1-point homogeneity.

A second concept, having to do with the redundancies among automorphisms, is also important. A subset $\mathscr{H}$ of automorphisms is said to be N-point unique if and only if any two members of $\mathscr{H}$ that agree at N distinct points necessarily are identical. And, $\mathscr{X}$ is said to be N-point unique if $\mathscr{A}$ is. If $\mathscr{X}$ is not N-point unique for any N, it is said to be ∞-point unique. If it is N-point unique for some N, it is said to be finitely unique.

The structure $\mathscr{X}$ is said to be of scale type (M, N) provided M is the largest value for which it is M-point homogeneous and N is the least value for which it is N-point unique. It is easy to see that for a continuum $M \leq N$.

The Stevens ratio scales are of type (1, 1), interval are of type (2, 2), and ordinal are of type (∞, ∞). The first question is: What else is mathematically possible? The answer is simple, although not simple to prove (*21*), when $\mathscr{X}$ is a relational structure on the continuum that is homogeneous and finitely unique. Then it is one of three scale types: (1, 1), (2, 2), or (1, 2); there are no other homogeneous, finitely unique scale types. In particular, $\mathscr{X}$ is isomorphic to a real structure for which the automorphism group is a subgroup of the affine transformations that includes all of the similarity ones.

So, within the framework of homogeneous, finitely unique structures on the continuum, Stevens had two of the three possibilities. An example of the (1, 2) scale type is the discrete interval scale whose group of transformations between representations is of the form $x \to k^n x + s$, where $k > 0$ is a fixed constant and n ranges over the integers (positive, negative, and zero). Outside of the above limitations, our knowledge of what is mathematically possible is incomplete. We do not have much information about the (M, ∞) cases, and although the $(0, N)$ cases have been mathematically characterized by Alper (in *20*), they have not entered in any systematic way into scientific applications. These $(0, N)$ cases range from structures with no automorphisms other than the identity to those with many, but not quite enough to be homogeneous; from those with little regularity of structure, to those that have major pieces that are highly regular. Examples of the latter are structures with an intrinsic zero (a fixed point of every automorphism) that are homogeneous on either side of the zero.

The remainder of the article focuses on additional results about the homogeneous, finitely unique cases, which include many of the most useful and applicable ones.

Target Numerical Representations

If homogeneity is coupled with a little additional qualitative structural information, powerful algebraic constraints result that greatly delimit the possible quantitative models of a qualitative situation (*22, 23*). The most fully studied cases involve concatenation structures of the form $\mathscr{X} = \langle X, \succsim, \circ \rangle$, where $\langle X, \succsim \rangle$ is a continuum and $\circ$ is a monotonic, binary operation on X. Such situations differ from physical ones in that the concatenation operation $\circ$ is not assumed to be either associative or commutative. They also differ from the physical ones in that $\circ$ may be intensive—$x \succsim y$ implies $x \succsim x \circ y \succsim y$ and $x \succsim y \circ x \succsim y$—rather than positive—$x \circ y \succ x$ and $x \circ y \succ y$ for all x, y. Nevertheless, under the assumption of homogeneity, $\mathscr{X}$ looks very much like a fundamental physical dimension. Before we make explicit how, it is useful to include a few remarks about the structure $\mathscr{X}$.

First, independent of whether $\langle x, \succsim \rangle$ is a continuum, it follows from homogeneity that $\mathscr{X}$ is either weakly positive ($x \circ x \succ x$ for all x) or weakly negative ($x \circ x \prec x$ for all x) or idempotent ($x \circ x \sim x$ for all x). This reflects the principle that all elements of a homogeneous structure "look alike." It also follows from the results on scale type that if $\mathscr{X}$ is finitely unique, then it is 1- or 2-point unique; in the latter case, it is necessarily intensive and idempotent. Furthermore, it can be shown that under very plausible conditions, such as $\circ$ being onto X and continuous in each variable, $\mathscr{X}$ is finitely unique.

The reason why such an $\mathscr{X}$ resembles a fundamental physical dimension is that it has a "unit" representation in the following sense. If $\mathscr{X}$ is extensive, then $\circ$ can be represented quantitatively as $+$ by a ratio scale of representations, $\mathscr{S}$. That is, for each φ in $\mathscr{S}$ and each x, y in X, $\varphi(x \circ y) = \varphi(x) + \varphi(y)$. This can be restated as follows: $\varphi(x \circ y) = \varphi(y) f[\varphi(x)/\varphi(y)]$, where $f(u) = 1 + u$. A unit representation for $\mathscr{X}$ is a quantitative structure of the form $\mathscr{R} = \langle \mathrm{Re}^+, \geq, \otimes \rangle$ that is isomorphic to $\mathscr{X}$ and such that there is a function f from Re^+ onto Re^+ with the following three properties: (i) f is strictly increasing; (ii) $f(t)/t$ is strictly decreasing; and (iii) if r, s are in Re^+, then $r \otimes s = s f(r/s)$.

Which of the three scale types a unit structure is can be described as follows. Consider the values of ρ for which $f(x^\rho) = f(x)^\rho$ obtains for all $x > 0$. Then (i) $\mathscr{X}$ is (1, 1) if and only if $\rho = 1$; (ii) $\mathscr{X}$ is (1, 2) if and only if, for some fixed $k > 0$ and all integers n, $\rho = k^n$; and (iii) $\mathscr{X}$ is (2, 2) if and only if it holds for all $\rho > 0$. In the first case, the set of isomorphisms from $\mathscr{X}$ onto $\mathscr{R}$ forms a ratio scale, in the second a discrete interval scale, and in the third an interval scale. The form of f has been characterized (*23*) in the (1, 2) case for f differentiable, and completely in the (2, 2) case, where it is the following generalization of a geometric mean: for some c, d in (0, 1).

$$r \otimes s = \begin{cases} r^c s^{1-c}, & \text{for } r > s \\ r, & \text{for } r = s \\ r^d s^{1-d}, & \text{for } r < s \end{cases}$$

This latter Luce and Narens called the dual bilinear form (*23*), and from it they generated a generalized version of subjective expected utility in which the decision maker exhibits a very bounded form of rationality. This model seems to accommodate many of the empirical anomalies that reject the classical utility theory (*23, 24*).

The question of generalizing the concept of a unit structure to much more general settings has recently been solved (*25*). Essential to doing this is the qualitative analogue of a translation $x \to x + s$ in the affine case. An automorphism of an ordered structure is said to be a translation if either it is the identity or has no fixed point. Then $\mathscr{R} = \langle R, \geq, R_j \rangle_{j \in J}$ is said to be a real unit structure provided R is a subset of Re^+ and there is a subset T of Re^+ such that (i) T is a group under multiplication, (ii) under multiplication T maps R into R, and (iii) the restriction of T to R is the set of translations of $\mathscr{R}$. It is easy to verify that the unit representations of concatenation structures meet these three conditions.

The key discovery is that any qualitative, homogeneous relational structure has such a unit representation provided that its set of translations exhibit the following high degree of regularity. If α and β are automorphisms, define $\alpha \succsim' \beta$ if and only if $\alpha(x) \succsim \beta(x)$ for all x in X. The property is that under $\succsim'$ and function composition, the translations form an Archimedean ordered group that is homogeneous, which by Hölder's (*26*) theorem is equivalent to their having an additive representation on Re^+. Thus, given a qualitative structure, one first studies the set of translations, determining if these two properties are met. Note that by the result quoted earlier, these conditions are met in any relational structure on a continuum that is finitely unique and homogeneous.

The reason for attending to homogeneous unit representations is that they are quite general and likely to appear in many behavioral science applications. Moreover, such structures have scales that provide strong forms of measurement. The general (1, 1) case is just as strong as the special case of extensive measurement used in the physical sciences. Thus, their existence frees behavioral scientists from being hobbled by the artificial constraint of measuring by using some variant of the very special unit structure $\langle \mathrm{Re}^+, \geq, + \rangle$.

Distribution in Conjoint Structures

Now we show that unit representations can provide the foundations for structures of several continuous variables that interrelate in exactly the same manner as the structure of physical dimensions of classical physics.

A number of important measurement structures involve a structure of the form $\mathscr{C} = \langle X \times P, \succsim \rangle$, where X and P are sets (factors) and $\succsim$ is a weak ordering (transitive and connected). In this case we may not assume that $\succsim$ is a total order since there will be many nonequal pairs that are equivalent in the attribute; these represent the trade-offs between the factors that leave the attribute unchanged. The most important property usually assumed is monotonicity, which in this context is often called "independence." $\mathscr{C}$ is said to be monotonic if and only if, for all x, y in X and p, q in P, both

$$(x, p) \succsim (x, q) \leftrightarrow (y, p) \succsim (y, q)$$

and

$$(x, p) \succsim (y, p) \leftrightarrow (x, q) \succsim (y, q)$$

This means that a natural weak order is induced on each component; these orders are denoted $\succsim_X$ and $\succsim_P$. In many physical situations, there is also a measurement structure on one or both components, for example, $\mathscr{X} = \langle X, \succsim_X, S_j \rangle_{j \in J}$. Historically, all of the examples have been concatenation structures, usually with the operation being extensive and an isomorphism φ_X onto the additive, positive, real numbers. An example of such a pair is the conjoint structure consisting of the ordering of mass-velocity pairs by kinetic energy, where mass and velocity are both extensive structures. The remarkable property of such measurement pairs, the property that underlies the structure of physical units, is that there is a function ψ_P on P such that the product $\varphi_X \psi_p$ preserves the order $\succsim$; moreover, if there is also an extensive structure on P with additive isomorphism φ_P, then there is a constant ρ such that $\varphi_X {\varphi_P}^{\rho}$ represents $\succsim$.

Of course, such a tight interlock exists only because there is some law relating the structures on the components to that of the conjoint structure. The questions to be answered are the following: First, what is the qualitative nature of that interlock? And, second, to what extent is it possible to generalize from extensive structures and still arrive at the same conclusion? The latter question is especially important to psychophysicists because, as a result of the lack of behavioral extensive structures, they have appeared to be barred from any possibility of adding fundamental measures to the system of physical measures. Such is not barred, however, if the product of powers of the measurements of fundamental attributes can also be achieved through nonextensive structures. With such an alternative possibility, psychophysicists may be able to model some variables, such as subjective sensory intensity, in a way that is consistent with the physical structure of units. We are not claiming to have accomplished this. However, we do claim that the research discussed below shows that the possibility exists.

Over a span of about 12 years, increasingly general results concerning the above two questions have been obtained (*16, 19, 20, 23, 25, 27, 28*); we present the current, most general formulation for the case of continuous variables. To do so, the qualitative interlocking property needs to be defined. Let $\mathscr{C} = \langle X \times P, \succsim \rangle$ be a conjoint structure and suppose x_i, y_i are in X, $i = 1, \ldots, n$. Then $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are said to be similar if and only if there exists p, q in P such that for $i = 1, \ldots, n$, $(x_i, p) \sim (y_i, q)$. A relation S of order n on X is said to distribute in $\mathscr{C}$ if and only if, whenever $\mathbf{x}$ is in S and $\mathbf{y}$ is similar to $\mathbf{x}$, then $\mathbf{y}$ is in S. A structure $\mathscr{X}$ on X is said to distribute in $\mathscr{C}$ if and only if each of its defining relations distributes in $\mathscr{C}$. Two additional concepts are helpful: $\mathscr{C}$ is said to be solvable if and only if for any three values the fourth exists such that $(x, p) \sim (y, q)$; and $\mathscr{C}$ is said to be complete if and only if $\langle X, \succsim_X \rangle$ and $\langle P, \succsim_P \rangle$ are continua.

Now, consider a conjoint structure $\mathscr{C}$ that is solvable, is complete, and has an ordered relational structure $\mathscr{X}$ on X. Then the following three propositions can be shown. (i) If $\mathscr{X}$ is homogeneous, finitely unique, and distributes in $\mathscr{C}$, then $\mathscr{C}$ has a multiplicative representation. (ii) If there are structures on both components that have representations as homogeneous real unit structures, then $\mathscr{C}$ has the product-of-powers representation. (iii) If $\mathscr{X}$ is homogeneous and finitely unique, then there is a conjoint structure within which it distributes.

Thus, nothing is really changed from classical physics if we replace the usual extensive structures by structures on continua that are homogeneous and finitely unique; the latter may or may not be based on concatenation operations.

Axiomatizations

Although we know a good deal about numerical representations of finitely unique, homogeneous structures, this by itself is of little help to the experimentalist who wishes to decide if an empirical system has a particular numerical representation and to estimate it for particular objects. To be testable, a property must be stated in terms of the defining, empirical primitives of the system, especially the ordering. It simply is not possible to verify statements about the set of automorphisms directly. One can reject a property such as homogeneity by, for example, showing that a specific pair of elements differ in an empirically specifiable property or that some element has a unique property. In particular, the existence of an upper bound—as in the cases of the velocity of light and the universal element in a probability structure—or the existence of a zero element will rule out homogeneity. But we know of no general way to demonstrate empirically either the homogeneity of the structure or the more demanding property that the set of translations forms a homogeneous, Archimedean ordered group, the condition that leads to unit representations. Of course, in special cases this can be accomplished by exploiting rather strong characteristics of empirical relations, for example, the associativity of a concatenation operation (*29*). Thus, it continues to be an important research topic to axiomatize, in a testable way, broad classes of homogeneous structures with unit representations, and to the extent possible to provide algorithms for constructing the representations.

Homogeneity can be tested in many cases that involve operations. A case in point is the general class of structures called positive concatenation structures (PCSs). These have monotonic operations on a continuum that are positive ($x \circ y \succ x$ and $x \circ y \succ y$) and restrictedly solvable ($x \succ y$ implies $x \succ y \circ z$ for some z). For PCSs, homogeneity is equivalent to the condition that for all positive integers n, $n(x \circ y) = nx \circ ny$, which is a testable property for each n (*22*). For example, if this fails for $n=2$, that is, $(x \circ y) \circ (x \circ y) \neq (x \circ x) \circ (y \circ y)$ for some x, y, then the PCS cannot be homogeneous.

We also know how to test for homogeneity with interval-scalable, monotonic operations. Such operations are highly restricted since they must have dual bilinear representations. Basically the approach to this problem is as follows. Define an operation $*$ that extends the given operation $\circ$ for $x \succ y$ throughout X and another $*'$ that extends $\circ$ for $x \prec y$ throughout X. Then the necessary and sufficient conditions for $\circ$ on a continuum to have a dual bilinear representation are that $*$ and $*'$ both be definable, both be right autodistributive $[(x*y)*z = (x*z)*(y*z)]$, and together satisfy generalized bisymmetry $[(x*y)*'(u*v) = (x*'u)*(v*'y)]$ (*30*).

The only other homogeneous structures with a monotonic operation on a continuum are idempotent and of type $(1, 1)$ or $(1, 2)$. Some of the $(1, 1)$ cases can be recoded as PCSs, and when this is possible we know how to axiomatize homogeneity for them. For the other cases, we do not have fully effective techniques of axiomatization. There is a mathematically informative generalization of the condition for PCSs, but it is not empirically testable because it entails having an unspecified translation as the starting point of an inductive property (*30*).

Since conjoint structures are weakly ordered cartesian products, they are not ordered relational structures as defined above; however, they can be recast in that form. So the concepts of homogeneity and uniqueness apply to them. Further, because they can be recoded in a natural way in terms of operations closely related to PCSs, their study is greatly simplified (*23, 28*). We cannot go into the details here.

Concluding Remarks

Because of the differences in their respective phenomena, physical and behavioral data require different mathematical representing structures and therefore different procedures of measurement. Processes that may allow behavioral attributes to have strong forms of measurement have been developed, and measurements of such attributes, if they exist, will act in much the same way as physical units. Moreover, it is mathematically feasible for them to be combined among themselves and with physical units in just the same way as physical units combine. We have also described the mathematical possibilities (scale types) for those strong forms of measurement involving homogeneous structures and have shown that although they are greatly limited in number they are far more general than the usual models used in physical measurement. Their inherent limitations naturally suggest strategies for scientific experimentation and discovery, since much of their description can be captured by qualitative axioms.

The results reported here do not cover some important situations in which there are distinguished elements (for example, upper or lower bounds, as in probability and relativistic velocity). It is not yet clear how best to classify them.

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The Chemistry of Self-Splicing RNA and RNA Enzymes

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Proteins are not the only catalysts of cellular reactions; there is a growing list of RNA molecules that catalyze RNA cleavage and joining reactions. The chemical mechanisms of RNA-catalyzed reactions are discussed with emphasis on the self-splicing ribosomal RNA precursor of *Tetrahymena* and the enzymatic activities of its intervening sequence RNA. Wherever appropriate, catalysis by RNA is compared to catalysis by protein enzymes.

THE ABILITY OF RNA TO ACT AS A BIOLOGICAL CATALYST has become well established in the last few years. The examples of such ribozymes fall into two categories. Self-splicing (*1–3*) and self-cleaving (*4–8*) RNAs exemplify intramolecular catalysis (*9*) in which the folded structure of the RNA mediates a reaction on another part of itself. In addition, RNA also acts as a

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