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Classification of Concatenation Measurement Structures according to Scale Type*†

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A relational structure is said to be of scale type (M, N) iff M is the largest degree of homogeneity and N the least degree of uniqueness (Narens, *Theory and Decision*, 1981, **13**, 1–70; *Journal of Mathematical Psychology*, 1981, **24**, 249–275) of its automorphism group. Roberts (in *Proceedings of the first Hoboken Symposium on graph theory*, New York: Wiley, 1984; in *Proceedings of the fifth international conference on graph theory and its applications*, New York: Wiley, 1984) has shown that such a structure on the reals is either ordinal or M is less than the order of at least one defining relation (Theorem 1.2). A scheme for characterizing N is outlined in Theorem 1.3. The remainder of the paper studies the scale type of concatenation structures $\langle X, \succsim, \circ \rangle$, where $\succsim$ is a total ordering and $\circ$ is a monotonic operation. Section 2 establishes that for concatenation structures with $M > 0$ and $N < \infty$ the only scale types are $(1, 1)$, $(1, 2)$, and $(2, 2)$, and the structures for the last two are always idempotent. Section 3 is concerned with such structures on the real numbers (i.e., candidates for representations), and it uses general results of Narens for real relational structures of scale type (M, M) (Theorem 3.1) and of Alper (*Journal of Mathematical Psychology*, 1985, **29**, 73–81) for scale type $(1, 2)$ (Theorem 3.2). For $M > 0$, concatenation structures are all isomorphic to numerical ones for which the operation can be written $x \circ y = yf(x/y)$, where f is strictly increasing and $f(x)/x$ is strictly decreasing (unit structures). The equation $f(x^\rho) = f(x)^\rho$ is satisfied for all x as follows: for and only for $\rho = 1$ in the $(1, 1)$ case; for and only for $\rho = k^n$, $k > 0$ fixed, and n ranging over the integers, in the $(1, 2)$ case; and for all $\rho > 0$ in the $(2, 2)$ case (Theorems 3.9, 3.12, and 3.13). Section 4 examines relations between concatenation and conjoint structures, including the operation induced on one component by the ordering of a conjoint structure and the concept of an operation on one component being distributive in a conjoint structure. The results, which are mainly of interest in proving other

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results, are mostly formulated in terms of the set of right translations of the induced operation. In Section 5 we consider the existence of representations of concatenation structures. The case of positive ones was dealt with earlier (Narens & Luce (*Journal of Pure & Applied Algebra* **27**, 1983, 197–233). For idempotent ones, closure, density, solvability, and Archimedean are shown to be sufficient (Theorem 5.1). The rest of the section is concerned with incomplete results having to do with the representation of cases with $M > 0$. A variety of special conditions, many suggested by the conjoint equivalent of a concatenation structure, are studied in Section 6. The major result (Theorem 6.4) is that most of these concepts are equivalent to bisymmetry for idempotent structures that are closed, dense, solvable, and Dedekind complete. This result is important in Section 7, which is devoted to a general theory of scale type (2, 2) for the utility of gambles. The representation is a generalization of the usual SEU model which embodies a distinctly bounded form of rationality; by the results of Section 6 it reduces to the fully rational SEU model when rationality is extended beyond the simplest equivalences. Theorem 7.3 establishes that under plausible smoothness conditions, the ratio scale case does not introduce anything different from the (2, 2) case. It is shown that this theory is closely related to, but somewhat more general, than Kahneman and Tversky's (*Econometrica* **47**, 1979, 263–291) prospect theory.

CONTENTS.

1. SCALE TYPE

1.1. *Introduction*

Until recently, most work in the theory of measurement has consisted of specifying axiomatically particular qualitative structures and then, first, showing that they have homomorphisms into some particular numerical structure—this is known as a *representation theorem*—and, second, specifying how the various homomorphisms are related to one another—this is known as the corresponding *uniqueness theorem.* Although in some cases [e.g., semiorders (Luce, 1956) or the unfolding model (Coombs, 1964)] it has proved difficult to give a universally acceptable formulation of the uniqueness theorem, for the vast majority of cases in the measurement literature it is an easy and straightforward matter. Almost always this consists of specifying a group of transformations that take one homomorphism into another. A notable exception to this occurred in Narens and Luce (1976), where extensive measurement was generalized to cases for which the qualitative concatenation operation need not be associative. These structures, which are called *positive concatenation structures* (PCSs), were shown to exhibit the following type of uniqueness: whenever two homomorphisms agree at a single point, then they are identical. Later Cohen and Narens (1979) showed that uniqueness for these structures could be specified in terms of a multiplicative subgroup of the real numbers, but that the nature of the subgroup varied with the structure. For some structures the subgroup consists only of the identity; for others the subgroup is the integers; and for still others it is a densely ordered subgroup that may consist of all positive real numbers. Thus, for the general case, there is no way to specify the uniqueness of positive concatenation structures in terms of a specific group of transformations.

A somewhat different approach to the uniqueness problem suggested by the above example is as follows: If φ is a homomorphism of a qualitative structure $\mathscr{X}$ into a numerical structure $\mathscr{R}$, then for each automorphism α of $\mathscr{X}$, $\varphi\alpha$ is another homomorphism of $\mathscr{X}$ into $\mathscr{R}$. In many important cases for measurement, e.g., when all homomorphisms are one-to-one and onto $\mathscr{R}$, the uniqueness theorem can be captured by stating how the automorphisms of $\mathscr{X}$ relate to one another. This is the approach employed by Cohen and Narens (1979) in dealing with positive concatenation structures and it was generalized by Narens (1981a, 1981b) to general relational structures. Narens (1981b) classified relational structures in terms of two properties of their automorphism groups: *degree of uniqueness*, which is defined to be the minimum number such that if two automorphisms of the structure agree at this many distinct points then they are identical, and the *degree of homogeneity*, which is the maximum number such that any two ordered sets of this number of elements can be mapped into each other by automorphisms of the structure. He established that ratio scalability is essentially captured when the degree of uniqueness = degree of homogeneity = 1 and that interval scalability corresponds to degree of uniqueness = degree of homogeneity = 2. He further showed that there are no structures for which the two degrees are finite, equal, and >2.

Luce and Cohen (1983) applied these concepts to conjoint measurement structures. They focussed their attention on the case where many of the automorphisms of the structure consist of functions operating separately on the components of the structure. For the cases where the degrees of uniqueness and homogeneity of these "factorizable" automorphisms agree on a component, they were able to uncover a good deal of information about the possible numerical based homomorphisms of the structure. However, when the degrees disagree, they, as was true in Narens (1981b), were unable to say much about the numerical automorphisms.

The purpose of the present paper is to carry out an analogous program of classification for general concatenation structures—i.e., ordered structures with a binary operation that is increasing in each variable—and to establish some close interconnections between these structures and conjoint ones. As we shall see, classifying concatenation structures according to these uniqueness and homogeneity concepts is quite revealing of their possible numerical representations, which turn out to be considerably more limited than one might at first guess. As we shall see, the work is not complete because, once again, we do not understand fully the possible transformation groups in which the degree of uniqueness and homogeneity differ. But at least the statement of what we do not understand is clear and unambiguous.

1.2. *General Definitions*

DEFINITION 1.1. Suppose X is a nonempty set. A relation of *order n on X* is, for $n > 0$, a subset of X^n or, for $n = 0$, a single element of X. A relation of *finite order* is one of order n for some finite n. Let J be an index set. Then $\mathscr{X} = \langle X, S_j \rangle_{j \in J}$ is called a *relational structure* iff each S_j is a relation of finite order. A relational structure may have uncountably many relations, but when J is countable or finite the notation $\langle X, S_1, S_2, ... \rangle$ is often used. If one relation of $\mathscr{X}$ is a weak ordering, i.e., a relation of order 2 that is transitive and connected, then $\mathscr{X}$ is said to be *weakly ordered.* Weak orders are usually denoted $\succsim$, and as is usual we define $\succ = \succsim \cap$ not $(\precsim)$ and $\sim = \succsim \cap \precsim$. *Endomorphisms* of $\mathscr{X}$ are homomorphisms of $\mathscr{X}$ into itself, and *automorphisms* of $\mathscr{X}$ are isomorphisms of $\mathscr{X}$ onto itself. We denote the group of automorphisms of $\mathscr{X}$ by $\mathscr{G}$, sometimes with added notation when several structures are involved.

The following two concepts are usually invoked with respect to the group of automorphisms, but there are occasions when they will be used in which the transformations either do not form a group or the transformations are not automorphisms or both.

DEFINITION 1.2. Suppose $\mathscr{X} = \langle X, S_j \rangle_{j \in J}$ is a weakly ordered relational structure and $\mathscr{H}$ is a set of order preserving maps from X into X, i.e., for all x, y in X and α in $\mathscr{H}$, $x \succsim y$ iff $\alpha(x) \succsim \alpha(y)$. Let M and N be nonnegative integers. The set $\mathscr{H}$ is said to be *M-point homogeneous* in $\mathscr{X}$ iff for every x_i, y_i in X, $i = 1,..., M$ for which $x_1 \prec x_2 \prec \cdots \prec x_M$ and $y_1 \prec y_2 \prec \cdots \prec y_M$, there exists an α in $\mathscr{H}$ such that $\alpha(x_i) \sim y_i$, $i = 1,..., M$. If $\mathscr{H}$ is M-point homogeneous for every nonnegative integer M, then $\mathscr{H}$ is said to be *∞-point homogeneous.* The set $\mathscr{H}$ is said to be *N-point unique* in $\mathscr{X}$ iff for every α, β in $\mathscr{H}$ and all x_i in X, $i = 1,..., N$, such that $x_1 \prec x_2 \prec \cdots \prec x_N$, if $\alpha(x_i) \sim \beta(x_i)$, $i = 1,..., N$, then for each x in X, $\alpha(x) \sim \beta(x)$. If there is no nonnegative integer for which $\mathscr{H}$ is N-point unique, then $\mathscr{H}$ is said to be *∞-point unique.* If M is the largest value for which $\mathscr{H}$ is M-point homogeneous and N is the smallest value for which $\mathscr{H}$ is N-point unique, then $\mathscr{H}$ is said to be of *type* (M, N). The structure $\mathscr{X}$ is said to have the degree of homogeneity and uniqueness of its automorphism group $\mathscr{G}$ and is said to be of *scale type* (M, N) if $\mathscr{G}$ is of type (M, N).

Observe that for either M or $N \geqslant 2$, the definitions of M-point homogeneity and N-point uniqueness make use of the ordering relation in a critical way. However, for $M, N \leqslant 1$, the ordering is immaterial and so these concepts make sense for unordered structures. In other literatures the concept of 1-point homogeneity is called "transitivity."

When $\succsim$ is a total order, the following two facts are easily shown: If X is infinite and $\mathscr{H}$ is M-point homogeneous for $M \geqslant 1$, then $\mathscr{H}$ is $(M-1)$-point homogeneous. And if X is infinite and $\mathscr{H}$ is of scale type (M, N), then $M \leqslant N$. Neither of these statements is true for the present definitions of homogeneity and uniqueness when X is finite. For if $M > |X|$, then the structure is vacuously M-point homogeneous whereas it need not be M'-point homogeneous for any $M' \leqslant |X|$. And if $X = \{1, 2, 3\}$, then $\langle X, \geqslant \rangle$ is 3-point homogeneous and 0-point unique. Roberts and Rosenbaum (1984) have begun studying the finite case, and for this case they propose modified definitions.

As we shall see when we examine structures having a relation of order 3 that is an operation, the possible values for the scale type are quite limited. The limitation on M is general, as is shown in Theorem 1.2, but we do not know of any comparably general result that places limits on N. For example, one can wonder if there are reasonable conditions under which the value of M limits that of N, e.g., to $N - M \leqslant 2$. For real numerical concatenation structures with $X = \mathrm{Re}$ or Re^+ and $\succsim\ = \ \geqslant$, we present below conditions that limit $N \leqslant 2$, but nothing yet has been established for the general case. There are additional general questions. If a numerical structure is of scale type (M, N), does its group of automorphisms include a subgroup of type (M, M)? And for the case of real numerical structures, where the automorphism group is a group of real transformations, does there exist an extension of $\mathscr{G}$ to a group of type (N, N)? As we shall see shortly, a good deal is known about the answers in the case where N is finite and the structure is *on* the real numbers.

1.3. *Relations Between Structure and Scale Type*

THEOREM 1.1. *Suppose $\mathscr{X} = \langle X, \succsim, S_j \rangle_{j \in J}$ is a weakly ordered relational structure and $\mathscr{G}$ is its automorphism group. If $\mathscr{G}$ is commutative, N-point unique for some $N > 0$, and for each x in X there exists γ in $\mathscr{G}$ such that $\gamma(x) \nsim x$, then $\mathscr{X}$ is of scale type* $(0, 1)$ or $(1, 1)$.

Proof. Let α, β be in $\mathscr{G}$ and suppose that for some x in X, $\alpha(x) \sim \beta(x)$. We first show there is no loss of generality in assuming $\alpha(x) \nsim x$. Suppose $\alpha(x) \sim x$. By hypothesis there exists γ in $\mathscr{G}$ such that $\gamma(x) \nsim x$. Since $\alpha(x) \sim \beta(x)$ and the automorphisms commute

$$\gamma\beta(x) \sim \gamma\alpha(x) \sim \alpha\gamma(x) \nsim x,$$

proving that $\gamma\alpha$ and $\gamma\beta$ have the desired properties. Since $\alpha(x) \nsim x$ and $\succsim$ is connected, either $\alpha(x) \succ x$ or $\alpha(x) \prec x$. Without loss of generality, assume the former. Then for all integers m and n, $m > n$ implies $\alpha^m(x) \succ \alpha^n(x)$. But for each integer n,

$$\beta\alpha^n(x) \sim \alpha^n\beta(x) \sim \alpha^n\alpha(x) \sim \alpha\alpha^n(x),$$

and so α and β agree at the points $\alpha^n(x)$ for $n = 1,..., N$ and $\alpha^1(x) \prec \alpha^2(x) \prec \cdots \prec \alpha^N(x)$. Since, by hypothesis, the structure is N-point unique, $\alpha(y) \sim \beta(y)$ for all y in X. Thus, $\mathscr{X}$ is 1-point unique, and since $M \leqslant N$, $\mathscr{X}$ is of scale type (0, 1) or (1, 1). Q.E.D.

The following result is due to Fred S. Roberts and was communicated to us in August of 1982. An improved version is given in Roberts and Rosenbaum (1985).

THEOREM 1.2 (Roberts). *Suppose* $\mathscr{X} = \langle X, \succsim, S_j \rangle_{j \in J}$ *is an ordered relational structure that is isomorphic to a real relational structure* $\mathscr{R} = \langle \text{Re}, \geqslant, R_j \rangle_{j \in J}$, *$M$ is a positive integer and the automorphism group of $\mathscr{X}$ is M-point homogeneous, and that the order of each S_j, j in J, is $\leqslant M$. Then $\mathscr{X}$ is ∞-point homogeneous and the automorphism group of $\mathscr{R}$ consists of all strictly monotonic increasing transformations and thus forms an ordinal scale for $\mathscr{X}$.*

Proof. Since $\mathscr{X}$ is M-point homogeneous and $M > 0$, none of the S_j can be individual constants, i.e., relations of order 0. Let φ be an isomorphism between $\mathscr{X}$ and $\mathscr{R}$ and let f be any strictly increasing function from Re onto Re. We show that $f * \varphi$, where $*$ denotes function composition, is also an isomorphism. Clearly, $f * \varphi$ is order preserving. Consider the relation S_i of $\mathscr{X}$ which is of order $k(i)$. If $(x_1, x_2,..., x_{k(i)})$ is in S_i, then for some permutation ρ of 1, 2,..., $k(i)$ they are ordered $x_{\rho(1)} \succsim x_{\rho(2)} \succsim \cdots \succsim x_{\rho[k(i)]}$ and so $\varphi(x_{\rho(1)}) \geqslant \varphi(x_{\rho(2)}) \geqslant \cdots \geqslant \varphi(x_{\rho[k(i)]})$. Since f is increasing $f * \varphi$ preserves the latter inequalities. However, since φ is onto, there exist $y_{\rho(j)}$ in X, $j = 1, 2,..., k(i)$, such that $\varphi(y_{\rho(j)}) = f\varphi(x_{\rho(j)})$. Since φ is order preserving, $y_{\rho(1)} \succsim y_{\rho(2)} \succsim \cdots \succsim y_{\rho[k(i)]}$. Since $k(i) \leqslant M$, by M-point homogeneity there exists α in $\mathscr{G}$ such that $\alpha(x_{\rho(j)}) \sim y_{\rho(j)}$. But $\varphi * \alpha$ is an isomorphism of $\mathscr{X}$ onto $\mathscr{R}$, and so $(x_1, x_2,..., x_{k(i)})$ in S_i iff $[\varphi\alpha(x_1), \varphi\alpha(x_2),..., \varphi\alpha(x_{k(i)})]$ in R_i iff $[\varphi(y_1), \varphi(y_2),..., \varphi(y_{k(i)})]$ in R_i iff $[f\varphi(x_1), f\varphi(x_2),..., f\varphi(x_{k(i)})]$ in R_i, which proves that $f * \varphi$ is an isomorphism. Thus, $\mathscr{X}$ has an ordinal scale representation, and so it is M'-point homogeneous for every M', i.e., ∞-point homogeneous. Q.E.D.

As an illustration of this result, consider a relational structure on the reals for which its relations are all of order 3 or less, e.g., $\langle \text{Re}, \geqslant, \circ \rangle$, where $\circ$ is a binary operation and so a relation of order 3. If the structure is not ∞-point homogeneous, then it must be either 0, 1, or 2-point homogeneous.

The following is an example of a structure of type (1, 1) in which all of the relations are of order 2.

EXAMPLE 1.1. For each r in Re^+, define the relation R_r by

$$R_r = \{(x, rx) \mid x \text{ in } \text{Re}^+\},$$

and let $\mathscr{R} = \langle \mathrm{Re}^+, \geqslant, R_r \rangle_{r \in \mathrm{Re}^+}$. Then $\mathscr{R}$ is a relational structure with infinitely many relations, all of which are of order 2. It is trivial that for each $r > 0$, $\alpha_r(x) = rx$ is an automorphism of $\mathscr{R}$, and so $\mathscr{R}$ is at least 1-point homogeneous. It is not more since, as we now show, $\mathscr{R}$ is 1-point unique. Suppose β is an automorphism of $\mathscr{R}$, and let $r = \beta(1)$. So $\alpha_r(1) = \beta(1)$. Consider any $s > 0$, then $s = \alpha_s(1)$. Since β is an automorphism and α_s is both an automorphism and a relation of $\mathscr{R}$,

$$\beta(s) = \beta[\alpha_s(1)] = \alpha_s[\beta(1)] = \alpha_s(r) = sr = \alpha_r(s),$$

whence $\beta \equiv \alpha_r$. So $\mathscr{R}$ is 1-point unique.

The next example is of a structure that has relations of orders 2 and 3 and is of scale type (2, 2).

EXAMPLE 1.2. Let $\mathscr{R} = \langle \mathrm{Re}, \geqslant, \circ \rangle$, where for each x, y in Re,

$$x \circ y = (x + y)/2.$$

We show that the structure is interval scalable and from this it follows easily that it is of scale type (2, 2). Suppose α is an automorphism, then

$$\alpha(x \circ y) = \alpha[\tfrac{1}{2}(x + y)] = \alpha(x) \circ \alpha(y) = \tfrac{1}{2}[\alpha(x) + \alpha(y)].$$

Set $y = 0$,

$$\alpha(\tfrac{1}{2}x) = \tfrac{1}{2}[\alpha(x) + \alpha(0)].$$

So,

$$\alpha(x) + \alpha(z) = 2\alpha[\tfrac{1}{2}(x + z)] = \alpha(x + z) + \alpha(0).$$

Since this holds for all x, z in Re and α is strictly increasing, the only solution (Aczél, 1966, p. 34) is

$$\alpha(x) = \alpha(0) + rx, \qquad r > 0,$$

which is an interval scale transformation. It is obvious that all interval scale transformations leave $\circ$ invariant.

It follows from a theorem of Narens (1981b) (see Theorem 3.1 below) that there are no similar structures of scale type (M, M) for $M > 2$, and so the pattern of the previous examples does not extend.

The definitions of M-pont homogeneity and N-point uniqueness use ordered sets. A similar concept using unordered sets is important in topological group theory: a group $\mathscr{H}$ of continuous, one-to-one transformations of a topological space onto itself is called *M-transitive* iff for all subsets $\{a_1, ..., a_M\}$ and $\{b_1, ..., b_M\}$, each consisting of M distinct points of the space, there exists a transformation T in $\mathscr{H}$ such that $T(a_i) = b_i$, $i = 1, ..., M$; and it is called *strictly M-transitive* iff it is M-transitive and for all transformations T, U in $\mathscr{H}$ if T and U agree at M distinct points of the space, then $T = U$. Consider a relational structure $\mathscr{X} = \langle X, \succsim, S_j \rangle_{j \in J}$, where $\succsim$ is a total ordering on X, then by using the interval topology on $\langle X, \succsim \rangle$ we may regard $\mathscr{X}$ as endowed with a topology. With respect to this topology, all automorphisms of $\mathscr{X}$ are continuous, one-to-one transformations of the space onto itself since they preserve $\succsim$. So the group $\mathscr{G}$ of automorphisms of $\mathscr{X}$ is such a group of transformations. Thus, $\mathscr{G}$ is 0-transitive iff it is 0-point homogeneous, and $\mathscr{G}$ is 1-transitive iff it is 1-point homogeneous. But for $M > 1$, the two concepts differ. M-transitivity implies M-point homogeneity but not vice versa. Since $\succsim$ is a total ordering, strict N-transitivity of $\mathscr{X}$ implies N-point uniqueness of $\mathscr{X}$.

The concepts of ∞-point homogeneity and uniqueness are really bundles of rather distinct uniqueness and homogeneity concepts. One can give descriptions of the distinct concepts within these bundles by using Cantor's set theoretic concept of order type. Since, for the purposes of this paper, such refinements are not needed, we do not pursue their detailed development.

We turn now to the question of uniqueness. We have not been able to establish a direct structural connection analogous to that for homogeneity in Theorem 1.2, but we are able to show that under a plausible condition the uniqueness must be finite.

DEFINITION 1.3. Suppose $\mathscr{X} = \langle X, \succsim, S_j \rangle_{j \in J}$ is a totally ordered relational structure.

(1) If A is a subset of X, then A is said to be *dense* in $\langle X, \succsim \rangle$ iff for each u, v in X, if $u \succ v$, then for some a in A, $u \succsim a \succsim v$.

(2) Suppose for some positive integer n, F is an n-ary operation on X. Then F is said to be *$\mathscr{X}$-invariant* iff for each $x_1, \ldots, x_n$ in X and each automorphism α of $\mathscr{X}$,

$$\alpha F(x_1, \ldots, x_n) = F[\alpha(x_1), \ldots, \alpha(x_n)].$$

(3) Suppose K is a set and for each k in K, F_k is a $n(k)$-ary operation $n(k) > 0$ and let $\mathscr{F} = \{F_k \mid k \text{ in } K\}$. Let $A \subseteq X$. Then, the *algebraic closure of A under $\mathscr{F}$* is the smallest set Y such that

(i) $A \subseteq Y$,

(ii) for each k in K and all $x_1, \ldots, x_{n(k)}$ in Y,

$$F_k[x_1, \ldots, x_{n(k)}] \text{ is in } Y.$$

We note that the algebraic closure of A exists.

THEOREM 1.3. *Suppose $\mathscr{X} = \langle X, \succsim, S_j \rangle_{j \in J}$ is a totally ordered relational structure, $\mathscr{F}$ is a set of $\mathscr{X}$-invariant operations on X, and the identity function is in $\mathscr{F}$. If for some integer N, the algebraic closure under $\mathscr{F}$ of each set of N distinct elements of X is dense in $\langle X, \succsim \rangle$, then $\mathscr{X}$ is N-point unique.*

Proof. Suppose not. Let $A = \{a_1, ..., a_N\}$ be a set of N distinct points and let α, β be automorphisms such that $\alpha \neq \beta$ and $\alpha(a_i) = \beta(a_i)$ for $i = 1, ..., N$. Define Y_k inductively as: $Y_0 = A$ and for integers $k \geqslant 0$

$$Y_{k+1} = \{F(x_1, ..., x_n) \mid x_1, ..., x_n \text{ are in } Y_k \text{ and } F \text{ is in } \mathscr{F}\}.$$

Let

$$Y = \bigcup_{k=0}^{\infty} Y_k.$$

Then Y is the algebraic closure of A under $\mathscr{F}$, and since $\mathscr{F}$ contains the identity function, $Y_k \subseteq Y_{k+1}$ for each nonnegative integer k. Observe that $\alpha = \beta$ on Y_0. Suppose $\alpha = \beta$ on Y_k, and let y be any element of Y_{k+1}. So, for some integer n and some F in $\mathscr{F}$, there exist $y_1, ..., y_n$ in Y_k such that $y = F(y_1, ..., y_n)$. By the inductive hypothesis, $\alpha(y_i) = \beta(y_i)$, $i = 1, ..., n$, and since F is $\mathscr{X}$-invariant,

$$\begin{aligned}\alpha(y) &= \alpha F(y_1, ..., y_n)\\ &= F[\alpha(y_1), ..., \alpha(y_n)]\\ &= F[\beta(y_1), ..., \beta(y_n)]\\ &= \beta F(y_1, ..., y_n)\\ &= \beta(y).\end{aligned}$$

Thus, $\alpha = \beta$ on Y_{k+1} and so, by induction, on Y.

Suppose $\alpha \neq \beta$, then there exists some x such that $\alpha^{-1}(x) \neq \beta^{-1}(x)$. Without loss of generality, suppose $\alpha^{-1}(x) \succ \beta^{-1}(x)$. By hypothesis, the closure of A under $\mathscr{F}$, Y, is dense in $\langle X, \succsim \rangle$, and so let y in Y be such that $\alpha^{-1}(x) \succsim y \succsim \beta^{-1}(x)$. If $\alpha^{-1}(x) \succsim y \succ \beta^{-1}(x)$, then taking inverses we have $x \succsim \alpha(y)$ and $\beta(y) \succ x$. But over Y, $\alpha = \beta$, which results in the contradiction $x \succ x$. The case $\alpha^{-1}(x) \succ y \succsim \beta^{-1}(x)$ is similar, so $\alpha \equiv \beta$, proving that $\mathscr{X}$ is N-point unique. Q.E.D.

To apply Theorem 1.3, one must select an appropriate set $\mathscr{F}$ of $\mathscr{X}$-invariant operations. Usually such an $\mathscr{F}$ turns out to be the primitive operation(s) of $\mathscr{X}$ together with some additional operations that are definable in terms of the primitives of $\mathscr{X}$. (It can be shown that any nontrivial operation that can be defined in terms of the primitives by use of the first-order predicate calculus is $\mathscr{X}$-invariant.) For example, it is easy to show that for a PCS (see Definition 2.1) with half elements, the half element function h defined by $h(x) = y$ iff $y \circ y = x$, is $\mathscr{X}$-invariant, and it is not difficult to show (see Lemmas 2.1 and 2.2 of Narens & Luce, 1976) that for each x in X, the algebraic closure of $\{x\}$ under $\{\circ, h, \iota\}$, where ι is the identity function, is dense in $\langle X, \succsim \rangle$. Thus, by Theorem 1.3, a PCS with half elements is 1-point unique.

From an axiomatic standpoint, the key assumption of Theorem 1.3 involving the density of a closure of a set under a set of $\mathscr{X}$-invariant operations is hardly elegant.

However, in many cases—as in the example just given—it is a consequence of more elegant axioms. Theorem 1.3 attempts to isolate what is common to such situations and shows what drives many uniqueness proofs.

2. General Concatenation Structures

2.1. *Definitions*

The classical model for a number of basic physical attributes, including mass and length, is extensive measurement. The basic idea is that objects exhibiting the attribute in question can be both ordered qualitatively and combined to form new objects that also exhibit the attribute. The structure is such that the order can be represented by numerical ordering and the operation by addition. The general concept of a concatenation structure is the generalization of extensive structures in which the operation is no longer modelled by $+$, but by some other binary numerical operation. In particular, three major assumptions are abandoned: associativity, which is substantially equivalent to additivity; positivity, which restricts the representation to the positive reals when $+$ is the operation; and the Archimedean property, which is one way of formulating that we are dealing with a single dimension that can be represented by $\geqslant$ on the reals. The following states explicitly the various concepts involved.

Definition 2.1. Let X be a nonempty set, $\succsim$ a binary (order 2) relation on X, $\circ$ a partial binary operation[1] on X, and $\mathscr{X} = \langle X, \succsim, \circ \rangle$. $\mathscr{X}$ is said to be a *concatenation structure* iff, for all w, x, y, z in X, the following five conditions hold:

(1) $\succsim$ is a total ordering;

(2) for some u, v in X, $u \succ v$;

(3) for some u, v in X, $u \circ v$ is defined;

(4) *local definability*: if $x \circ y$ is defined and $x \succsim w$ and $y \succsim z$, then $w \circ z$ is defined;

(5) *monotonicity*:

(i) if $x \circ z$ and $y \circ z$ are defined, then $x \succsim y$ iff $x \circ z \succsim y \circ z$;

(ii) if $w \circ x$ and $w \circ y$ are defined, then $x \succsim y$ iff $w \circ x \succsim w \circ y$.

If condition (1) is weakened to assuming that $\succsim$ is a weak ordering and conditions (2–5) hold, we say that $\mathscr{X}$ is a *weakly ordered* concatenation structure.

[1] A partial binary operation on X is a function from a subset of $X \times X$ into X. The expression $x \circ y$ is said to be *defined* iff (x, y) is in the domain of $\circ$. Note that a partial binary operation is a relation of order 3.

In the following definitions, $\mathscr{X} = \langle X, \succsim, \circ \rangle$ is said to have the property denoted by the italicized word provided that the defining conditions hold for all x, y, z in X for which the indicated operations are defined:

Closed iff $\circ$ is an operation, i.e., $x \circ y$ is defined for all x, y in X.
Idempotent iff $x \circ x = x$.
Weakly positive iff $x \circ x \succ x$.
Weakly negative iff $x \circ x \prec x$.
Positive iff $x \circ y \succ x, y$.
Negative iff $x \circ y \prec x, y$.
Intern iff whenever $x \succ y$, then $x \succ x \circ y \succ y$ and $x \succ y \circ x \succ y$.
Intensive iff $\mathscr{X}$ is both intern and idempotent.
Associative iff $x \circ (y \circ z) = (x \circ y) \circ z$.
Half elements iff for each x in X there exists a u in X such that $x = u \circ u$.

Restrictedly solvable iff whenever $x \succ y$, there exists u in X such that either $x \succ y \circ u \succ y$ or $x \succ x \circ u \succ y$. (Note that if $\mathscr{X}$ is positive and restrictedly solvable, the latter inequality canot arise.)

Solvable iff given any three of w, x, y, z the fourth exists such that $w \circ x = y \circ z$.

Dedekind complete iff $\langle X, \succsim \rangle$ is Dedekind complete, i.e., every nonempty bounded subset of X has a least upper bound in X.

Suppose $\mathscr{X}$ is a concatenation structure and J is an interval of integers. The sequence $\{x_j\}_{j \in J}$, x_j in X, is said to be a *standard sequence* iff J is an interval whose first element is 1 and x_j is defined inductively by: $x_1 = x$ and if j in J then $x_j = x_{j-1} \circ x$ is defined. A sequence $\{x_j\}_{j \in J}$ is said to be a *difference sequence* iff for some u, v in X, $u \neq v$, and all j such that $j, j-1$ are in J; $x_j \circ u$ and $x_{j-1} \circ v$ are defined and $x_j \circ u = x_{j-1} \circ v$. A sequence $\{x_j\}_{j \in J}$ is said to be *bounded* iff for some p, q in X, $p \succsim x_j \succsim q$ for all j in J.

$\mathscr{X}$ is said to be *Archimedean* iff either

(i) $\mathscr{X}$ is positive, restrictedly solvable, and either every bounded standard sequence is finite or every bounded difference sequence is finite, or

(ii) $\mathscr{X}$ is solvable and every bounded difference sequence is finite.

$\mathscr{X}$ is said to be a PCS[2] iff it is positive, restrictedly solvable and Archimedean in the sense that every bounded standard sequence is finite.

$\mathscr{X}$ is said to be *extensive* iff it is an associative PCS.

Comment. We have not formulated the concept of "Archimedean" in an exhaustive way, but rather for the classes of structures that have been or will be shown below to have Archimedean representations in the sense of numerical comparability. A positive structure may be Archimedean in two senses, but an idem-

[2] Narens and Luce (1976) called this a "positive concatenation structure" and abbreviated it PCS. We use just the abbreviation in an attempt to avoid confusion with a concatenation structure that is positive but need not be either restrictedly solvable or Archimedean.

potent one can be Archimedean only in the second sense given in the above definition.

A number of simple relations hold among these concepts, including the following: A concatenation structure that is positive (negative) is weakly positive (negative); a closed concatenation structure that is idempotent is intern, i.e., intensive; an associative, idempotent structure is degenerate.

The following theorem shows the relationship between Dedekind completeness and the Archimedean condition.

THEOREM 2.1. *Suppose* $\mathscr{X} = \langle X, \succsim, \circ \rangle$ *is a concatenation structure that is Dedekind complete. Then the following two statements are true*:

(i) *If* $\mathscr{X}$ *is positive and for each* x, y *such that* $x \succ y$ *there exists* z *such that* $z \circ y = x$, *then* $\mathscr{X}$ *is Archimedean in the sense that each bounded standard sequence is finite.*

(ii) *If* $\mathscr{X}$ *is solvable, then* $\mathscr{X}$ *is Archimedean.*

Proof. (i) Suppose not. Let $\{x_i\}$ be a bounded, infinite standard sequence. Then $x_1 = x, x_2 = x \circ x, x_3 = x_2 \circ x, \ldots$, and by positivity $x_1 \prec x_2 \prec x_3 \prec \cdots$. By Dedekind completeness, let a be the l.u.b. of $\{x_i\}$. Then $a \succ x_1 = x$. By hypothesis, let z be such that $a = z \circ x$. By positivity, $a \succ z$. Therefore, by the choice of a, let x_j be such that $a \succ x_j \succ z$. By positivity,

$$x_{j+1} = x_j \circ x \succ z \circ x = a,$$

contradicting the choice of a.

(ii) Suppose $\mathscr{X}$ is solvable and not Archimedean. We may suppose that there exists a bounded infinite sequence $\{x_i\}$ and elements u, v of X such that $x_{i+1} \circ u = x_i \circ v$. Without loss of generality, we may suppose $x_1 \prec x_2 \prec \cdots$, which by monotonicity implies $v \succ u$. By Dedekind completeness, let a be the l.u.b. of $\{x_i\}$. By solvability, let y be such that $a \circ u = y \circ v$. Since $v \succ u$, it follows from monotonicity that $a \succ y$. Since a is the l.u.b. of the sequence, let j be such that $a \succ x_j \succ y$. Then by monotonicity,

$$x_{j+1} \circ u = x_j \circ v \succ y \circ v = a \circ u,$$

and so by monotonicity, $x_{j+1} \succ a$, which contradicts the choice of a. Q.E.D.

2.2 *Homogeneous Structures*

THEOREM 2.2. *Suppose* $\mathscr{X}$ *is a concatenation structure whose automorphism group is* 1-*point homogeneous. Then,*

(i) $\mathscr{X}$ *is closed*;

(ii) $\mathscr{X}$ *is either idempotent, weakly positive, or weakly negative*;

(iii) *if $\mathscr{X}$ is N-point unique for some finite N, then either $N=1$ or $\mathscr{X}$ is idempotent and $N=1$ or 2.*

Proof. (i) Since there exist u, v such that $u \circ v$ is defined, it follows by local definability that $w \circ w$ is defined for $w = \min(u, v)$. Consider any x, y in X and let $z = \max(x, y)$. By 1-point homogeneity there exists α in $\mathscr{G}$ such that $\alpha(w) = z$, and so $z \circ z$ is defined since by monotonicity and local definability $\alpha(w \circ w) = \alpha(w) \circ \alpha(w) = z \circ z$. By local definability $x \circ y$ is defined, proving $\circ$ is an operation.

(ii) Suppose for some x, $x \circ x = x$. Then for each y, there exists α in $\mathscr{G}$ such that $\alpha(x) = y$. By monotonicity

$$y \circ y = \alpha(x) \circ \alpha(x) = \alpha(x \circ x) = \alpha(x) = y,$$

proving that $\mathscr{X}$ is idempotent. The proofs in the other two cases are similar.

(iii) Suppose $\mathscr{X}$ is N-pont unique. Since $\mathscr{X}$ is 1-point homogeneous, $N \geqslant 1$. Suppose $\mathscr{X}$ is not idempotent, then by (ii) it is either weakly positive or negative. Without loss of generality, suppose the former. Note that by weak positivity,

$$x \prec x \circ x \prec (x \circ x) \circ (x \circ x) \prec [(x \circ x) \circ (x \circ x)] \circ [(x \circ x) \circ (x \circ x)] \prec \cdots.$$

Now, suppose α, β are in $\mathscr{G}$ and agree at x. Then, by induction on the fact that

$$\alpha(x \circ x) = \alpha(x) \circ \alpha(x) = \beta(x) \circ \beta(x) = \beta(x \circ x),$$

they agree at all points within the above sequence of inequalities. Thus, by N-point uniqueness, $\alpha \equiv \beta$, and so $N=1$. Next, suppose $\mathscr{X}$ is idempotent and so is intern. Let x, y in X and α, β in $\mathscr{G}$ be such that $x \succ y$, $\alpha(x) = \beta(x)$, and $\alpha(y) = \beta(y)$. Since $x \succ x \circ y \succ y$, and

$$\alpha(x \circ y) = \alpha(x) \circ \alpha(y) = \beta(x) \circ \beta(y) = \beta(x \circ y),$$

by using induction it follows that α and β agree at N distinct points and so are identical, proving $N \leqslant 2$. Q.E.D.

This result establishes that if a finite uniqueness condition holds, then each homogeneous concatenation structure falls into one of three scale types: (1, 1), (1, 2), or (2, 2). This means that we should try either to understand or to rule out those cases that are not unique for any finite N. Theorem 1.3 shows that the density of the algebraic closure of $\circ$ is sufficient to do so. And in the next section, another sufficient condition for real concatenation structures is presented, namely, continuty of the operation.

The several cases that remain after ∞-point uniqueness is ruled out are summarized in the simple tree, where M is the degree of homogeneity of the structure in Fig. 1.

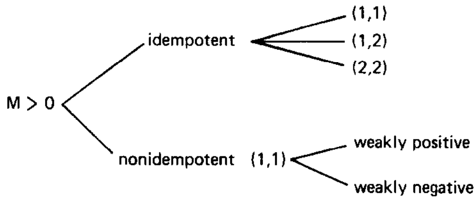

FIGURE 1

3. FUNCTIONAL CHARACTERIZATION OF REAL CONCATENATION STRUCTURES

3.1. *Introduction*

This section is devoted to an attempt to construct all real concatenation structures for the several scale types (M, N). We say "attempt" because we do not have satisfactory results for the $M = 0$ case.

The following results of Narens (1981a, 1981b) are essential to the developments in this section. In the statement of these results, the *similarity group* refers, as is usual, to the multiplicative group of positive reals, and the *affine group* to the group of real transformations $\{rx + s \mid r \text{ in } \mathrm{Re}^+ \text{ and } s \text{ in } \mathrm{Re}\}$.

THEOREM 3.1. *Suppose $\mathscr{X} = \langle X, \succsim, S_j \rangle_{j \in J}$ is a totally ordered relational structure for which $\langle X, \succsim \rangle$ and $\langle \mathrm{Re}^+, \geqslant \rangle$ are isomorphic. Then the following three statements are true*:

(1) *If $\mathscr{X}$ is of scale type* $(1, 1)$, *then $\mathscr{X}$ is isomorphic to a real relational structure $\mathscr{R} = \langle \mathrm{Re}^+, \geqslant, R_j \rangle_{j \in J}$ whose automorphism group is the similarity group and the isomorphisms of $\mathscr{X}$ onto $\mathscr{R}$ form a ratio scale.*

(2) *If $\mathscr{X}$ is of scale type* $(2, 2)$, *then $\mathscr{X}$ is isomorphic to a real relational structure $\mathscr{R} = \langle \mathrm{Re}^+, \geqslant, R_j \rangle_{j \in J}$ whose automorphism group is the affine group and the isomorphisms of $\mathscr{X}$ onto $\mathscr{R}$ form an interval scale.*

(3) *$\mathscr{X}$ cannot be of type (M, M) for $M > 2$.*

Proof. Theorem 2.6 of Narens (1981a) and Theorems 2.2 and 2.3 of Narens (1981b). Q.E.D.

This result is related to, but apparently different from, ones that can be found in the mathematical literature. The following references of related results were brought to our attention by K. Strambach. Hölder (1901) established that a real transformation group that preserves order and is 1-point unique can be Archimedean ordered, and so is isomorphic to a subgroup of the similarity group; see Fuchs (1966). A brief proof can be found in Salzmann (1958). Brouwer (1909) studied real transformation groups that are order preserving, locally compact, and continuous, and showed that if such a group is M-point homogeneous, $M \geqslant 1$ (in this literature, this property is called "transitivity") and 2-point unique, then it is isomorphic to the affine group. Strambach (1969) studied groups of order preserving homeomorphisms (both the map and its inverse are continuous) that are 2-point unique, and he showed the equivalence of three properties which are distinct from either 1- or 2-point homogeneity. Other references that include some related material are Glass (1981) and Rosenstein (1982).

The section is subdivided as follows. In Section 3.2, we show that under very reasonable conditions, real concatenation structures are 2-point unique. Thus, for these real structures we have the four cases mentioned earlier: $M=0$, $(1, 1)$, $(2, 2)$, and $(1, 2)$. We do not know much about the $M=0$ case (Section 3.3). Thanks to Theorem 3.1 we can fully characterize the $(1, 1)$ and $(2, 2)$ cases (Sects. 3.4 and 3.6). Section 3.5 is concerned with $(1, 1)$ idempotent structures that are related to positive structures by doubling functions. For the $(1, 2)$ case, the following result due to Alper (1985) gives us the necessary understanding to characterize the $(1, 2)$ real concatenation structures.

THEOREM 3.2. *Suppose* $\mathscr{X} = \langle X, \succsim, S_j \rangle_{j \in J}$ *is a totally ordered structure,* $\langle X, \succsim \rangle$ *and* $\langle \mathrm{Re}, \geqslant \rangle$ *are isomorphic, and* $\mathscr{X}$ *is of scale type* $(1, 2)$. *Then there exists a real relational structure* $\mathscr{R} = \langle \mathrm{Re}, \geqslant, R_j \rangle_{j \in J}$ *such that* $\mathscr{X}$ *and* $\mathscr{R}$ *are isomorphic, the automorphism group of* $\mathscr{R}$ *is a subgroup of the affine group for which all of the translations* $x \to x + r$, r *in* Re, *are included.*

Proof. Alper (1985). Q.E.D.

We must at this point enter into an aside about the domains of representations and their corresponding automorphism groups. If $\mathscr{X} = \langle X, \succsim, \circ \rangle$ is an arbitrary concatenation structure such that $\langle X, \succsim \rangle$

(i) has no maximal or minimal element,

(ii) has a countable dense subset (Definition 1.3), and

(iii) is Dedekind complete (Definition 2.1),

then for some real operations $\oplus$ and $\otimes$, $\mathscr{X}$ is isomorphic to the real concatenation structures $\langle \mathrm{Re}, \geqslant, \oplus \rangle$ and $\langle \mathrm{Re}^+, \geqslant, \otimes \rangle$ (Cantor, 1895). Thus, in general the choice of Re or Re^+ as the domain of a real concatenation representation is arbitrary. Physicists almost always use Re^+, in which case the $(2, 2)$ representation has as its automorphism group not the affine group but rather transformations $x \to \sigma x^\rho$, where $\sigma > 0$ and $\rho > 0$. Stevens (1957) called such scales "logarithmic interval," which has come to be abbreviated "log interval," because a logarithmic transformation of a Re^+ representation yields an interval scale representation on Re. To maintain a consistent terminology, it seems appropriate to call this group of automorphisms the *log affine* group. It is important to realize that the unit in the $(2, 2)$ case, i.e., the parameter that is comparable to that of the similarity transformations of the $(1, 1)$ case, is the σ of the log affine transformation, not ρ which is the parameter that multiplies the variable $\log x$ in the interval scale representation.

Put another way, if we formulate the (1, 1) case as a representation on Re, the automorphisms become the *translation* group $x \to x + s$, where $s = \log \sigma$. Following the convention of physics, we shall mostly use the Re^+ representation, but we shall also refer to the automorphisms of the similarity group as translations. When we come, in Section 7, to discuss the utility of gambles, we will shift to Re, which is the convention followed in economics and psychology.

3.2. *Uniqueness Results*

For measurement theory, PCSs (Definition 2.1) are by far the most important structures that satisfy positivity. The following result of Cohen and Narens (1979) nicely characterizes uniqueness for this class of structures:

THEOREM 3.3. *Suppose $\mathscr{X}$ is a* PCS. *Then the following are true*:

(1) *$\mathscr{X}$ is either* 0 *or* 1-*point unique.*

(2) *If the operation is not closed, then $\mathscr{X}$ is* 0-*point unique, i.e., the identity is the only automorphism.*

Proof. Theorems 2.4 and 2.5 of Cohen and Narens (1979). Q.E.D.

Note that it follows from Part 2 of this result that if $\langle X, \succsim \rangle$ has a maximal element, then $\mathscr{X}$ is 0-point unique. The following result is comparable for real structures with an intern operation:

THEOREM 3.4. *Suppose $[a, b]$, $a < b$, is a closed interval in* Re *and $\circ$ is an intern operation on $[a, b]$. Then $\mathscr{R} = \langle [a, b], \geqslant, \circ \rangle$ is* 0-*point unique.*

Proof. Suppose α is an automorphism different from the identity ι. Since α and ι are strictly increasing functions of Re onto Re, they are continuous and, therefore, so is $h = \alpha - \iota$. Let p be in $[a, b]$ and $\alpha(p) \neq p$. Since $\alpha(a) = a$ and $\alpha(b) = b$, p is in the open interval (a, b) and $h(p) \neq 0$. Since h is continuous, there is an open interval U about p such that $U \subseteq [a, b]$ and $h(q) \neq 0$ for all q in U. Let (c, d) be the largest open interval about p such that $U \subseteq (c, d) \subseteq [a, b]$ and $h(q) \neq 0$ for all q in (c, d). Suppose $h(c) \neq 0$. Then $h(c) \neq h(a)$ and so $c \neq a$. Thus $a < c$, and so we can find an open interval V about c such that $V \subseteq [a, b]$ and $h(q) \neq 0$ for all q in V. But in that case, $(c, d) \cup V$ would be an open interval about p meeting the defining conditions of (c, d) and larger than (c, d). As this is impossible, $h(c) = 0$. Similarly, $h(d) = 0$. Since $c < d$ and $\circ$ is intern, $c \circ d$ is in (c, d), and so

$$0 \neq h(c \circ d) = \alpha(c \circ d) - c \circ d = \alpha(c) \circ \alpha(d) - c \circ d = c \circ d - c \circ d = 0,$$

which is impossible. So $\alpha \equiv \iota$. Q.E.D.

Note that Theorem 3.4 only assumes $\circ$ to be intern; it does not assume that $\mathscr{R}$ is a concatenation structure.

We turn now to a sequence of three theorems that establish 2-point uniqueness. The first, which is a consequence of Theorem 3.4, applies to real intensive structures that meet a special condition. This is then used in Theorem 3.6 to formulate a condition that applies to nonintensive structures as well. And finally, Theorem 3.7 shows that this condition is satisfied by any real concatenation operation that is continuous.

THEOREM 3.5. *Suppose* $\mathscr{R} = \langle \text{Re}, \geqslant, \circ \rangle$ *is a real intensive concatenation structure with the following property: for each* u, v *in* Re, *if* $u < v$, *there exist* w, x, y, z *in* Re *such that*

$$w < u < x,\ y < v < z \qquad \textit{and} \qquad u < w \circ y,\ x \circ z < v.$$

Then $\mathscr{R}$ is *2-point unique.*

Proof. Suppose not. Then there are $r < s$ in Re and distinct automorphisms α, β of $\mathscr{R}$ such that $\alpha(r) = \beta(r)$ and $\alpha(s) = \beta(s)$. So $\gamma = \beta^{-1}\alpha$ has the property $\gamma(r) = r$ and $\gamma(s) = s$. Let $\mathscr{S} = \langle [r, s], \geqslant, \circ' \rangle$, where $\circ'$ is the restriction of $\circ$ to $[r, s]$. Applying Theorem 3.4 to $\mathscr{S}$, the restriction of γ to $\mathscr{S}$, which is an automorphism of $\mathscr{S}$, must agree with the identity ι on $[r, s]$. Since $\gamma \neq \iota$, let p in Re be such that $\gamma(p) \neq p$. First, assume $p < r$. Let

$$u = \text{lub}\{z \mid p \leqslant z < r \quad \text{and} \quad \gamma(z) \neq z\}.$$

Then for all x in $(u, s]$, $\gamma(x) = x$. Hence since α and β are continuous, so is γ and it agrees with ι on $(u, s]$, so $\gamma(u) = u$. By hypothesis, let y in $[u, s]$ and $w < u$ be such that $t = w \circ y$ is in $[u, s]$. Then since γ is an automorphism of $\mathscr{R}$,

$$\gamma(t) = \gamma(w \circ y) = \gamma(w) \circ \gamma(y).$$

But $\gamma(t) = t$ and $\gamma(y) = y$, so

$$w \circ y = \gamma(t) = \gamma(w) \circ \gamma(y) = \gamma(w) \circ y,$$

which by monotonicity yields $\gamma(w) = w$. Therefore, by Theorem 3.4, $\gamma = \iota$ on $[w, s]$. But by the choice of w and u, $[w, s]$ must include a z for which $\gamma(z) \neq z$, which is impossible. Thus, we assume $p > s$, and the argument is similar, using the other half of the hypothesis. Q.E.D.

THEOREM 3.6. *Suppose* $\mathscr{R} = \langle \text{Re}, \geqslant, \circ \rangle$ *is a real concatenation structure with a closed operation and with half elements that satisfies the property: for each* u, v *in* Re, *if* $u < v$, *then there exist* w, x, y, z *in* Re *such that*

$$w < u < x,\ y < v < z \qquad \textit{and} \qquad u \circ u < w \circ x,\ z \circ y < v \circ v. \tag{3.1}$$

Then $\mathscr{R}$ *is 2-point unique.*

Proof. Since $\mathscr{R}$ has half elements, let h be the function from Re to Re defined by $h(x) \circ h(x) = x$. Note that for each automorphism α of $\mathscr{R}$ and each x in Re,

$$\alpha(x) = \alpha[h(x) \circ h(x)] = \alpha h(x) \circ \alpha h(x),$$

and so by definition of h, $\alpha h(x) = h[\alpha(x)]$. If for each x, y in Re we define the operation $*$ *by* $x * y = h(x \circ y)$, then

$$\begin{aligned}\alpha(x * y) &= \alpha[h(x) \circ h(y)] \\ &= \alpha h(x) \circ \alpha h(y) \\ &= h[\alpha(x)] \circ h[\alpha(y)] \\ &= \alpha(x) * \alpha(y).\end{aligned}$$

Since the automorphisms of $\mathscr{R}$ preserve $\geqslant$, we conclude that they are also automorphisms of $\mathscr{S} = \text{Re}, \geqslant, *\rangle$. Now, suppose x, y are in Re and $x < y$. From the fact that by monotonicity of $\circ$, $x \circ x < x \circ y < y \circ y$ and that for all z in Re $h(z \circ z) = z$, we may conclude

$$x < h(x \circ y) = x * y < y,$$

and so $*$ is an intern operation. Since h and $\circ$ are strictly monotonic, it follows that $\mathscr{S}$ satisfies monotonicity, and so $\mathscr{S}$ is an intensive structure. By applying h to the right inequalities of Eq. (3.1), we see that for each u, v in Re, if $u < v$, then there exist w, x, y, z in Re such that

$$w < u < x,\ y < v < z \qquad \text{and} \qquad u < w * y,\ z * x < v.$$

So, by Theorem 3.5, $\mathscr{S}$ satisfies 2-point uniqueness and, since each automorphism of $\mathscr{R}$ is one of $\mathscr{S}$, so does $\mathscr{R}$. Q.E.D.

The next result, which is a consequence of Theorem 3.6, seems to be the one that will usually be invoked in applications to assure that a real concatenation structure is 2-point unique, since the property of continuity of the operation is usually acceptable to scientists.

THEOREM 3.7. *Suppose* $\mathscr{R} = \langle \text{Re}, \geqslant, \circ \rangle$ *is a closed concatenation structure and* $\circ$ *is onto* Re. *If the operation* $\circ$ *is continuous, then* $\mathscr{R}$ *satisfies 2-point uniqueness.*

Proof. We first show that $\mathscr{R}$ has half elements. Let w be in Re and choose w_1 and w_2 such that $w_1 < w < w_2$. Since $\circ$ is onto, there are u, v, x, y in Re such that

$$u \circ v = w_1 < w < w_2 = x \circ y.$$

Let $p = \min\{u, v\}$ and $q = \max\{x, y\}$, then

$$p \circ p \leqslant w_1 < w < w_2 \leqslant q \circ q.$$

By the monotonicity of $\circ$, $p < q$. Define the function f by $f(s) = s \circ s$. Since $\circ$ is continuous, so is f. Since $f(p) < w < f(q)$, by the intermediate value theorem of the calculus, for some r with $p < r < q$, $r \circ r = f(r) = w$, and so r is the half element of w.

According to Theorem 3.6, to prove the result it suffices to show for each u, v in Re, $u < v$, there exists w, x, y, z meeting Eq. (3.1). As the proofs of the w, x and y, z pairs are similar, we only do the former. Let y be such that $u < y < v$. Since $\circ$ is monotonic, $u \circ u < u \circ y < v \circ v$. Let $\varepsilon > 0$ be such that

$$u \circ u < u \circ y - \varepsilon < u \circ y < u \circ y + \varepsilon < v \circ v.$$

Since $\circ$ is continuous, there exists $\delta > 0$ such that for each w, x in Re, if $|w - u| < \delta$ and $|x - y| < \delta$, then $|w \circ x - u \circ y| < \varepsilon$. So choose any $w < u$ such that $|w - u| < \delta$ and set $x = y$, and we have

$$w < u < y < v \qquad \text{and} \qquad u \circ u < w \circ y < v \circ v.$$

Using a similar proof for x and z, we conclude 2-point uniqueness. Q.E.D.

3.3. *0-Point Homogeneity*

We do not at this time have a theory for 0-point homogeneous structures, not even concatenation ones. All we have are techniques for generating a large variety of examples, but we have nothing approaching a satisfactory classification scheme. The following are some examples: Let $\circ_1$ be the operation and Re^+ defined by

$$x \circ_1 y = x + y + x^2 y^2.$$

Cohen and Narens (1979, Example 4.2, p. 225) showed that $\langle \mathrm{Re}^+, \geqslant, \circ_1 \rangle$ is a PCS of scale type (0, 0). The structure $\mathscr{R} = \langle \mathrm{Re}, \geqslant, + \rangle$ is, perhaps, the best known example of one having an invariant element under its automorphisms, namely 0, and it is of scale type (0, 1). The similarity group is the automorphism group of $\mathscr{R}$, as it is of the (1, 1) substructures $\langle \mathrm{Re}^+, \geqslant, + \rangle$ and $\langle \mathrm{Re}^-, \leqslant, + \rangle$ which, incidentally, are both PCSs. Thus $\mathscr{R}$ is decomposable into well defined, well-behaved substructures together with relations involving a single point 0. To understand such a structure with an invariant element, it appears sufficient to understand the structures on either side of the invariant element and the behavior of the element itself. Using this strategy, one can reduce the study of a structure with finitely many invariant elements to the study of structures with no invariant elements. Unfortunately, there are structures with no invariant elements that are 0-point homogeneous, and we do not understand much about these. An example (Cohen and Narens, 1979, Example 3.1, p. 207) is $\langle \mathrm{Re}^+, \geqslant, \circ_2 \rangle$, where

$$x \circ_2 y = x + y + (xy)^{1/2}\{2 + \sin[(\tfrac{1}{2}) \log xy]\},$$

which is a PCS with automorphisms $x \to x \exp(2\pi n)$, $n = 0, 1, \ldots$, that are a group of type (0, 1). The structure has no invariant elements.

3.4. *Scale Type* (1, 1): *The Ratio Scale Case*

The following concept, which appeared in Cohen and Narens (1979) is essential to understanding concatenation structures that are 1-point homogeneous.

DEFINITION 3.1. A real structure $\langle \mathrm{Re}^+, \geqslant, \circ, f \rangle$ is said to be a *real unit structure* iff $\langle \mathrm{Re}^+, \geqslant, \circ \rangle$ is a closed concatenation structure and f is a function from Re^+ onto Re^+ such that

(i) f is strictly monotonic increasing,

(ii) f/ι, where ι is the identity function, is strictly decreasing, i.e., if $x < y$, then $f(x)/x > f(y)/y$, and

(iii) for all x, y in Re^+,

$$x \circ y = yf(x/y). \tag{3.2}$$

THEOREM 3.8. *If $\mathscr{R} = \langle \mathrm{Re}^+, \geqslant, \circ, f \rangle$ is a real unit structure, then the following statements are true*:

(1) $\lim_{x \to \infty} f(x)/x$ *exists and* $= k \leqslant f(1)$.

(2) $\mathscr{R}$ *has half elements.*

(3) *If $\mathscr{R}$ is weakly positive, then* $f(1) > 1$.

(4) *If $\mathscr{R}$ is positive, then $f(x) > x$ and $k \geqslant 1$, and $\mathscr{R}$ is Archimedean iff $f^{(n)}(1)$ is unbounded; a sufficient condition for the latter statement is $k > 1$.*

(5) *If $\mathscr{R}$ is positive and restrictedly solvable, then $k = 1$.*

(6) *If $\mathscr{R}$ is weakly negative, then $f(1) < 1$ and it is impossible for $\mathscr{R}$ to be restrictedly solvable.*

(7) *If $\mathscr{R}$ is negative, then $f(x) < x$.*

(8) *Suppose $\mathscr{R}$ is idempotent. Then $f(1) = 1$; and for each $x < 1$, $x < f(x) < 1$; and for each $x > 1$, $x > f(x) > 1$.*

Proof. (1) Since f/ι strictly is decreasing and >0, the limit k exists, and for $x > 1$, $f(x)/x < f(1)/1$ and so $k \leqslant f(1)$.

(2) For each x in Re^+, let $y = x/f(1)$, then $y \circ y = yf(y/y) = x$, proving that half elements exist.

(3) Since $\mathscr{R}$ is weakly positive, $x < x \circ x = xf(1)$, whence $f(1) > 1$.

(4) Since $\mathscr{R}$ is positive, $x < x \circ y = yf(x/y)$. So, setting $z = x/y$, we have $z < f(z)$, and so $k \geqslant 1$. Observe that

$$2x = x \circ x = xf(x/x) = xf(1) = xf[f(1)\, x/x],$$

and so by induction for $n > 2$,

$$nx = (n-1)\, x \circ x = xf[(n-1)\, x/x] = xf[f^{(n-1)}(1)\, x/x] = xf^{(n)}(1).$$

Thus, nx is unbounded iff $f^{(n)}(1)$ is. Suppose $k > 1$. Then from the fact that f/ι is strictly decreasing, $f(x) > kx$, and so $f(1) > k$. Since f is strictly increasing, by induction $f^{(n)}(1) > k^n$, and so $f^{(n)}(1)$ is unbounded.

(5) For each $\varepsilon > 0$, restricted solvability implies there exists $\delta(y, \varepsilon)$ such that $y < y \circ \delta(y, \varepsilon) < y(1+\varepsilon)$. Using part (iii) of the definition of a unit structure, letting $\delta = \delta(y, \varepsilon)$, and dividing by y,

$$1 < f(y/\delta)/(y/\delta) < 1 + \varepsilon.$$

Now, let $\varepsilon \to 0$, then since $\delta \to 0$ it follows that $k = 1$.

(6) As in part (3), $f(1) < 1$. Were restricted solvability to hold, an argument analogous to that used in part (5) shows $k = 1 > f(1)$, which is impossible by part (1).

(7) The proof is analogous to the first part of part (4).

(8) Suppose x is in Re^+. Since $x = x \circ x = xf(1)$, we see $f(1) = 1$. Suppose $x < 1$. For any z in Re^+, let $y = xz$, then $y < y \circ z = zf(y/z) = zf(x)$, and so $x < f(x)$. Since $f(1) = 1$ and f is strictly increasing, $f(x) < 1$. Similarly, if $x > 1$ then $x > f(x) > 1$. Q.E.D.

The following result generalizes slightly Theorems 3.3 and 3.4 of Cohen and Narens (1979).

THEOREM 3.9. *Suppose* $\mathscr{R} = \langle \mathrm{Re}^+, \geqslant, \circ \rangle$ *is a real, closed concatenation structure. Then,* $\mathscr{R}$ *is of scale type* (1, 1) *iff* $\mathscr{R}$ *is isomorphic to a real unit structure* $\langle \mathrm{Re}^+, \geqslant, \circ', f \rangle$ *with the following property: if for all* $x > 0$ *and some* $\rho > 0$, $f(x^\rho) = f(x)^\rho$, *then* $\rho = 1$.

Proof. Suppose $\mathscr{R}$ is a closed concatenation structure of scale type (1, 1). Since, by Theorem 3.1, $\mathscr{R}$ is isomorphic to a real structure with the similarity group its automorphism group, we may without loss of generality assume $\mathscr{R}$ is that structure. Thus, $\circ$ must satisfy the functional equation that for all $x, y, r > 0$,

$$r(x \circ y) = rx \circ ry.$$

Define the function f by $f(z) = z \circ 1$, which is well defined since $\circ$ is an operation. Then by setting $r = 1/y$, we see that

$$(x \circ y)/y = (x/y) \circ 1 = f(x/y),$$

which establishes part (iii) of the unit representation. By monotonicity of the operation,

$$\begin{aligned} x/y \geqslant x'/y \quad & \text{iff } x \geqslant x' \\ & \text{iff } x \circ y \geqslant x' \circ y \\ & \text{iff } yf(x/y) \geqslant yf(x'/y) \\ & \text{iff } f(x/y) \geqslant f(x'/y), \end{aligned}$$

proving that f is strictly increasing. And

$$\begin{aligned} x/y \leqslant x/y' \quad & \text{iff } y \geqslant y' \\ & \text{iff } x \circ y \geqslant x \circ y' \\ & \text{iff } yf(x/y) \geqslant y'f(x/y') \\ & \text{iff } f(x/y)/(x/y) \geqslant f(x/y')/(x/y') \end{aligned}$$

proving that f/ι is strictly decreasing.

Now suppose that for some $\rho \neq 1$, $f(x^\rho) = f(x)^\rho$. We show that this implies that $\alpha(x) = x^\rho$ is an automorphism, which is impossible since it is not a similarity. Obviously, α is increasing and onto, and

$$\begin{aligned} \alpha(x \circ y) &= y^\rho f(x/y)^\rho \\ &= y^\rho f(x^\rho/y^\rho) \\ &= \alpha(y)\, f[\alpha(x)/\alpha(y)] \\ &= \alpha(x) \circ \alpha(y), \end{aligned}$$

and so α is indeed an automorphism, which is impossible. So such $\rho \neq 1$ cannot exist.

Conversely, it is easily verified that any real unit structure is a closed concatenation structure for which the similarities are all automorphisms, and so the automorphism group $\mathscr{G}$ contains a (1, 1) subgroup. Since $\circ'$ is onto and is continuous, by Theorem 3.7, the structure is 2-point unique. Thus, by Theorem 3.2, $\mathscr{G}$ is a subgroup of the log affine group. If $\alpha(x) = \sigma x^\rho$ is in $\mathscr{G}$, then by the above argument $f(x^\rho) = f(x)^\rho$ and so $\rho = 1$, proving 1-point uniqueness. Thus, the unit structure is of scale type (1, 1). Q.E.D.

3.5. *Idempotent Structures with Doubling Functions*

A major part of the literature on measurement concerns the construction of numerical representations for particular qualitative structures. Most of Krantz *et al.* (1971) is devoted to this sort of enterprise, focusing on structures with additive and polynomial representations. Narens and Luce (1976) and Cohen and Narens (1979) carried out a similar program for PCSs (see Definition 2.1). The key to the known constructions is positivity and the Archimedean axiom, and so these methods do not extend to idempotent structures, which are nonpositive and not Archimedean in the sense of PCSs. It would be nice to have a general constructive device for idempotent structures that is similar to the standard sequence approach used with PCSs, but none has appeared in the literature. Something along these lines is developed in Section 5. An alternative tack was suggested by Narens and Luce (1976) for those idempotent structures that can be mapped onto a PCS by means of a function having the following properties:

DEFINITION 3.2. Suppose $\mathscr{X} = \langle X, \succsim, * \rangle$ is an intensive structure and δ is a function from $A \subseteq X$ into X. Then δ is said to be a *doubling function* iff for every x, y in X,

(i) δ is strictly monotonic increasing;

(ii) if $x \succsim y$ and x is in A, then y is in A;

(iii) if $x \succ y$, then there is u in X such that $y * u$ is in A and $x \succ \delta(y * u)$;

(iv) if $x * y$ is in A, then $\delta(x * y) \succ x, y$;

(v) let $x_n, n = 1, 2, \ldots$, be such that $x_1 - x$ and if x_{n-1} is in A, then $x_n \sim (x_{n-1}) * x$; then either there exists an integer n such that x_n is not defined or $x_n \succsim y$.

Let $\mathscr{X}$ and δ be as in Definition 3.2. Narens and Luce (1976, Theorem 3.1) proved that the operation $\circ$ defined by $x \circ y = \delta(x * y)$ is such that $\langle X, \succsim, \circ \rangle$ is a PCS with half elements. They also proved that any PCS with half elements yields an intensive structure with a doubling function equal to the inverse of the half element function. The question of uniqueness of the doubling function was left unresolved, and later it was answered by Michael Cohen but not published. Although he dealt with partial operations, we present his result only for the simpler case of an operation.

THEOREM 3.10 (Cohen). *Suppose $\mathscr{X} = \langle X, \succsim, * \rangle$ is a closed, idempotent concatenation structure having a doubling function δ with domain A. If δ' is another doubling function with domain A', then δ' differs from δ at most at one point, which is maximal in the structure. The PCS $\langle X, \succsim, \circ \rangle$, where $\circ$ is defined by $x \circ y = \delta(x * y)$ when $x * y$ is in the domain A, and $\mathscr{X}$ have the same group of automorphisms.*

Proof. By Theorem 3.1 of Narens and Luce (1976), let $\circ$ and $\circ'$ be the PCS operations induced by δ and δ', respectively. For any c in $\delta(A \cap A')$, define h by $h(c) = \delta'\delta^{-1}(c)$. We show that h has three properties:

(i) h is strictly increasing. This follows from the fact that δ and δ' are strictly increasing.

(ii) If a, b are in $A \cap A'$, then $a \circ b$ is in $\delta(A \cap A')$ and $h(a \circ b) \succ a, b$. The former is obvious and the latter follows from

$$h(a \circ b) = \delta'\delta^{-1}\delta(a * b) = \delta'(a * b) = a \circ' b \succ a, b.$$

(iii) If $a \succ b$, then there is a c such that $b \circ c$ is in $\delta(A \cap A')$ and $a \succ h(b \circ c)$.

To show this, observe that by Definition 3.2(iii), there exist d and d' such that $a \succ \delta(b * d)$ and $a \succ \delta'(b * d')$. Choose $c = \min(d, d')$, then by Definition 3.2(ii) c is in $A \cap A'$. Thus

$$h(b \circ c) = \delta'\delta^{-1}\delta(b * c) = \delta'(b * c) \prec a.$$

We now use these properties to show that h is the identity over $\delta(A \cap A')$. Suppose a is in $\delta(A \cap A')$ and $h(a) \succ a$. By property (iii) of h, there exists c such that $h(a) \succ h(a \circ c)$, which by property (i) implies $a \succ a \circ c$, contrary to the positivity of $\circ$. Suppose $a \succ h(a)$. By restricted solvability there is some b such that $a \succ h(a) \circ b$. Since a is in $\delta(A \cap A')$ then so is $h(a) \circ b$ by Definition 3.2(i) and (ii). So by properties (i) and (ii) of h, $h(a) \succ h[h(a) \circ b] \succ h(a)$, which is impossible. Therefore, $h(a) \sim a$.

Now, without loss of generality, suppose a' is in $A' - A$, in which case $A - A' = \varnothing$. If a is in A, we show a is in A'. If $a' \succsim a$, this is trivially so. If $a \succ a'$, then a' is in A, contrary to choice. Now, suppose a' and a'' are both in $A' - A$ and $a' \succ a''$. Let $b' = \delta'(a')$ and $b'' = \delta'(a'')$. By the monotonicity of δ', $b' \succ b'' \succ \delta'(c)$ for all c in A. By Definition 3.2(iii), there is a d in X such that $b'' * d$ is in A and so, by what was shown above in A'. Using the identity of δ and δ' over $A \cap A'$, $b' \succ \delta(b'' * d) = \delta'(b'' * d)$. But by Definition 3.2(iv), $\delta'(b'' * d) \succ b''$, which contradicts the fact that for $b'' * d$ in A, $b'' \succ \delta'(b'' * d)$. So $a' \sim a''$.

Next, we show that $b' = \delta'(a')$ is maximal in X. Suppose not, then there is c in X such that $c \succ b'$. By Definition 3.2(iii) and (iv), there is d in A such that $b' * d$ is in A and $c \succ \delta(b' * d) \succ b'$. But

$$\delta'(b' * d) = \delta(b' * d) \succ b' = \delta'(a'),$$

and so by monotonicity, $b' * d \succ a'$. Thus, by Definition 3.2(ii), a' is in A, contrary to choice.

As Narens and Luce (1976, Theorem 3.3) showed, if α is an automorphism of $\mathscr{X}$, then $\alpha^{-1}\delta\alpha$ is also a doubling function, and so by its uniqueness $\alpha\delta = \delta\alpha$. We use this to show that α is also an automorphism of the induced PCS:

$$\alpha(x \circ y) = \alpha\delta(x * y) = \delta\alpha(x * y) = \delta[\alpha(x) * \alpha(y)] = \alpha(x) \circ \alpha(y).$$

Since $\delta\alpha\,\delta^{-1} = \alpha\delta\,\delta^{-1} = \alpha$, we see that $\alpha\delta^{-1} = \delta^{-1}\alpha$, and so if α is an automorphism of $\langle X, \succsim, \circ \rangle$, then

$$\alpha(x * y) = \alpha\,\delta^{-1}(x \circ y) = \delta^{-1}\alpha(x \circ y) = \delta^{-1}[\alpha(x) \circ \alpha(y)] = \alpha(x) * \alpha(y),$$

proving that α is an automorphism of $\mathscr{X}$. Q.E.D.

Since the automorphism group of a 1-point homogeneous PCS is, by Theorem 3.3, of scale type (1, 1), the same is true of its associated intensive structure. Thus, by Theorem 3.1, they both have real ratio scale representations and by Theorem 3.9 we know that they can be put in the form of real unit structures. Thus, we can ask under what conditions such real structures have doubling functions.

THEOREM 3.11. *Suppose $\mathscr{R} = \langle \mathrm{Re}^+, \geqslant, *, f \rangle$ is a real unit structure that is idempotent and $k = \lim_{x \to \infty} f(x)/x$. The necessary and sufficient conditions for $\mathscr{R}$ to have a doubling function are*

(1) $\lim_{x\to 0} f(x) \geqslant k$, *and*

(2) *defining* $f_n(k)$ *inductively by* $f_1(k)=1$ *and* $f_n(k)=f[f_{n-1}(k)/k]$, *then* $\lim_{n\to\infty} f_n(k)=\infty$.

In this case, the doubling function is $\delta=\iota/k$.

Proof. First, assume that the conditions hold. We show that $\delta=\iota/k$ is a doubling function with domain Re^+.

(i) and (ii) δ is obviously strictly increasing and defined on Re^+.

(iii) Suppose $x>y$ and let z be such that $1<z<x/y$. By the definition of k and the fact (Definition 3.1) f/ι is decreasing, let $v>0$ be such that $kz>f(v)/v$, and let $u=y/v$. Then

$$\delta(y*u)=uf(y/u)/k=yf(v)/vk<yz<x.$$

(iv) Since f/ι is decreasing, we know that $f(x/y)/(x/y)>k$ and so $\delta(x*y)=yf(x/y)/k>x$. By assumption (1) and the fact f is increasing, $f(x)>k$ and so $\delta(x*y)=yf(x/y)/k>y$.

(v) Suppose x_n is a sequence of the type defined in Definition 3.2(v).

Then by induction $x_n=xf_n(k)/k$. By assumption (2), $f_n(k)$ is unbounded, therefore so is x_n and the Archimedean property holds.

Conversely, suppose that a doubling function δ exists. We first establish that it is linear. Since $\mathscr{R}$ is homogeneous, its induced PCS is (1, 1) by Theorems 3.10 and 3.3, and thus by Theorems 3.1 and 3.8 we know that its operation $\circ$ satisfies $x\circ y=yg(x/y)$ for some g meeting conditions (i)–(iii) of Definition 3.1. Thus,

$$yg(x/y)=x\circ y=\delta(x*y)=\delta[yf(x/y)].$$

Setting $y=x$ and noting $f(1)=1$ (Theorem 3.8 (8)), we obtain

$$xg(1)=\delta[xf(1)]=\delta(x).$$

By Theorem 3.8 (3), $g(1)>1$. Let $k=1/g(1)$. We next show that $\lim_{x\to\infty} f(x)/x=k$. By Definition 3.2(iv), $x<\delta(x*z)=zf(x/z)/k$, and so for all $x,z>0$, $k<f(x/z)/(x/z)$, proving that $\lim_{x\to\infty} f(x)/x\geqslant k$. Select any $z,\varepsilon>0$ and let $x=z+\varepsilon>z$. By Definition 3.2(iii), there exists a v such that

$$z+\varepsilon=x>\delta(z*v)=(z*v)/k=vf(y/v)/k,$$

and so dividing by z,

$$1+\varepsilon/z>f(z/v)/(z/v)\,k.$$

Since z and ε are arbitrary and the limit exists, we see $k\geqslant\lim_{x\to\infty} f(x)/x$, thereby proving their equality.

To show property (1), note that by the positivity of the doubling function (Definition 3.2(iv)), for all $x, y > 0$, $y < \delta(x^*y) = yf(x/y)/k$, and so $f(x/y) > k$. Since f is increasing, the limit exists and $\geqslant k$.

To show property (2), we let x_n be as in Definition 3.2(v) and note that, as in the first part of the proof, $x_n = xf_n(k)/k$, and so the Archimedean property of the doubling function implies $f_n(k)$ is unbounded. Q.E.D.

This result definitely does not encompass all real (1, 1) idempotent structures. For example, those of the form

$$f(x) = ax + bx^c + 1 - a - b,\ 0 < c < 1,$$

satisfy the first property of Theorem 3.11 only for $a \leqslant (1-b)/2$. The reason is that $k = a$ and $\lim_{x \to 0} f(x) = 1 - a - b$. So there is much more to be understood about ratio scale idempotent structures.

3.6. *Scale Type* (2, 2): *The Interval Scale Case*

THEOREM 3.12. *Suppose $\mathscr{R} = \langle \mathrm{Re}^+, \geqslant, \circ \rangle$ is a real, closed concatenation structure. Then the following statements are equivalent:*

(1) *$\mathscr{R}$ is of scale type* (2, 2) *with the log affine automorphism group.*

(2) *There exists a real unit structure $\langle \mathrm{Re}^+, \geqslant, \circ', f \rangle$ that is isomorphic to $\langle \mathrm{Re}^+, \geqslant, \circ \rangle$ and f has the property that for all $x, \rho > 0$*

$$f(x^\rho) = f(x)^\rho. \tag{3.3}$$

(3) *There exist real constants c, d, $0 < c, d < 1$, such that*

$$\begin{aligned} x \circ y &= x^c y^{1-c} && \text{if } x > y \\ &= x && \text{if } x = y \\ &= x^d y^{1-d} && \text{if } x < y. \end{aligned} \tag{3.4}$$

Proof. (1) implies (2). Since $\circ$ is invariant under the log affine group, which includes the similarity group, which is (1, 1), it follows from Theorem 3.9 that $\mathscr{R}$ is a real unit structure. So the invariance

$$\alpha(x \circ y) = \alpha(x) \circ \alpha(y)$$

may be reformulated, using Definition 3.1 (iii), as

$$\sigma[yf(x/y)]^\rho = \sigma y^\rho f[(x/y)^\rho],$$

from which Eq. (3.3) follows directly.

The converse, (2) implies (1), is obvious.

(2) implies (3). Observe that Eq. (3.3) is really two independent equations since for $\rho > 0$, $x \geqslant 1$ iff $x^\rho \geqslant 1$. By Theorem 1, p. 66 of Aczél (1966), both solutions are power functions with, let us say, exponents c and d. The fact that f must be monotonic increasing and f/ι monotonic decreasing forces $0 < c, d < 1$. Using Definition 3.1 (iii) yields Eq. (3.4).

The converse, (3) implies (2), is trivial. Q.E.D.

As was noted earlier, economics and psychology follow the convention of writing (2, 2) cases in interval scale fashion rather than as a log interval scale. In the case just discussed, taking logarithms yields the following representation: for all u, v in Re, there exist constants c, d, where $0 < c, d < 1$, such that

$$\begin{aligned} u \circ v &= cu + (1 - c)\, v \qquad && \text{if } u > v, \\ &= u, && \text{if } u = v, \\ &= du + (1 - d)\, v && \text{if } u < v. \end{aligned} \tag{3.5}$$

As this representation seems to be important (see Sect. 7 below), it probably deserves an identifying name. Since it generalizes the familiar linear representation of the idempotent, bisymmetric structure in which $c = d$ (see Krantz *et al.*, 1971, Sect. 6.9), we refer to it as the *dual bilinear representation.*

3.7. *Scale Type* (1, 2)

For continuous real concatenation structures, the only 1-point homogeneous case that can arise with $N > M$ is (1, 2) (Theorem 3.7), and we also know that it is isomorphic to a subgroup of the affine group (Theorem 3.2).

One example of such a (1, 2) group is the set of transformations $x \to \sigma x^\rho$, where $\rho = k^n$ for some constant $k > 0$ and every integer n, and $\sigma > 0$. It is easily verified that this is a group, that it is 1-point but not 2-point homogeneous, and that it is 2-point but not 1-point unique. We refer to it as the *discrete log affine* group.

THEOREM 3.13. *Suppose* $\mathscr{R} = \langle \mathrm{Re}^+, \geqslant, \circ \rangle$ *is a real, closed concatenation structure. Then the following statements are equivalent*:

(1) $\mathscr{R}$ *is of scale type* (1, 2).

(2) *There is a real unit structure* $\langle \mathrm{Re}^+, \geqslant, \circ', f \rangle$ *such that* $\mathscr{R}$ *is isomorphic to* $\langle \mathrm{Re}^+, \geqslant, \circ' \rangle$ *and there exists a constant* $k > 0$ *such that Eq.* (3.3) (*Theorem* 3.12) *holds iff for some integer* n, $\rho = k^n$.

(3) *The automorphism group of* $\mathscr{R}$ *is isomorphic to the discrete log affine group.*

Proof. (1) implies (2). By Theorem 3.2, the automorphism group $\mathscr{G}$ of this representation is a subgroup of the log affine group and the (1, 1) subgroup corresponds to the multiplicative positive reals. So, they are of the form σx^ρ, where $\sigma > 0$ and ρ is in subgroup $\mathscr{H}$ of the positive reals. $\mathscr{H}$ cannot be the identity since, by Theorem 3.9, that would mean $\mathscr{R}$ is (1, 1), not (1, 2). If $\mathscr{H}$ is dense in the reals, then by the continuity of f—it is onto and strictly increasing—it would follow that Eq. (3.3) is true for all $\rho > 0$, in which case by Theorem 3.12, $\mathscr{R}$ would be (2, 2), not (1, 2). So $\mathscr{H}$ must be discrte, in which case it is generated by some $k > 0$ and so the elements of $\mathscr{H}$ are of the form k^n, n an integer.

It is trivial to show that (2) implies (3) and (3) implies (1). Q.E.D.

The next and last result of the section provides a characterization of differentiable, real unit structures that satisfy the properties of Theorem 3.13. It makes it quite clear that such structures exist since the characterization is easily realized.

THEOREM 3.14. *$\mathscr{R} = \langle \mathrm{Re}^+, \geqslant, \circ \rangle$ is a real, closed, idempotent concatenation structure and $\langle \mathrm{Re}^+, \geqslant, \circ, f \rangle$ is a real unit structure with the following two properties:*

(i) *f is differentiable everywhere except at $x = 1$, and*

(ii) *there exists a unique $k > 1$ such that, for all $x > 0$, Eq. (3.3) holds iff $\rho = k^n$, n an integer,*

iff there exist two functions q_1 and q_2 on the closed interval $[0, \log k]$ with the following properties, where x is in $[0, \log k]$ and $i = 1, 2$:

(a) $q_i(0) = q_i(\log k)$,

(b) $0 < q_i(x)$,

(c) *q_i is not constant,*

(d) *q_i is differentiable on $(0, \log k)$,*

(e) $\lim_{x \downarrow 0} q_i'(x) = \lim_{x \uparrow \log k} q_i'(x)$,

(f) $0 < q(x) + q'(x) < 1$,

such that if p_i is the periodic function of period $\log k$ that agrees with $\log q_i$ on $[0, \log k]$ and $h_i = \iota + p_i$, then

$$
\begin{aligned}
f(x) &= \exp \exp h_1(\log \log x) && \text{if } x > 1 \\
&= 1 && \text{if } x = 1 \\
&= 1/\exp \exp h_2(\log \log 1/x) && \text{if } x < 1.
\end{aligned}
\tag{3.6}
$$

Proof. Suppose q_i satisfy properties (a)–(f) and f is defined by Eq. (3.6). We establish that f defines a unit structure (Definition 3.1) that exhibits properties (i) and (ii) of the theorem. By assumptions (d) and (e), the definitions of p_i, h_i, and f, it is clear that f is differentiable except possibly at 1. To show property (ii), let $c = \log k$ and observe that

$$h_i(x + c) = x + c + p_i(x + c) = x + c + p_i(x) = h_i(x) + c.$$

So, for $x > 1$,

$$\begin{aligned} f(x^k) &= \exp\exp h_1(\log\log x^k) \\ &= \exp\exp h_1(\log k \log x) \\ &= \exp\exp h_1(c + \log\log x) \\ &= \exp\exp[c + h_1(\log\log x)] \\ &= \exp k \exp h_1(\log\log x) \\ &= [\exp\exp h_1(\log\log x)]^k \\ &= f(x)^k. \end{aligned}$$

By essentially the same calculation, the result holds for $x < 1$. From this is follows immediately that Eq. (3.3) holds if $\rho = k^n$. Suppose it holds for some ρ not of this form. Then, by a well-known argument, it holds for a set of ρ's that is dense in Re^+ and so by the continuity of f it holds for all $\rho > 0$. This implies that $h_i(x + a) = h_i(x) + a$ for each $a > 0$, in which case p_i is periodic with every period, i.e., a constant, and so q_i is a constant, violating assumption (c).

Next we show that f is increasing. Since $\exp x > 0$, $\exp\exp x > 1$, and so $f(x) > 1$ if $x > 1$ and $f(x) < 1$ if $x < 1$. So it suffices to show that f is increasing on each side of 1. For $x > 1$, f is increasing

iff h_1 is increasing
iff $h_1'(x) > 0$ [by (d) and (e) and definitions of p_1 and h_1]
iff $1 + p_1'(x) > 0$ [definition of h_1]
iff $1 + p_1'(x) > 0$ for x in $[0, c]$ [by the periodicity of p_1]
iff $1 + q_1'(x)/q_1(x) > 0$ [$p_1 = \log q_1$ for x in $[0, c]$]
iff $q_1(x) + q_1'(x) > 0$ [$q_1(x) > 0$],

which condition is assumed in property (f). For $x < 1$ the reasoning is similar, using p_2, since the two inversions $1/x$ and $1/\exp\exp h_2$ maintain the relation that f is increasing iff h_2 is increasing.

Finally, we show that f/ι is decreasing. For $x > 1$, f/ι is decreasing

iff $f(\exp\exp x)/\exp\exp x$ is decreasing
iff $\exp\exp h_1(x)/\exp\exp x$ is decreasing
iff $\exp[\exp h_1(x) - \exp x]$ is decreasing
iff $e^x[\exp p_1(x) - 1]$ is decreasing [because $h_1 = \iota + p_1$]
iff $e^x[p_1'(x)\exp p_1(x) + \exp p_1(x) - 1] < 0$ [because p_1 is differentiable]
iff $[\exp p_1(x)][p_1'(x) + 1] < 1$ [$e^x > 0$]
iff $[\exp p_1(x)][p_1'(x) + 1] < 1$ for x in $[0, c]$ [because p_1 is periodic]
iff $q_1(x)[1 + q_1'(x)/q_1(x)] < 1$ [definition of p_1]
iff $q_1(x) + q_1'(x) < 1$,

which is true by property (f). Again, the argument is similar for $x < 1$.

Conversely, suppose f is a unit representation satisfying properties (i) and (ii). Define

$$h_1(x) = \log\log f(\exp\exp x), \qquad x > 1,$$

$$h_2(x) = \log\log 1/f(1/\exp\exp x), \qquad x < 1.$$

Using property (ii) it is easy to verify that for $c = \log k$,

$$h_i(x+c) = h_i(x) + c.$$

This is Abel's equation with the particular solution $h_i = \iota$ and so, as is well known, the general solution is $h_i = \iota + p_i$, where p_i is periodic with period c. By property (i) p_i is everywhere differentiable. Define q_i on $[0, c]$ by $q_i = \exp p_i$. Properties (a), (b), (d), and (e) are immediate. Property (c) follows from the fact that k is unique. And property (f) was shown above to follow from the facts that f is increasing and f/ι is decreasing. Q.E.D.

4. Relations between Concatenation and Conjoint Structures

4.1. *Definitions*

Conjoint structures of the form $\langle A \times P, \succsim \rangle$, where A and P are sets and $\succsim$ is a weak ordering that is independent (monotonic) in a sense defined below, have at least three important ties to concatenation structures. For one, when reasonable solvability conditions are satisfied, the conjoint structure induces on each component a concatenation structure which is closely related to a PCS. For a second, many measurement contexts are modeled as conjoint structures together with one or more operations defined either on the conjoint structure itself or on one of its components. Moreover, in many such situations the operations combine either with the conjoint ordering itself or with the orderings induced by it on A and/or P to form concatenation structures that nicely interlock with the conjoint structure in such a way that their automorphism groups are closely related. An example is the conjoint structure of pairs consisting of substances and volumes, for which there is an ordering and concatenation of masses on the whole structure and a concatenation of volumes on the second component. The third tie comes from the fact that it is always possible to construct from a concatenation structure a conjoint one that encodes the same information. This recoding sometimes suggests alternative ways to study the concatenation structure. The first two possibilities are explored in this section, and the third is considered in the next two sections. Some of the results are already known in part, but as they have been cast in a somewhat different and less informative guises in earlier publications, we believe that some partial repetition is not amiss. In each case, we make clear what is new and what is old.

The following is a precise statement of what a binary conjoint structure is and of some of the solvability conditions that will be used.

DEFINITION 4.1. Suppose A and P are nonempty sets and $\succsim$ is a binary relation on $A \times P$. Then, $\mathscr{C} = \langle A \times P, \succsim \rangle$ is a *conjoint structure* iff for each a, b in A and p, q in P, the following three conditions are satisfied:

(1) $\succsim$ is a nontrivial weak ordering;

(2) *independence* holds, i.e.,

(i) $ap \succsim bp$ iff $aq \succsim bq$,

(ii) $ap \succsim aq$ iff $bp \succsim bq$.

Observe that independence permits one to define *induced weak orders* on A and P, which are denoted $\succsim_A$ and $\succsim_P$, respectively.

(3) $\succsim_A$ and $\succsim_P$ are total orderings.

In addition,

(4) $\mathscr{C}$ is said to satisfy the *Thomsen condition* iff for all a, b, e in A and p, q, x in P,

$$ax \sim eq \quad \text{and} \quad ep \sim bx \quad \text{imply} \quad ap \sim bq.$$

(5) For a_0 in A and p_0 in P, $\mathscr{C}$ is said to be A-*solvable relative to* $a_0 p_0$ iff

(i) for each a in A, there exists $\pi(a)$ in P such that

$$ap_0 \sim a_0 \pi(a),$$

(ii) for each ap in $A \times P$, there exists $\xi(a, p)$ in A such that

$$\xi(a, p)\, p_0 \sim ap.$$

(6) $\mathscr{C}$ is said to be *unrestrictedly A-solvable* iff for each a in A and p, q in P there exists b in A such that $ap \sim bq$. The definition of *unrestrictedly P-solvable* is similar.

(7) $\mathscr{C}$ is said to be *dense* iff whenever $ap \succ bp$, there exists q in P such that $ap \succ bq \succ bp$.

(8) Let J be an (infinite or finite) interval of integers. Then a sequence $\{a_j\}_{j \in J}$, a_j in A, is said to be a *standard sequence on A* iff there exist p, q in P such that (i) it is not the case that $p \sim_P q$ and (ii) for all $j, j+1$ in J, $a_{j+1}\, p \sim a_j q$. The sequence $\{a_j\}_{j \in J}$ is said to be *bounded* iff for some c, d in A, $c \succsim a_j \succsim d$ for all j in J. $\mathscr{C}$ is said to be *Archimedean* iff every bounded standard sequence on A is finite.

Note that assumption (3) is inessential and it is made only as a matter of convenience.

4.2. *Induced Concatenation Structures*

DEFINITION 4.2. Suppose that $\mathscr{C} = \langle A \times P, \succsim \rangle$ is a conjoint structure that is A-solvable relative to $a_0 p_0$ in $A \times P$. The *induced operation* on A, $*$, is defined as follows, for each a, b in A,

$$a * b = \xi[a, \pi(b)],$$

where π and ξ are defined in Definition 4.1 (5).

DEFINITION 4.3. Let X be a nonempty set, $\succsim$ a binary relation on X, $\circ$ a binary operation on X, and x_0 an element of X. Then $\mathscr{X} = \langle X, \succsim, \circ, x_0 \rangle$ is said to be a *total concatenation structure* iff the following five conditions hold:

(1) $\langle X, \succsim, \circ \rangle$ is a concatenation structure.

(2) The restriction of $\mathscr{X}$ to $X^+ = \{x \mid x \text{ in } X \text{ and } x \succ x_0\}$ is a PCS.

(3) The restriction of $\mathscr{X}$ to $X^- = \{x \mid x \text{ in } X \text{ and } x \prec x_0\}$ but with the converse order $\precsim$ is a PCS, i.e., $\langle X^-, \precsim, \circ \rangle$ is a PCS.

(4) For all x in X, $x \circ x_0 \sim x_0 \circ x \sim x$.

(5) For x in X^+ and y in X^-, there exist u, v in X such that $u \circ y$ and $v \circ x$ exist and $u \circ y \succ x$ and $y \succ v \circ x$.

THEOREM 4.1. *Suppose $\langle A \times P, \succsim \rangle$ is an Archimedean conjoint structure that is A-solvable relative to $a_0 p_0$ in $A \times P$. Let $*$ be the operation induced on A relative to $a_0 p_0$. Then $\langle A, \succsim_A, *, a_0 \rangle$ is a total concatenation structure. Moreover, any closed total concatenation structure is isomorphic to one induces by some A-solvable conjoint structure.*

Proof. Luce and Cohen (1983, Theorems 2 and 3). Q.E.D.

The main significance of this result is that we know just what properties of $*$ follow from the very weak assumptions of a conjoint structure and A-solvability. The question is how best to use $*$ to study conjoint structures. One useful device is a certain class of transformations defined in terms of $*$.

DEFINITION 4.4. Suppose $\mathscr{X} = \langle X, \succsim, \circ \rangle$ is a concatenation structure. For x in X, the transformation $\tau_x = \iota \circ x$, where ι is the identity transformation on X, is said to be a *right translation* of $\mathscr{X}$. (It is the mapping of X into X such that $\tau_x(y) = y \circ x$. A similar concept of left translation can be defined, but it will not be used.) The class of all right translations is denoted $\mathscr{T}$ or, if two or more operations are involved, by $\mathscr{T}(\circ)$.

THEOREM 4.2. *Suppose $\mathscr{C} = \langle A \times P, \succsim \rangle$ is a conjoint structure that is A-solvable relative to $a_0 p_0$ in $A \times P$, $*$ is the corresponding induced operation, $\mathscr{I}_A = \langle A, \succsim_A, * \rangle$, and $\mathscr{T}$ is the class of right translations of $*$. Then the following statements are true:*

(1) *$\mathcal{T}$ satisfies 1-point uniqueness in $\mathcal{I}_A$.*

(2) *$\mathcal{T}$ satisfies 1-point homogeneity in $\mathcal{I}_A$ iff $\mathcal{C}$ is unrestrictedly P-solvable.*

(3) *$\mathcal{T}$ is closed under function composition iff $*$ is a closed associative operation.*

(4) *$\mathcal{T}$ is commutative iff $\mathcal{C}$ satisfies the Thomsen condition, in which case $*$ is both associative and commutative.*

(5) *$\mathcal{T}$ is a group iff $\mathcal{C}$ is unrestrictedly P-solvable and $*$ is a closed associative operation.*

(6) *If $\mathcal{T}$ is a group and $\mathcal{C}$ is Archimedean, then $\mathcal{C}$ satisfies the Thomsen condition and $\mathcal{T}$ is an Archimedean ordered group.*

Proof. (1) Using the monotonicity of $*$ (Theorem 4.1),

$$\tau_a(c) = \tau_b(c) \text{ iff } c * a \sim_A c * b \text{ iff } a \sim_A b \text{ iff } a = b,$$

which proves 1-point uniqueness.

(2) Suppose $\mathcal{T}$ is 1-point homogeneous, and let a, b in A and q in P be given. So there exists c in A such that

$$a * c = \tau_c(a) = b * \pi^{-1}(q),$$

which is equivalent to $a\pi(c) \sim bq$. Thus, unrestricted P-solvability holds. Conversely, suppose a, b are in A and let p solve $bp_0 \sim ap \sim a * \pi^{-1}(p)\, p_0$. By independence (Definition 4.1 (2)), $\tau_{\pi^{-1}(p)}(a) = b$, which establishes 1-point homogeneity.

(3) Assume $\mathcal{T}$ is closed under function composition, and let a, b, c be in A. By the closure of $\mathcal{T}$, there exists d in A such that $\tau_c \tau_b = \tau_d$, and in fact

$$d = a_0 * d = \tau_d(a_0) = \tau_c \tau_b(a_0) = (a_0 * b) * c = b * c.$$

And so,

$$a * (b * c) = a * d = \tau_d(a) = \tau_c \tau_b(a) = (a * b) * c,$$

proving that $*$ is a closed associative operation. Conversely, suppose $*$ is associative, then

$$\tau_a \tau_b(c) = (c * b) * a = c * (b * a) = \tau_{a * b}(c),$$

which establishes the closure of $\mathcal{T}$.

(4) Suppose $\mathcal{T}$ is commutative and $ax \sim eq$ and $ep \sim bx$. These are equivalent to

$$a * \pi^{-1}(x) = e * \pi^{-1}(q) \qquad \text{and} \qquad e * \pi^{-1}(p) = b * \pi^{-1}(x).$$

Freely using the commutativity of $\mathscr{T}$, i.e., $(a * b) * c = \tau_c \tau_b(a) = \tau_b \tau_c(a) = (a * c) * b$, and the monotonicity of $*$.

$$\begin{aligned}[a * \pi^{-1}(p)] * \pi^{-1}(x) &= [a * \pi^{-1}(x)] * \pi^{-1}(p) \\ &= [e * \pi^{-1}(q)] * \pi^{-1}(p) \\ &= [e * \pi^{-1}(p)] * \pi^{-1}(q) \\ &= [b * \pi^{-1}(x)] * \pi^{-1}(q) \\ &= [b * \pi^{-1}(q)] * \pi^{-1}(x),\end{aligned}$$

and so by the monotonicity of $*$, $ap \sim bq$.

Conversely, suppose the Thomsen condition holds. Apply it to the definitions:

$$a * b,\ p_0 \sim a,\ \pi(b) \qquad \text{and} \qquad a,\ \pi(c) \sim a * c,\ p_0,$$

to get $a * b, \pi(c) \sim a * c, \pi(b)$, which is equivalent to $(a * b) * c = (a * c) *$. This establishes the commutativity of $\mathscr{T}$. Observe that from this property,

$$a * b = (a_0 * a) * b = (a_0 * b) * a = b * a,$$

which establishes the commutativity of $*$. And using both properties.

$$a * (b * c) = (b * c) * a = (b * a) * c = (a * b) * c,$$

which is the associativity of $*$.

(5) Suppose $\mathscr{T}$ is a group. Since it is closed under function composition, part (3) implies $*$ is associative. We next establish that $\mathscr{T}$ is 1-point homogeneous and so, by part (2), $\mathscr{C}$ is unrestrictedly P-solvable. Select a, b in A. Note that $\tau_a(a_0) = a$ and $\tau_b(a_0) = b$, and so $\tau_b \tau_a^{-1}(a) = b$. Since $\mathscr{T}$ is a group, $\tau_b \tau_a^{-1}$ is in $\mathscr{T}$.

Conversely, the closure of $*$ insures that $\mathscr{T}$ is closed. We show that if τ_a is in $\mathscr{T}$, then τ_a^{-1} is also in $\mathscr{T}$. Suppose $\tau_a(b) = c$, then by 1-point homogeneity of $\mathscr{T}$ (part (2)), there exists τ_d in $\mathscr{T}$ such that $\tau_d(c) = b$ and so $\tau_d \tau_a(b) = b = \iota(b)$. By the closure of $\mathscr{T}$, $\tau_d \tau_a$ is in $\mathscr{T}$, and so by 1-point uniqueness (part (1)), $\tau_d \tau_a = \iota$, whence $\tau_d = \tau_a^{-1}$.

(6) Suppose $\mathscr{T}$ is a group and $\mathscr{C}$ is Archimedean. Then by part (5) and Theorem 5 of Luce and Cohen (1983), $\mathscr{C}$ satisfies the Thomsen condition. $\mathscr{T}$ can be ordered as follows: for each a, b in A,

$$\begin{aligned}\tau_a \succsim' \tau_b \text{ iff for all } c \text{ in } A, &\quad \tau_a(c) \succsim_A \tau_b(c), \\ \text{iff for all } c \text{ in } A, &\quad c * a \succsim_A c * b, \\ \text{iff } a \succsim_A b. &\end{aligned}$$

From the associativity of $*$, we note that $\tau_a^n = \iota * na$. Since $\langle A, \succsim_A, * \rangle$ is a total concatenation structure (Theorem 4.1), it follows that for each $a, b \succ_A a_0$, there exists an n such that $na \succsim_A b$, and thus, $\tau_a^n \succsim' \tau_b$, proving that $\mathscr{T}$ is Archimedean. Q.E.D.

Part (4) of the above theorem is essentially the same statement as Theorem 5 of Luce and Cohen (1983), except that they assumed $\mathscr{C}$ to be Archimedean.

4.3. *The Distributive Interlock*

Narens and Luce (1976) extended and developed an idea in Narens (1976) concerning the interlock between a concatenation operation on one component of a conjoint structure and the conjoint structure itself. This interlock appears to be what holds together the various scales of classical physics and leads to the mathematical structure that underlies the important structure of physical units. It, therefore, appears to be of great importance in measurement theory. The purpose of this subsection is to develop more fully its algebraic analysis which was initiated in Narens (1981a) and was carried out more fully in Luce and Cohen (1983).

DEFINITION 4.5. Suppose $\mathscr{C} = \langle A \times P, \succsim \rangle$ is a conjoint structure and $\circ$ is a partial operation on A. Then $\circ$ is said to be *distributive* in $\mathscr{C}$ iff for all a, b, c, d in A for which $a \circ b$ and $c \circ d$ are defined and all p, q in P,

$$ap \sim cq \quad \text{and} \quad bp \sim dq \quad \text{imply} \quad a \circ b, p \sim c \circ d, q.$$

DEFINITION 4.6. Suppose $\mathscr{C} = \langle A \times P, \succsim \rangle$ is a conjoint structure and α is an automorphism of $\mathscr{C}$. Then α is said to be *factorizable* iff there are maps β from A onto A and γ from P onto P such that for al a, b in A and p, q in P,

$$\alpha(a, p) = (\beta(a), \gamma(p)).$$

THEOREM 4.3. *Suppose $\mathscr{C} = \langle A \times P, \succsim \rangle$ is a conjoint structure that is A-solvable relative to $a_0 p_0$ in $A \times P$, and let * be the induced operation and $\mathscr{T}$ the corresponding set of right translations of *. Suppose that $\mathfrak{A} = \langle A, \succsim_A, \circ \rangle$ is a closed concatenation structure, $\mathscr{G}$ is its group of automorphisms, and $\mathscr{E}$ its set of endomorphisms. Then the following four statements are true:*

(1) *If $\circ$ is distributive in $\mathscr{C}$, then $\mathscr{T} \subseteq \mathscr{E}$.*

(2) *$\mathscr{T} \subseteq \mathscr{G}$ iff $\circ$ is distributive in $\mathscr{C}$ and $\mathscr{C}$ is unrestrictedly A-solvable.*

(3) *$\mathscr{T} = \mathscr{G}$ iff $\mathscr{T} \subseteq \mathscr{G}$ and $\mathscr{G}$ satisfies 1-point uniqueness.*

(4) *Suppose $\mathscr{C}$ is Archimedean. Then, $\mathscr{T} = \mathscr{G}$ iff $\mathscr{T} \subseteq \mathscr{G}$, $\mathscr{T}$ satisfies 1-point homogeneity, $*$ is associative (and commutative), and for α in $\mathscr{G}$, both (α, ι_P) and $(\iota_A, \pi\alpha\pi^{-1})$ are factorizable automorphisms of $\mathscr{C}$.*

Proof. (1) $\tau_a = \iota * a$ is order preserving because $*$ is monotonic. Applying distributivity to the definitions

$$x * a, p_0 \sim x, \pi(a) \qquad \text{and} \qquad y * a, p_0 \sim y, \pi(a)$$

yields

$$(x \circ y) * a, p_0 \sim x \circ y, \pi(a) \sim (x * a) \circ (y * a), p_0,$$

whence

$$\tau_a(x \circ y) = \tau_a(x) \circ \tau_a(y),$$

proving τ_a is an endomorphism.

(2) Suppose $\circ$ is distributive in $\mathscr{C}$ and $\mathscr{C}$ is unrestrictedly A-solvable. If τ is in $\mathscr{T}$, then by unrestricted A-solvability it is onto; since it is order preserving, it is one-to-one; and by part (1), it is an endomorphism. So $\mathscr{T} \subseteq \mathscr{G}$.

Conversely, suppose $\mathscr{T} \subseteq \mathscr{G}$. Suppose $ap \sim cq$ and $bp \sim dq$. Observe,

$$\begin{aligned} ap \sim cq \text{ iff } \tau_{\pi^{-1}(p)}(a) &= a * \pi^{-1}(p) \\ &= c * \pi^{-1}(q) \\ &= \tau_{\pi^{-1}(q)}(c). \end{aligned}$$

Similarly,

$$bp \sim dq \text{ iff } \tau_{\pi^{-1}(p)}(b) = \tau_{\pi^{-1}(q)}(d).$$

Since $\tau_{\pi^{-1}(p)}$ and $\tau_{\pi^{-1}(q)}$ are automorphisms and $\circ$ is monotonic,

$$\begin{aligned} \tau_{\pi^{-1}(p)}(a \circ b) &= \tau_{\pi^{-1}(p)}(a) \circ \tau_{\pi^{-1}(p)}(b) \\ &= \tau_{\pi^{-1}(q)}(c) \circ \tau_{\pi^{-1}(q)}(d) \\ &= \tau_{\pi^{-1}(q)}(c \circ d), \end{aligned}$$

which is equivalent to $a \circ b, p \sim c \circ d, q$, proving that $\circ$ is distributive. To show unrestricted A-solvability, suppose a in A and p, q in P. Since $\tau_{\pi^{-1}(p)}$ is in $\mathscr{T}$, its inverse exists. Let

$$a = \tau^{-1}_{\pi^{-1}(p)} \tau_{\pi^{-1}(q)}(b).$$

Then,

$$a * \pi^{-1}(p) = \tau_{\pi^{-1}(p)}(a) = \tau_{\pi^{-1}(q)}(b) = b * \pi^{-1}(q),$$

whence $ap \sim bq$, proving solvability.

(3) Suppose $\mathscr{T} = \mathscr{G}$, then by Theorem 4.2 (1), $\mathscr{G}$ is 1-point unique. Conversely, suppose $\mathscr{T} \subseteq \mathscr{G}$ and let τ be in $\mathscr{G}$. Denoting $a = \tau(a_0)$, then

$$\tau_a(a_0) = a_0 * a = a = \tau(a_0),$$

but since $\mathscr{T} \subseteq \mathscr{G}$ and $\mathscr{G}$ is 1-point unique, $\tau = \tau_a$, and so $\mathscr{T} = \mathscr{G}$.

(4) Suppose $\mathscr{T} = \mathscr{G}$ and $\mathscr{C}$ is Archimedean. Let a, b be in A. From the facts $\tau_a(a_0) = a$, $\tau_b(a_0) = b$, and τ_a^{-1} is in $\mathscr{T} = \mathscr{G}$, we see $\tau_b \tau_a^{-1}(a) = b$, which establishes 1-point homogeneity. By parts (4) and (6) of Theorem 4.2 and the fact that $\mathscr{T} = \mathscr{G}$ is Archimedean, it follows that $\mathscr{C}$ satisfies the Thomsen condition and $*$ is

associative and commutative. If τ is in $\mathscr{T}=\mathscr{G}$, then τ is necessarily of the form $\iota_A * a = a * \iota_A$, and so by Theorem 9.1 of Luce and Cohen (1983),

$$(a * \iota_A, \iota_P *_P p_0) = (\iota_A * a, \iota_P) = (\tau, \iota_P)$$

is a factorizable automorphism of $\mathscr{C}$. Since

$$\begin{aligned}\pi\tau\pi^{-1}(p) &= \pi\{\iota_A[\pi^{-1}(p)] * a\}\\ &= \pi[\pi^{-1}(p) * a]\\ &= \pi\xi[\pi^{-1}(p), \pi(a)]\\ &= p *_P \pi(a) \qquad \{\text{since } p *_P q = \pi\xi[\pi^{-1}(p), \pi^{-1}(\mathrm{q})]\\ &= [\iota_P *_P \tau(a)](p),\end{aligned}$$

and so by Theorem 9.1 of Luce and Cohen (1983),

$$(\iota_A, \pi\tau\pi^{-1}) = (a_0 * \iota_A, \iota_P *_P \pi(a))$$

is a factorizable automorphism of $\mathscr{C}$.

We establish the converse. From the 1-point homogeneity of $\mathscr{T}$ and the associativity of $*$, Theorem 4.2 (2) and (4) imply $\mathscr{T}$ is a group. That together with the fact that $\mathscr{C}$ is Archimedean imply, by Theorem 4.2 (6), that $\mathscr{C}$ satisfies the Thomsen condition. Let τ be in $\mathscr{G}$. Since (τ, ι_P) is a factorizable automorphism and the Thomsen condition holds, Theorem 9.2 of Luce and Cohen (1983) yields $\tau = \tau(a_0) * \tau^*$, where τ^* is an automorphism of the structure $\langle A, \succsim_A, * \rangle$. By Theorem 8 of Luce and Cohen (1983),

$$\iota_P = (\pi\tau^*\pi^{-1}) *_P \iota_P(p_0) = \pi\tau^*\pi^{-1},$$

and so $\tau^* = \iota_A$. Since $*$ is commutative,

$$\tau = \tau(a_0) * \iota_A = \iota_A * \tau(a_0),$$

proving that τ is in $\mathscr{T}$. Q.E.D.

COROLLARY 1. *If $\mathscr{T} \subseteq \mathscr{G}$, $\mathscr{C}$ is unrestrictedly P-solvable, $\mathscr{G}$ is N-point unique, $N \geqslant 1$, and $\circ$ is not idempotent, then $\mathscr{T} = \mathscr{G}$.*

Proof. By Theorem 4.2 (2), $\mathscr{T}$ satisfies 1-point homogeneity and since $\mathscr{T} \subseteq \mathscr{G}$, so does $\mathscr{G}$. By Theorem 2.1 (3), $\mathscr{G}$ must be 1-point unique since, otherwise, $\circ$ is idempotent, contrary to assumption. By part (3) of the present theorem, $\mathscr{T} = \mathscr{G}$.

COROLLARY 2. *If $\mathscr{I}_A$ is a PCS and $\mathscr{T} \subseteq \mathscr{G}$, then $\mathscr{T} = \mathscr{G}$.*

Proof. By Theorem 2.1 of Cohen and Narens (1979), the automorphism group of a PCS is 1-point unique and so, by part (3) of this theorem, $\mathscr{T} = \mathscr{G}$. Q.E.D.

Assuming $\mathscr{I}_A$ to be a PCS, Luce and Cohen (1983) proved in their Theorem 10 that part (2) of the present Theorem 4.3 is equivalent to $\mathscr{T} = \mathscr{G}$, and they proved substantially all of part (4) of the present Theorem 4.3.

Some obvious questions, to which we do not know the answers, are these: Suppose $\mathscr{T} \subseteq \mathscr{G}$. Under what conditions is $\mathscr{G}$ 2-point unique? Given that it is 2-point unique, then when is it 1-point homogeneous and when is it 2-point homogeneous? And what conditions on $\mathscr{G}$ force $\mathscr{T}$ to be a group? In particular, is $\mathscr{G}$ being of type (2, 2) sufficient? Special cases of these questions are considered in the corollary to Theorem 5.4.

The next result is of interest in light of Theorem 5.4 below and is the key to proving a later important result (Theorem 6.4).

THEOREM 4.4. *Suppose X is a nonempty set, $\succsim$ is a binary relation on X, $\circ$ is a binary operation on X, and $\succsim'$ and $\succsim''$ are binary relations on $X \times X$ that satisfy the following conditions:*

(1) *$\mathscr{X} = \langle X, \succsim, \circ \rangle$ is a closed, totally ordered concatenation structure.*

(2) *$\mathscr{C}' = \langle X \times X, \succsim' \rangle$ and $\mathscr{C}'' = \langle X \times X, \succsim'' \rangle$ are both unrestrictedly solvable conjoint structures.*

(3) *$\mathscr{C}'$ is Archimedean and satisfies the Thomsen condition.*

(4) *The induced orderings of $\succsim'$ and $\succsim''$ are both equal to $\succsim$.*

(5) *The operation $\circ$ is distributive in both $\succsim'$ and $\succsim''$.*

Let $*'$ and $*''$ denote the operations induced by $\succsim'$ and $\succsim''$, respectively, relative to x_0 in X. Then, $*' = *''$, and so $\mathscr{C}''$ also satisfies the Thomsen condition and is Archimedean.

Proof. Let $\mathscr{G}$ be the group of automorphisms of $\mathscr{X}$ and $\mathscr{T}'$ and $\mathscr{T}''$ the set of right translations of $*'$ and $*''$. By assumption (3) and Theorem 5 of Luce and Cohen (1983), $*'$ is associative. This together with assumptions (2) and (3) and Theorem 4.2 (5) and (6) imply $\mathscr{T}'$ is an Archimedean ordered group. By Theorem 4.2 (1) and (2), both $\mathscr{T}'$ and $\mathscr{T}''$ are of type (1, 1). By assumptions (1), (2), (4), and (5), Theorem 4.3 (2) implies that both $\mathscr{T}'$ and $\mathscr{T}'' \subseteq \mathscr{G}$. By Theorem 3.2, $\mathscr{G}$ is isomorphic to a subgroup of the affine group and $\mathscr{T}'$ is isomorphic to the translations of the affine group. Since $\mathscr{T}'' \subseteq \mathscr{G}$ and the elements of $\mathscr{T}''$ are not dilations, necessarily $\mathscr{T}'' \subseteq \mathscr{T}'$. Thus, for each y in X and τ''_y in $\mathscr{T}''$, there exists a z in X such that τ'_z is in $\mathscr{T}'$ and $\tau'_z = \tau''_y$. Thus for these choices of y and z

$$z = x_0 *' z = \tau'_z(x_0) = \tau''_y(x_0) = x_0 *'' y = y,$$

and so, for all x_0, y in X, $x_0 *' y = x_0 *'' y$. Since $\mathscr{C}'$ is Archimedean, so is $*' = *''$, whence $\mathscr{C}''$ is Archimedean. By Theorem 5 of Luce and Cohen (1983), $\mathscr{C}''$ satisfies the Thomsen condition iff $*''$ is associative, which it is because it is identical to $*'$ and that is associative because, by assumption (3), $\mathscr{C}'$ satisfies the Thomsen condition. Q.E.D.

One might conjecture that the conclusion to Theorem 4.4 could be strengthened to $\succsim' = \succsim''$. This is false, as will be shown in Theorem 6.4.

4.4. *Some Examples of Distributive Structures*

EXAMPLE 4.1. Any real unit structure $\langle \mathrm{Re}^+, \geqslant, \circ \rangle$, where for x, y in Re^+, $x \circ y = yf(x/y)$, is distributive in the conjoint structure $\langle \mathrm{Re}^+ \times \mathrm{Re}^+, \succsim \rangle$, where $(x, y) \succsim (u, v)$ iff $xy \geqslant uv$. For suppose $xu = x'v$ and $yu = y'v$, then $x/y = x'/y'$ and so

$$(x \circ y)\, u = yuf(x/y) = y'vf(x'/y') = (x' \circ y')\, v.$$

EXAMPLE 4.2. The ratio scale concatenation structure $\langle \mathrm{Re}^+ \cup \{0\}, \geqslant, \circ \rangle$, where $\circ$ is defined for $x, y \geqslant 0$ by $x \circ y = (x + y)/2$, is distributive in the conjoint structure $\langle (\mathrm{Re}^+ \cup \{0\}) \times \mathrm{Re}^+, \succsim \rangle$, where $\succsim$ is defined as follows: for $x, y \geqslant 0$ and $u, v > 0$,

$$(x, u) \succsim (y, v) \qquad \text{iff} \quad xu + u^2 \geqslant yv + v^2.$$

We verify that $\circ$ is distributive. Suppose

$$xu + u^2 = x'v + v^2 \qquad \text{and} \qquad yu + u^2 = y'v + v^2,$$

then adding and dividing by 2 yields

$$[(x + y)\, u/2] + u^2 = [(x' + y')\, v/2] + v^2.$$

It is easy to verify that the conjoint structure satisfies the independence axioms.

Example 4.2 is interesting because the conjoint structure is not additive nor can it be transformed into an additive representation. This can be seen by directly verifying that the Thomsen condition fails, e.g., $(5, 1) \sim (1, 2)$ and $(1, 3) \sim (11, 1)$, but not $(5, 3) \sim (11, 2)$.

Several theorems show that ratio-scale concatenation structures that are distributive in an unrestrictedly solvable conjoint structure force the latter to be additive. The above example shows that unrestricted solvability is not a minor, technical restriction since we have a ratio-scale concatenation structure that is distributive in a nonadditive conjoint structure which is not unrestrictedly solvable. The existence of distributive structures that violate unrestricted solvability opens up new possibilities for conjoint measurement: they are algebraically rich, have many properties in common with classical physical measurement, but are not additive. At this time, not much is known about this interesting class of structures.

5. AXIOMATIZATIONS OF IDEMPOTENT CONCATENATION STRUCTURES

5.1. *What Has Been Done*

The earliest result in measurement theory was the axiomatization of extensive structures (Definition 2.1) in the late 19th century. Improved axiomatizations appeared sporadically to the present. For a summary of such results until 1971 see Krantz *et al.* (1971, Chap. 3). For the extensive case, the representation is usually stated in terms of addition, and it is usually constructed by taking limits of approximations by standard sequences to the object being measured. Associativity of the partial operation and the Archimedean property are essential to this process. The method is constructive in the sense that approximate numerical measures can, in fact, be constructed by following the techniques of the proof. Indeed, meter sticks marked into millimeters and the series of weights for a pan balance are both finite portions of standard sequences which can be used to measure lengths and weights to a certain accuracy.

Narens and Luce (1976) generalized these results appreciably by showing the existence of numerical representations for structures having nonassociative operations. Their method of "construction" was not direct and does not lead to practical methods for computing approximate measures. The proof entails showing that the axioms of a PCS (Definition 2.1) are adequate to establish the existence of a countable, order dense subset. By the Cantor–Birkhoff theorem, which states that each ordered set with a countable dense subset is homomorphic to $\langle \text{Re}, \geqslant \rangle$, this was sufficient to find a scale, and then in terms of it a numerical operation can be defined so as to represent the qualitative one. On the assumption that half elements exist, they showed that the scale is 1-point unique, but the entire uniqueness question was not completely clarified until Cohen and Narens (1979). They showed that the group of automorphisms of a PCS is Archimedean ordered, and so by Hölder's theorem it is isomorphic to a subgroup of the additive reals. Among these groups, only the dense ones have a unit representation; and only some of these are 1-point homogeneous. Interestingly, they were able to establish a structural condition equivalent to 1-point homogeneity, namely, that the n-copy operators are automorhisms, i.e., for each positive integer n and every x, y in X, $n(x \circ y) = nx \circ ny$. This is useful since it offers an empirical way to check if the structure is homogeneous.

When we turn to idempotent structures, the results are considerably less complete. The major result is the axiomatization of bisymmetric structures (Pfanzagl, 1959a, b), leading to the (2, 2) unit representation with $c = d$ (Theorem 3.12). One method of proof involves transforming the bisymmetric structure into an equivalent conjoint one, showing that it satisfies the Thomsen condition, and so in turn, reducing it to an extensive structure. We generalize this strategy below. The only other result in this area of which we are aware is in Narens and Luce (1976), where it was pointed out that when an idempotent structure has a doubling function (Definition 3.2), then that structure is very similar to a PCS and its representation leads to one for the idempotent structure. As we have seen, for 1-point homogeneous structures, doubling functions exist for only some (1, 1) cases, and when it exists it is unique and is an automorphism of the structure (Sect. 3.5). No axiomatic treatment has been provided for the remaining (1, 1) cases, the (1, 2) ones, or the (2, 2) ones with $c \neq d$. As we shall see in Section 7, good reasons exist in utility theory for understanding the latter case more fully.

This section extends the axiomatic results about intensive structures. In particular, two things are accomplished. First, Theorem 5.1 shows that every solvable, Archimedean, idempotent concatenation structure has a numerical representation. This result is on a par with the representation for PCSs found in Luce and Narens (1976), and together they pretty much close the issue of the existence of representations for concatenation structures that are solvable. The major remaining open question is the degree to which the solvability conditions can be weakened. The current method of proof follows the scheme used previously in the bisymmetric case, reducing the problem to the representation of general conjoint structures.

Having established existence, the next question to be considered is the scale type. It is easy to show that the structures in question are at most 2-point unique. However, establishing general conditions for at least 1-point homogeneity is another matter. Recall that Cohen and Narens did this for PCSs by showing 1-point homogeneity to be equivalent to the condition that the n-copy operators are automorphisms. Since our strategy is a chain of reductions—idempotent to conjoint to total concatenation to PCS—, one might conjecture that we might also trace through conditions for 1-point homogeneity. Theorem 5.2 establishes exactly why this will not work. Nonetheless, one can still ask whether a concept corresponding to n-copy operators exists for idempotent structures and whether it can be used to formulate a condition equivalent to homogeneity. As we shall see in the next section (Theorems 6.3 and 6.4), this is possible only in the case of solvable bisymmetric structures.

Our second accomplishment in this section, which is embodied in Theorem 5.4, is the formulation of a condition for the existence of unit structures. It is not, however, a condition stated explicitly in terms of the primitives of the structure, but rather a scheme for developing an axiomatization. Although it appears interesting and potentially useful, we have not successfully applied it to any important problem.

5.2. *Representations for Solvable, Dense, Archimedean Structures*

THEOREM 5.1. *Suppose $\mathscr{X} = \langle X, \succsim, \circ \rangle$ is a concatenation structure that is totally ordered, closed, solvable, dense, and Archimedean* (*Definition* 2.1). *Then $\mathscr{X}$ is either* 1- *or* 2-*point unique and it is isomorphic to a real closed structure $\langle R, \geqslant, \otimes \rangle$ where $R \subseteq \mathrm{Re}^+$.*

Proof. Let $\mathscr{C} = \langle X \times X, \succsim' \rangle$ be defined as follows: for each u, v, x, y in X,

$$uv \succsim' xy \qquad \text{iff} \quad u \circ v \succsim x \circ y. \tag{5.1}$$

Note that the orderings induced on the components, X, of $\mathscr{C}$ by $\succsim'$ are both equal to $\succsim$. Following the proof of Krantz *et al.* (1971, p. 298), but omitting the proof of double cancellation which is the only part invoking bisymmetry, and noting that density of $\mathscr{X}$ implies density of $\mathscr{C}$, it follows that $\mathscr{C}$ is a conjoint structure that is dense, unrestrictedly solvable in each component, and Archimedean. In the corollary to Theorem 2 of Luce and Cohen (1983), it is shown that there are real mappings φ and $\psi = \varphi\pi^{-1}$, where π is given in Definition 4.1 (5), and a real operation $\otimes'$ such that $\varphi \otimes' \psi$ represents $\mathscr{C}$. If we define $\otimes$ on Re^+ by: for all $r, s > 0$,

$$r \otimes s = r \otimes' \varphi\pi^{-1}\varphi^{-1}(s), \tag{5.2}$$

then we have,

$$\begin{aligned} u \circ v \succsim x \circ y \quad & \text{iff } uv \succsim' xy \\ & \text{iff } \varphi(u) \otimes' \varphi\pi^{-1}(v) \geqslant \varphi(x) \otimes' \varphi\pi^{-1}(y) \\ & \text{iff } \varphi(u) \otimes \varphi(v) \geqslant \varphi(x) \otimes \varphi(y), \end{aligned} \tag{5.3}$$

proving that φ and $\otimes$ yield a representation.

To show that $\mathscr{X}$ is at most 2-point unique, it is sufficient to show that each automorphism of $\mathscr{X}$ that leaves two distinct elements of X invariant is the identity. Suppose α is the automorphism and a and b are distinct elements for which $\alpha(a) = a$ and $\alpha(b) = b$. Suppose $\varphi, \otimes$ is a representation of $\mathscr{X}$, R is the domain of φ and $R \subseteq \mathrm{Re}^+$. Define $\otimes'$ by Eq. (5.2). Then by Eq. (5.3), $\varphi \otimes' \psi$, where $\psi = \varphi\pi^{-1}$, forms a representation of the equivalent conjoint structure $\mathscr{C}$. Since α is an automorphism of $\mathscr{X}$, it follows that $\varphi', \otimes$, where $\varphi' = \varphi\alpha$ is also a representation for $\mathscr{X}$, and φ and φ' agree at a and b. To show that α is the identity, it is sufficient to show $\varphi = \varphi'$. By Eq. (5.2) and letting $\psi' = \psi\varphi^{-1}\varphi'$,

$$\begin{aligned} \varphi'(x) \otimes \varphi'(y) &= \varphi'(x) \otimes' \varphi\pi^{-1}\varphi^{-1}\varphi'(y) \\ &= \varphi'(x) \otimes' \psi\varphi^{-1}\varphi'(y) \\ &= \varphi'(x) \otimes' \psi'(y). \end{aligned}$$

Thus, $\varphi' \otimes' \psi'$ is also a representation for $\mathscr{C}$. By Theorem 2 of Luce and Cohen (1983), this means that φ and φ' both represent the total concatenation structure induced by $\mathscr{C}$ relative to (a, a). Since φ and φ' also agree at b, it follows from Theorem 1 of Luce and Cohen that $\varphi = \varphi'$. Q.E.D.

To gain some idea why it is difficult to formulate the concept of 1-point homogeneity in idempotent $\mathscr{X}$ even though it has been done for PCSs, it is useful to trace through the relations among the automorphisms of the several structures involved.

DEFINITION 5.1. A nontrivial automorphism is called a *dilation* iff it has a fixed point. Any other nontrivial automorphism is called a *translation*.

Recall that a (1, 1) subgroup of the affine group corresponds to the translations, not the dilations. The next result shows that it is the dilations, not the translations, that are maintained as automorphisms through the chain from idempotent to total concatenation structures. The translations appear only as isomorphisms among the induced total concatenation structures.

THEOREM 5.2. *Suppose $\mathscr{X} = \langle X, \succsim, \circ \rangle$ is an idempotent concatenation structure meeting the conditions of Theorem* 5.1, *$\mathscr{C}$ is the conjoint structure defined by Eq.* (5.1), *and $\mathscr{I}(a)$ is the total concatenation structure defined in terms of $\mathscr{C}$ and a in X (Definition* 4.3 *and Theorem* 4.1). *Then the following statements are true*:

(1) *A mapping α of X onto X is an automorphism of $\mathscr{X}$ iff (α, α) is a factorizable automorphism of $\mathscr{C}$.*

(2) *α is a dilation of $\mathscr{X}$ with fixed point a in X iff α is an automorphism of $\mathscr{I}(a)$ and $\alpha\pi = \pi\alpha$, where π is defined in Definition* 4.1 (5).

(3) *An n-copy operator of $\mathscr{I}(a)$ is an automorphism of $\mathscr{I}(a)$ iff it is a dilation of $\mathscr{X}$.*

(4) *If τ is a nontrivial translation of $\mathscr{X}$, then for each a in X, τ is an isomorphism between $\mathscr{I}(a)$ and $\mathscr{I}[\tau(a)]$ and τ is not an automorphism of any of the induced structures.*

Proof. (1) If α is an automorphism of $\mathscr{X}$, then from Eq. (5.1) and using both properties of an automorphism,

$$\begin{aligned} uv \succsim' xy \quad & \text{iff } u \circ v \succsim x \circ y \\ & \text{iff } \alpha(u \circ v) \succsim \alpha(x \circ y) \\ & \text{iff } \alpha(u) \circ \alpha(v) \succsim \alpha(x) \circ \alpha(y) \\ & \text{iff } \alpha(u)\, \alpha(v) \succsim' \alpha(x)\, \alpha(y), \end{aligned}$$

and so (α, α) is a factorizable automorphism of $\mathscr{C}$. Conversely, if (α, α) is a factorizable automorphism, then α is order preserving since

$$\begin{aligned} x \succsim y \quad & \text{iff } x \circ u \succsim y \circ u \\ & \text{iff } xu \succsim' yu \\ & \text{iff } \alpha(x)\, \alpha(u) \succsim' \alpha(y)\, \alpha(u) \\ & \text{iff } \alpha(x) \circ \alpha(u) \succsim \alpha(y) \circ \alpha(u) \\ & \text{iff } \alpha(x) \succsim \alpha(y). \end{aligned}$$

And α preserves $\circ$ since, by idempotency,

$$\begin{aligned} x \circ y = u \quad & \text{iff } x \circ y = u \circ u \\ & \text{iff } xy \sim' uu \\ & \text{iff } \alpha(x)\, \alpha(y) \sim' \alpha(u)\, \alpha(u) \\ & \text{iff } \alpha(x) \circ \alpha(y) = \alpha(u) \circ \alpha(u) = \alpha(u) \\ & \text{iff } \alpha(x) \circ \alpha(y) = \alpha(x \circ y). \end{aligned}$$

(2) Suppose α is a dilation, then by part (1) and the definition of a dilation (α, α) is a factorizable automorphism of $\mathscr{C}$ and so, by Theorem 8 of Luce and Cohen (1983), α is an automorphism of $\mathscr{I}(a)$ and $\alpha\pi = \pi\alpha$. The converse follows also from Theorem 8 of Luce and Cohen and part (1).

(3) By Theorem 7 of Luce and Cohen, if α_n is the n-copy operator of $\mathscr{I}(a)$, it is an automorphism of $\mathscr{I}(a)$ iff (α_n, α_n) is a factorizable automorphism of $\mathscr{C}$, and so the conclusion follows from parts (1) and (2).

(4) Suppose τ is a translation of $\mathscr{X}$. By Part 1 and the definition of a translation (Definition 5.1), (τ, τ) is a factorizable automorphism of $\mathscr{C}$ and so, by Theorem 8 of Luce and Cohen (1983), it establishes an isomorphism between $\mathscr{I}(a)$ and $\mathscr{I}[\tau(a)]$. It is not an automorphism of any of the induced structures since if it were, it would have the identity element a of that structure as a fixed point, making it a dilation, contrary to assumption. Q.E.D.

5.3. *Almost Homogeneous and Unique Structures*

The subgroup of dilations at a single point a, $\mathscr{I}(a)$, is necessarily 0-point homogeneous since a dilation cannot map the points below a into those above a and vice versa. Nonetheless, it is quite possible for the dilations to exhibit a good deal of homogeneity and/or uniqueness on either side of the fixed point a. We study this case briefly here because it is used in the next subsection as well as in Section 6.3.

DEFINITION 5.2. Suppose $\mathscr{X} = \langle X, \succsim, R_j \rangle_{j \in J}$ is a totally ordered relational structure. Let $\mathscr{G}$ be its group of automorphisms and, for each a in X, let $\mathscr{G}(a)$ be the subgroup of dilations at a. Let M and N be nonnegative integers. Then, $\mathscr{G}(a)$ is said to be *almost M-point homogeneous* (*almost N-point unique*) iff the restriction of $\mathscr{G}(a)$ to $X^+ = \{x \mid x \text{ in } X \text{ and } x \succ a\}$ and to $X^- = \{x \mid x \text{ in } X \text{ and } x \prec a\}$ is, in each case, M-point homogeneous (N-point unique). If for some a in X, $\mathscr{G}(a) = \mathscr{G}$ and $\mathscr{G}$ is nontrivial, then a is said to be an *intrinsic zero* of $\mathscr{X}$.

THEOREM 5.3. *Suppose $\mathscr{X} = \langle X, \succsim, R_j \rangle_{j \in J}$ is a totally ordered relational structure, $\langle X, \succsim \rangle$ is dense, and $\mathscr{G}$ is the group of automorphisms of $\mathscr{X}$. Then,*

(1) *$\mathscr{G}$ is 2-point homogeneous iff, for each a in X, $\mathscr{G}(a)$ is almost 1-point homogeneous.*

(2) *$\mathscr{G}$ is 2-point unique iff, for each a in X, $\mathscr{G}(a)$ is almost 1-point unique.*

Proof. (1) Suppose $\mathscr{G}$ is 2-point homogeneous. Then for either $x, y \succ a$ or $x, y \prec a$, there exists α in $\mathscr{G}$ such that $\alpha(a) = a$ and $\alpha(x) = y$. Thus, α is in $\mathscr{G}(a)$ and that subgroup is almost 1-point homogeneous.

Conversely, suppose for each a in X $\mathscr{G}(a)$ is almost 1-point homogeneous. We first establish that $\mathscr{X}$ does not have a minimal element. For suppose m were minimal, then since $x \succsim m$ and for all α in $\mathscr{G}$, $\alpha(x) \succsim \alpha(m) \succsim m$, we conclude $\alpha(m) = m$ because α is onto X. But that is impossible since for $a \succ m$, $\mathscr{G}(a)$ is almost

1-point homogeneous and so m must map under automorphisms into every element $\prec a$. Consider any u, v, x, y in X with $u \succ v$ and $x \succ y$. Select $a \prec \min(y, v)$, which exists since there is no minimal element. Without loss of generality, suppose $y \precsim v$. Then, x, $u \succ y$ and there exists α in $\mathscr{G}(a)$ such that $\alpha(y) = v$. Furthermore,

$$\alpha^{-1}(u) \succ \alpha^{-1}(v) = y.$$

Thus, by hypothesis, there exists β in $\mathscr{G}(y)$ such that $\beta(x) = \alpha^{-1}(u)$. So $\alpha\beta(x) = u$ and $\alpha\beta(y) = \alpha(y) = v$, proving $\mathscr{G}$ is 2-point homogeneous.

(2) Suppose $\mathscr{G}$ is 2-point unique. Let α, β in $\mathscr{G}(a)$ be such that for some $x \succ a$, $\alpha(x) = \beta(x)$. Since $\alpha(a) = a = \beta(a)$, it follows from 2-point uniqueness that $\alpha = \beta$, and so $\mathscr{G}(a)$ is almost 1-point unique.

Conversely, consider α, β in $\mathscr{G}$ and x, y in $X, x \neq y$, such that $\alpha(x) = \beta(x)$ and $\alpha(y) = \beta(y)$. Without loss of generality, suppose $x \succ y$. Since $\alpha^{-1}\beta(y) = y$, $\alpha^{-1}\beta$ is in $\mathscr{G}(y)$. Since that subgroup is almost 1-point unique, the fact that $\alpha^{-1}\beta(x) = x = \iota(x)$ implies $\alpha^{-1}\beta = \iota$, proving 2-point uniqueness. Q.E.D.

5.4. *A Characterization of Unit Representations*

As was remarked above (Sect. 5.1), 1-point homogeneity in a PCS is equivalent to its n-copy operators being automorphisms; however we showed that no direct use of this fact leads to understanding 1-point homogeneity in idempotent structures (Theorem 5.2). Theorem 6.3 of the next section provides a directly comparable result for a special class of idempotent structures called "self distributive" (Definition 6.1 (4)). However, as Theorem 6.4 makes clear, these structures are really bisymmetric, and so the remaining idempotent structures require an entirely different qualitative concept to establish their homogeneity. This subsection offers a scheme for characterizing all structures with unit structure representations, but we are not sure it will prove of practical use. So far, we have not been able to realize it in any specific example.

THEOREM 5.4. *Suppose $\mathscr{X} = \langle X, \succsim, \circ \rangle$ is a closed concatenation structure. Then the following three statements are equivalent*:

(1) *$\mathscr{X}$ has a unit representation onto* Re$^+$;

(2) *There exists a binary relation $\succsim'$ on $X \times X$ such that*

(i) *$\mathscr{C} = \langle X \times X, \succsim' \rangle$ is conjoint structure that is unrestrictedly solvable and Archimedean*;

(ii) *each ordering induced by $\succsim'$ on X is $\succsim$*;

(iii) *the Thomsen condition holds in $\mathscr{C}$*;

(iv) *$\circ$ is distributive in $\mathscr{C}$.*

(3) *For each a in X there exists a unique operation $*_a$ on X such that*

(i) $\langle X, \succsim, *_a, a\rangle$ *is a total concatenation structure;*

(ii) $*_a$ *is associative and commutative;*

(iii) *if $\mathscr{T}(*_a)$ is the right translations of $*_a$ and $\mathscr{G}$ is the automorphism group of $\mathscr{X}$, then $\mathscr{T}(*_a) \subseteq \mathscr{G}$.*

(iv) $\mathscr{T}(*_a)$ *is 1-point homogeneous.*

Proof. (1) implies (2). Suppose φ is the scale that maps $\mathscr{X}$ into the unit representation f. Define $\succsim'$ by: for all u, v, x, y in X,

$$uv \succsim' xy \quad \text{iff} \quad \varphi(u)\,\varphi(v) \geqslant \varphi(x)\,\varphi(y). \tag{5.4}$$

$\mathscr{C} = \langle X \times X, \succsim'\rangle$ is obviously a conjoint structure that is unrestrictedly solvable, Archimedean (statement (2)(i)) and for which statements (2)(ii) and (2)(iii) hold. Suppose $xp \sim' uq$ and $yp \sim' vq$. By Eq. (5.4), $\varphi(x)\,\varphi(p) = \varphi(u)\,\varphi(q)$ and $\varphi(y)\,\varphi(p) = \varphi(v)\,\varphi(q)$, from which $\varphi(x)/\varphi(y) = \varphi(u)/\varphi(v)$. Thus,

$$\varphi(p)\,\varphi(y)\,f[\varphi(x)/\varphi(y)] = \varphi(q)\,\varphi(v)\,f[\varphi(u)/\varphi(v)],$$

which is equivalent to $(x \circ y)\,p \sim' (u \circ v)\,q$, proving that $\circ$ is distributive in $\mathscr{C}$.

(2) implies (1). Suppose such a $\mathscr{C}$ exists. Then by Theorem 6.1 of Krantz *et al.* (1971, p. 257), it has a representation satisfying Eq. (5.4). By assumption (2ii), φ is order preserving. Define f as follows: for each $t > 0$, select x, y in X such that $t = \varphi(x)/\varphi(y)$ and set $f(t) = \varphi(x \circ y)/\varphi(y)$. To show that f is well defined suppose $t = \varphi(u)/\varphi(v) = \varphi(x)/\varphi(y)$. So, by Eq. (5.4), we may select p and q such that $xp \sim' uq$ and $yp \sim' vq$. Since $\circ$ is distributive, $(x \circ y)\,p \sim' (u \circ v)\,q$. Thus, $\varphi(x \circ y)\,\varphi(p) = \varphi(u \circ v)\,\varphi(q)$, and so $\varphi(x \circ y)/\varphi(y) = \varphi(u \circ v)/\varphi(v)$. By its choice, f is a unit representation.

(2) implies (3). Let $*_a$ be the operation induced on X by $\mathscr{C}$ relative to a. By Theorem 2 of Luce and Cohen (1983), $\langle X, \succsim, *_a, a\rangle$ is a total concatenation structure. By Theorem 4.2 (4), $*_a$ is associative and commutative. By Theorem 4.3 (2), $\mathscr{T}(*_a) \subseteq \mathscr{G}$. And by Theorem 4.2 (2), $\mathscr{T}(*_a)$ is 1-point homogeneous.

(3) implies (2). By Theorem 3 of Luce and Cohen (1983), there exists $\succsim'$ on $X \times X$ such that $\mathscr{C} = \langle X \times X, \succsim'\rangle$ is a conjoint structure whose induced total concatenation structure relative to a is isomorphic to $\langle X, \succsim, *_a, a\rangle$. By Theorem 4.3 (2) and assumption (3.iii), $\circ$ is distributive in $\mathscr{C}$ and $\mathscr{C}$ is solvable relative to the first component. By Theorem 4.2 (2) and assumption (3.iv), $\mathscr{C}$ is also solvable relative to the second component. Q.E.D.

COROLLARY. *Suppose $\mathscr{X} = \langle X, \succsim, \circ\rangle$ is a closed concatenation structure that has a unit representation onto* Re^+. *For a in S, let $*_a$, $\mathscr{T}(*_a)$, and $\mathscr{C}$ be as in Theorem* 5.4. *Then the following statements are true:*

(1) $\mathscr{T}(*_a)$ *is an Archimedean ordered group of scale type* (1, 1).

(2) $\mathscr{T}(*_a) \cap \mathscr{G}(*_a) = \{\iota\}$.

(3) *$\mathscr{G}(*_a)$ is almost 1-point homogeneous and almost 1-point unique.*

(4) *If $\mathscr{G}(*_a) \subseteq \mathscr{G}(\circ)$, then $\mathscr{G}(\circ)$ is 2-point but not 1-point unique.*

(5) *Let $\mathscr{H} = \bigcup_{a \in X} \mathscr{G}(*_a)$, then $\mathscr{H} \subseteq \mathscr{G}(\circ)$ iff $\mathscr{X}$ is of scale type (2, 2).*

Proof. There is no loss of generality in assuming $\mathscr{X}$ itself is its own real unit representation.

(1) This assertion follows from parts (1), (2), (4), and (5) of Theorem 4.2.

(2) Suppose α is in $\mathscr{T}(*_a) \cap \mathscr{G}(*_a)$. Then $\alpha(a) = a$ and for some z in X and all x in X, $\alpha(x) = x *_a z$. Thus,

$$z = a *_a z = \alpha(a) = a,$$

and so $\alpha = \iota$.

(3) Since $\mathscr{C}$ is defined in terms of multiplication, one verifies that $x *_a y = xy/a$. So α in $\mathscr{G}(*_a)$ satisfies the functional equation $\alpha(xy/a) = \alpha(x *_a y) = \alpha(x) *_a \alpha(y) = \alpha(x)\,\alpha(y)/a$, and the only strictly increasing solutions are $\alpha(x) = x^\rho a^{1-\rho}$, $\rho > 0$. For $x, y > a$ or $x, y < a$, the equation $y = \alpha(x)$ has the solution $\rho = \log(y/a)/\log(x/a)$, proving that $\mathscr{G}(*_a)$ is almost 1-point homogeneous. If for some x in X, $x \neq a$, and α, β in $\mathscr{G}(*_a)$, $\alpha(x) = \beta(x)$, then by what was shown above there are $\rho, \sigma > 0$ such that $x^\rho a^{1-\rho} = x^\sigma a^{1-\sigma}$, i.e., either $x = a$, contrary to choice, or $\rho = \sigma$. Thus, $\mathscr{G}(*_a)$ is almost 1-point unique.

(4) Suppose $\mathscr{G}(*_a) \subseteq \mathscr{G}(\circ)$ and $\mathscr{G}(\circ)$ is 1-point unique. A contradiction will be shown. If $x, y > a$, by part (3) there exists α in $\mathscr{G}(*_a) \subseteq \mathscr{G}(\circ)$ such that $\alpha(x) = y$. But since $\mathscr{T}(*_a)$ is 1-point homogeneous, it contains an element τ such that $\tau(x) = y$. By part (3)(iii) of Theorem 5.4, τ is in $\mathscr{G}(\circ)$, so by its 1-point uniqueness, $\alpha = \tau$. But that is impossible by part (2) of this corollary. Since $\mathscr{X}$ is a unit structure, $\mathscr{G}(\circ)$ is 2-point unique.

(5) Suppose $\mathscr{H} \subseteq \mathscr{G}(\circ)$. By part (3) and Theorem 5.3, $\mathscr{H}$ is 2-point homogeneous and unique. Thus, so is $\mathscr{G}(\circ)$ and so $\mathscr{X}$ is of scale type (2, 2). Conversely, if $\mathscr{X}$ is of scale type (2, 2), then its automorphisms are all of the transformations of the form σx^ρ, and by the proof of part (3) we know that those of $\mathscr{G}(*_a)$ are also of this form, and so $\mathscr{G}(*_a) \subseteq \mathscr{G}(\circ)$. Q.E.D.

6. Bisymmetry and Related Properties

6.1. *Definitions*

The literature on concatenation structures has been concerned largely with structures satisfying additional constraints, two of the most familiar being associativity and bisymmetry. We have previously made use of associativity, and now we turn to bisymmetry and to a number of quite closely related concepts, some of which are concatenation reformulations via Eq. (5.1) of conjoint measurement concepts. They are formulated as follows:

DEFINITION 6.1. Suppose $\mathscr{X} = \langle X, \succsim, \circ \rangle$ is a closed concatenation structure. Then, the following conditions are said to hold in $\mathscr{X}$, where p, q, u, v, x, y are in X,

(1) *Thomsen* iff

$$x \circ u = v \circ q \quad \text{and} \quad v \circ p = y \circ u \quad \text{imply} \quad x \circ p = y \circ q.$$

(2) *Bisymmetry* iff

$$(u \circ v) \circ (x \circ y) = (u \circ x) \circ (v \circ y).$$

(3) *Autodistributivity* (Aczél, 1966, p. 293) iff

$$(x \circ y) \circ u = (x \circ u) \circ (y \circ u) \quad \text{and} \quad v \circ (x \circ y) = (v \circ x) \circ (v \circ y).$$

Separately, these are referred to, respectively, as *right* and *left autodistributivity*. If it holds just for some fixed $u = v$, then it is referred to as *autodistributivity relative* to u.

(4) *Self distributivity* iff

$$x \circ p = u \circ q \quad \text{and} \quad y \circ p = v \circ q \quad \text{imply} \quad (x \circ y) \circ p = (u \circ v) \circ q.$$

(5) *Solvable relative to a* in X iff for each x in X there exist u, v in X such that $u \circ a = x = a \circ v$. Denote the right solution to $y \circ a$ by $\pi_a(y)$ and the left solution to $x \circ y$ by $\xi_a(x, y)$. Then the *induced operation* $*_a$ is defined by $x *_a y = \xi_a[x, \pi_a(y)]$. The set of *right translations relative to a* is denoted $\mathscr{T}(*_a)$. The set of all *n-copy operators* of $*_a$ is denoted $\Theta(*_a)$.

(6) *Dual bisymmetry* iff for some a $\mathscr{X}$ is solvable relative to a and the induced operation $*_a$ satisfies

$$(u \circ v) *_a (x \circ y) = (u *_a x) \circ (v *_a y).$$

(7) For each p, q in X, the *difference function* $\delta_{p,q}$ is defined as follows: $\delta_{p,q}(x) = y$ iff $y \circ p = x \circ q$. The set of all difference functions with p fixed is denoted $\mathscr{D}(p)$.

The theorems of this section describe how the concepts just defined relate to one another as we introduce increasingly stronger assumptions about $\mathscr{X}$, how ultimately they collapse into a single concept when $\mathscr{X}$ is solvable, idempotent, dense, and Dedekind complete (Theorem 6.4).

6.2. *Relations Among Concepts for Solvable Structures*

We first examine how several of the above concepts can be recast in the equivalent conjoint structure.

THEOREM 6.1. *Suppose $\mathscr{X} = \langle X, \succsim, \circ \rangle$ is a closed concatenation structure, $\mathscr{C} = \langle X \times X, \succsim' \rangle$, where $\succsim'$ is defined by Eq.* (5.1), *and $\circ$ is treated as an operation on the first component of $\mathscr{C}$. Then, the following five statements are true*:

(1) *$\circ$ in $\mathscr{X}$ satisfies the Thomsen condition iff $\mathscr{C}$ satisfies the Thomsen condition. If, in addition, $\mathscr{X}$ is solvable relative to a, then the induced operation $*_a$ is associative and commutative.*

(2) *$\circ$ in $\mathscr{X}$ is self distributive iff $\circ$ in $\mathscr{C}$ is distributive.*

(3) *Suppose $\mathscr{X}$ is solvable relative to b for all b in X, and let $\mathscr{G}$ be its group of automorphisms. If $\circ$ is self distributive and satisfies the Thomsen condition, then $\mathscr{T}(*_a)$ is a subgroup of $\mathscr{G}$.*

(4) *If in addition to all of the assumptions of (3), $\mathscr{X}$ is Archimedean, then $\mathscr{T}(*_a)$ is an Archimedean totally ordered group, where the order $\succsim'$ is defined by $\tau_x \succsim' \tau_y$ iff $x \succsim y$.*

(5) *If $\mathscr{X}$ is solvable relative to a, then $\mathscr{T}(*_a) = \mathscr{D}(a)$.*

Proof. (1) The equivalence is trivial. Observe that the induced operation of $\mathscr{X}$ is identical to that of Definition 4.2 relative to *aa*, and so by Theorem 4.2 (4) we know $*_a$ is associative and commutative.

(2) Trivial.

(3) By Theorem 4.2 (4)(5) and part (2) of this theorem, $\mathscr{T}(*_a)$ is a group. By Theorem 4.3 (2), $\mathscr{T}(*_a) \subseteq \mathscr{G}$.

(4) Since $\mathscr{X}$ is Archimedean, so is $\mathscr{C}$, and by Theorem 4.2 (6), $\mathscr{T}(*_a)$ is an Archimedean ordered group.

(5) Fix a and let q be any element of X. Then,

$$\begin{aligned} \delta_{a,q}(x) \circ a &= x \circ q && \text{(Definition of } \delta_{a,q}) \\ &= \xi_a(x, q) \circ a && \text{(Definition of } \xi_a) \\ &= [x *_a \pi_a^{-1}(q)] \circ a && \text{(Definition of } *_a). \end{aligned}$$

By monotonicity, the last expression holds iff $\delta_{a,q}(x) = x *_a \pi_a^{-1}(q)$. Since $\mathscr{X}$ is solvable relative to a and π_a^{-1} is onto X, it follows that $\mathscr{D}(a) = \mathscr{T}(*_a)$. Q.E.D.

THEOREM 6.2. *Suppose $\mathscr{X}$ is a closed concatenation structure that is solvable relative to a in X. Let $*_a$ be the induced operation, $\mathscr{G}$ the set of automorphisms of $\mathscr{X}$, and $\mathscr{E}$ the set of endomorphisms of $\mathscr{X}$. Then the network of implications shown in Fig. 2 holds.*

Proof. The numbered arrows of the diagram will be proved in order.

(1) Krantz *et al.* (1971, p. 298).

(2)
$$\begin{aligned} a \circ \pi_a(x \circ y) &= (x \circ y) \circ a && \text{(Definition of } \pi_a) \\ &= (x \circ a) \circ (y \circ a) && \text{(Autodistributivity relative to } a) \\ &= [a \circ \pi_a(x)] \circ [a \circ \pi_a(y)] && \text{(Definition of } \pi_a) \\ &= a \circ [\pi_a(x) \circ \pi_a(y)], \end{aligned}$$

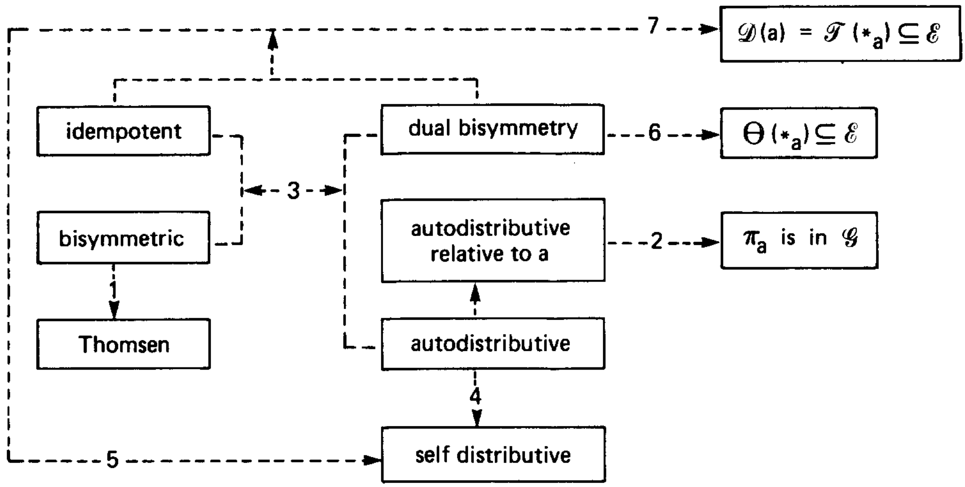

FIGURE 2

and so by monotonicity, π_a is an endomorphism. By solvability it is onto, and by monotonicity it is one-to-one, so it is an automorphism.

(3) Assume idempotence and bisymmetry, then we first prove autodistributivity:

$$\begin{aligned}(x \circ y) \circ u &= (x \circ y) \circ (u \circ u) && \text{(Idempotence)}\\ &= (x \circ u) \circ (y \circ u) && \text{(Bisymmetry).}\end{aligned}$$

The left-sided case is similar. Next, we show dual bisymmetry:

$$\begin{aligned}[(u \circ v) *_a (x \circ y)] \circ a &= (u \circ v) \circ \pi_a(x \circ y) && \text{(Definition of } *_a)\\ &= (u \circ v) \circ [\pi_a(x) \circ \pi_a(y)] && \text{(Part (2))}\\ &= [u \circ \pi_a(x)] \circ [v \circ \pi_a(y)] && \text{(Bisymmetry)}\\ &= [(u *_a x) \circ a] \circ [(v *_a y) \circ a] && \text{(Definition of } *_a)\\ &= [(u *_a x) \circ (v *_a y)] \circ a && \text{(Autodistributivity)}\end{aligned}$$

whence, by monotonicity, dual bisymmetry holds.

Conversely, we first show idempotence.

$$\begin{aligned}(x \circ x) *_a (a \circ a) &= (x *_a a) \circ (x *_a a) && \text{(Dual bisymmetry)}\\ &= x \circ x && (a \text{ is the identity of } *_a)\\ &= (x \circ x) *_a a,\end{aligned}$$

and so, by the monotonicity of $*_a$, $a \circ a = a$. Using this and autodistributivity relative to both a and x,

$$(x \circ x) \circ a = (x \circ a) \circ (x \circ a) = x \circ (a \circ a) = x \circ a,$$

and so, by the monotonicity of $\circ$, $x \circ x = x$. To show bisymmetry, consider,

$$\begin{aligned}
(u \circ v) \circ (x \circ y) &= (u \circ v) \circ [\pi_a \pi_a^{-1}(x) \circ \pi_a \pi_a^{-1}(y)] \\
&= (u \circ v) \circ \pi_a[\pi_a^{-1}(x) \circ \pi_a^{-1}(y)] && \text{(Part 3)} \\
&= \{(u \circ v) *_a [\pi_a^{-1}(x) \circ \pi_a^{-1}(y)]\} \circ a && \text{(Definition of } *_a) \\
&= \{[u *_a \pi_a^{-1}(x)] \circ [v *_a \pi_a^{-1}(y)]\} \circ a && \text{(Dual bisymmetry)} \\
&= \{[u *_a \pi_a^{-1}(x)] \circ a\} \circ \{[v *_a \pi_a^{-1}(y)] \circ a\} && \text{(Autodistributivity)} \\
&= (u \circ x) \circ (v \circ y) && \text{(Definition of } *_a).
\end{aligned}$$

(4) Suppose $x \circ p = u \circ q$ and $y \circ p = v \circ q$, then by monotonicity and right autodistributivity,

$$(x \circ y) \circ p = (x \circ p) \circ (y \circ p) = (u \circ q) \circ (v \circ q) = (u \circ v) \circ q.$$

(6) Suppose $x \circ p = u \circ q$ and $y \circ p = v \circ q$. Applying the definition of $*_a$ to these yields,

$$[x *_a \pi_a^{-1}(p)] \circ a = [u *_a \pi_a^{-1}(q)] \circ a$$
$$[y *_a \pi_a^{-1}(p)] \circ a = [v *_a \pi_a^{-1}(q)] \circ a.$$

From this, monotonicity, idempotence, and dual bisymmetry,

$$\begin{aligned}
(x \circ y) *_a \pi_a^{-1}(p) &= (x \circ y) *_a [\pi_a^{-1}(p) \circ \pi_a^{-1}(p)] \\
&= [x *_a \pi_a^{-1}(p)] \circ [y *_a \pi_a^{-1}(p)] \\
&= [u *_a \pi_a^{-1}(q)] \circ [v *_a \pi_a^{-1}(q)] \\
&= (u \circ v) *_a [\pi_a^{-1}(q) \circ \pi_a^{-1}(q)] \\
&= (u \circ v) *_a \pi_a^{-1}(q),
\end{aligned}$$

and so, by the definition of $*_a$, $(x \circ y) \circ p = (u \circ v) \circ q$.

(6) Let θ_n be the n-copy operator of $*_a$. Observe,

$$\theta_1(x \circ y) = x \circ y = \theta_1(x) \circ \theta_1(y),$$

and proceeding by induction and using dual bisymmetry,

$$\begin{aligned}\theta_n(x \circ y) &= \theta_{n-1}(x \circ y) *_a (x \circ y)\\ &= [\theta_{n-1}(x) \circ \theta_{n-1}(y)] *_a (x \circ y)\\ &= [\theta_{n-1}(x) *_a x] \circ [\theta_{n-1}(y) *_a y]\\ &= \theta_n(x) \circ \theta_n(y),\end{aligned}$$

proving that the n-copy operator is an endomorphism.

(7) Let $\tau_b(x) = x *_a b$, then

$$\begin{aligned}\tau_b(x \circ y) &= (x \circ y) *_a b\\ &= (x \circ y) *_a (b \circ b)\\ &= (x *_a b) \circ (y *_a b)\\ &= \tau_b(x) \circ \tau_b(y),\end{aligned}$$

and so the right translations of $*_a$ are endomorphisms. By Theorem 6.1 (5), they are the same set as $\mathscr{D}(a)$. Q.E.D.

Our next result, which depends upon adding solvability to the assumptions about $\mathscr{X}$, answers the question, raised at the end of Section 5.2, as to the idempotent analog of the result for PCSs that 1-point homogeneity is equivalent to the n-copy operators being automorphisms.

THEOREM 6.3. *Suppose* $\mathscr{X} = \langle X, \succsim, \circ \rangle$ *is a closed, solvable concatenation structure. Then the following are true*:

(1) $\mathscr{D}(a)$ *is a subgroup of the automorphisms of* $\mathscr{X}$ *iff* $\mathscr{X}$ *is self distributive. In this case, for each a in X,* $\mathscr{D}(a)$ *is* 1*-point homogeneous.*

(2) *If* $\circ$ *satisfies both the Thomsen condition and right autodistributivity, then* $\circ$ *is bisymmetric.*

Proof. (1) By Theorem 6.1 (5), for each a in X, $\mathscr{D}(a) = \mathscr{T}(*_a)$. Let $\mathscr{C}$ be defined by Eq. (5.1). By Theorem 4.3 (2), $\mathscr{D}(a) = \mathscr{T}(*_a)$ is a subgroup of the automorphisms of $\mathscr{X}$ iff $\circ$ is distributive in $\mathscr{C}$ and $\mathscr{C}$ is unrestrictedly solvable in the first coordinate. By Theorem 6.1 (2), $\circ$ is distributive in $\mathscr{C}$ iff $\mathscr{X}$ is self distributive, and the fact that $\mathscr{X}$ is solvable implies that $\mathscr{C}$ is unrestrictedly solvable in both coordinates. Thus, the condition holds. In this case, by Theorem 4.2 (2), $\mathscr{T}(*_a)$ is 1-point homogeneous because $\mathscr{C}$ is unrestrictedly solvable in the second coordinate.

(2) Let u, v, x, y be in X. By solvability, there exist p, q in X such that $p \circ y = u \circ v$ and $u \circ x = q \circ y$, and so by Thomsen condition, $p \circ x = q \circ v$. Thus,

$$\begin{aligned}(u\circ v)\circ(x\circ y) &= (p\circ y)\circ(x\circ y) && \text{(Definition of } p \text{ and monotonicity)}\\ &= (p\circ x)\circ y && \text{(Right autodistributivity)}\\ &= (q\circ v)\circ y && \text{(Above)}\\ &= (q\circ y)\circ(v\circ y) && \text{(Right autodistributivity)}\\ &= (u\circ x)\circ(v\circ y) && \text{(Definition of } q \text{ and monotonicity),}\end{aligned}$$

which proves bisymmetry. Q.E.D.

6.3. *Equivalences in Solvable, Idempotent, Dense, Dedekind Complete Structures*

The remainder of this section is devoted to showing that in the presence of the further restrictions of idempotence, density, and Archimedean, all of the properties in Theorem 6.2, save the Thomsen condition and the fact that π_a is an automorphism, collapse into the single concept of bisymmetry.

THEOREM 6.4. *Suppose $\mathscr{X} = \langle X, \succsim, \circ \rangle$ is a closed concatenation structure that is idempotent, solvable, and Dedekind complete. Let $\mathscr{G}$ be its group of automorphisms. Then the following statements are equivalent*:

(1) *$\mathscr{X}$ is bisymmetric.*

(2) *$\mathscr{X}$ is right autodistributive.*

(3) *$\mathscr{X}$ is self distributive.*

(4) *For each a in X, $\Theta(*_a) \subseteq \mathscr{G}$.*

(5) *$\mathscr{D}(a) = \mathscr{T}(*_a)$ is a subgroup of $\mathscr{G}$.*

Proof. To show the equivalence of (1), (2), and (3), it suffices, by Theorem 6.2, to show (3) implies (1). So suppose $\mathscr{X}$ is self distributive. By Theorem 5.1, there exists a numerical representation φ and $\mathscr{X}$ is either 1- or 2-point unique. Since a closed, idempotent concatenation structure is intern, it is dense, and so by Theorem 6.3, self distribution implies $\mathscr{X}$ is 1-point homogeneous. Thus, $\mathscr{X}$ has a unit representation (Theorems 3.2 and 3.9) f and, so by Theorem 5.4, there exists a conjoint structure fulfilling the properties of $\succsim'$ in Theorem 4.4. Define $\succsim''$ by

$$uv \succsim'' xy \qquad \text{iff} \quad u\circ v \succsim x\circ y.$$

Since $\circ$ is self-distributive, Theorem 6.1 (2) establishes that $\succsim''$ fulfills the conditions of $\succsim''$ in Theorem 4.4. Since φ and f yield the unit representation of $\circ$, $\varphi\varphi$ represents $\succsim'$. Also, there exist ψ_1 and ψ_2 such that $\psi_1\psi_2$ represents $\succsim''$, and each is monotonic in $\succsim$ since $\succsim$ is the order induced on the first component by both $\succsim'$ and $\succsim''$. We show that it is possible to select $\psi_1 = \varphi$. Because $\circ$ is distributive in $\succsim''$, we have

$$\psi_1(x)/\psi_1(y) = \psi_1(u)/\psi_1(v)$$

implies

$$\psi_1(x \circ y)/\psi_1(y) = \psi_1(u \circ v)/\psi_1(v),$$

and so, as in Theorem 5.4, we can define a function g such that ψ_1 and g constitute a unit representation. By the uniqueness of unit representations (see Cohen and Narens, 1979, Theorem 3.5), for some $\sigma, \rho > 0$, $\psi_1 = \sigma\varphi^\rho$. So setting $\psi = \psi_2^{1/\rho}$, we see that $\varphi\psi$ represents $\succsim''$. Moreover, by considering $u \circ x \succsim u \circ y$, we see that there is a strictly increasing function h such that $\psi = h(\varphi)$. Thus, we have two representations of $\circ$,

$$\begin{aligned} u \circ v \succsim x \circ y \quad & \text{iff } \varphi(u)\, h[\varphi(v)] \geqslant \varphi(x)\, h[\varphi(y)], \\ & \text{iff } \varphi(v)\, f[\varphi(u)/\varphi(v)] \geqslant \varphi(y)\, f[\varphi(x)/\varphi(y)], \end{aligned}$$

and so for some strictly increasing F,

$$F\{\varphi(x)\, h[\varphi(y)]\} = \varphi(y)\, f[\varphi(x)/\varphi(y)]. \tag{6.1}$$

If we select $y = a$ such that $h[\varphi(a)] = 1$, and set $A = \varphi(a)$, then

$$F[\varphi(x)] = Af[\varphi(x)/A]. \tag{6.2}$$

Now, substituting Eq. (6.2) into Eq. (6.1) and setting $W = \varphi(x)/A$, $Y = \varphi(y)/A$, and $H(z) = h(zA)$, we obtain

$$f[WH(Y)] = Yf(W/Y). \tag{6.3}$$

Observe that by setting $W = Y$ in Eq. (6.3) and noting that $f(1) = 1$ because the structure is idempotent,

$$H(W) = f^{-1}(W)/W. \tag{6.4}$$

If we set $U = W/Y$ and $V = f^{-1}(Y)$ and substitute Eq. (6.4) into Eq. (6.3), then f is characterized by

$$f(UV) = f(U)\, f(V),$$

which, since f is strictly increasing, is well known to have as its unique solution

$$f(U) = U^c, \qquad c > 0.$$

Since f is a unit representation, f/ι is decreasing so $c < 1$. This defines the (2, 2) bilinear representation with $c = d$, and so the structure is bisymmetric.

(1) is equivalent to (4). Suppose $\mathscr{X}$ is bisymmetric. By Theorem 6.2, we know that the n-copy operator of $*_a$, θ_n, is an endomorphism of $\mathscr{X}$. In the bisymmetric case, there is a real representation $\langle \mathrm{Re}^+, >, \otimes \rangle$ under φ with, for r, s in Re^+, $r \otimes s = r^c s^{1-c}$ for some c in (0, 1). It is easy to verify that the real induced operation $*_a$ is given by $r *_a s = rs/A$, where $A = \varphi(a)$, and so by induction $\theta_n(r) = r(r/A)^{n-1}$.

Thus, it is onto the reals and 1 : 1, hence it is an automorphism, and so, then, is the n-copy operator in $\mathscr{X}$.

Conversely, if the n-copy operators of $*_a$ are automorphisms, then their restrictions to X^+ and X^- are the automorphisms of the PCSs that make up the two parts of the total concatenation structure $\langle X, \succsim, *_a, a\rangle$ (see Definition 4.3). Thus, by Theorem 3.1 of Cohen and Narens (1979), these PCSs are 1-point homogeneous and 1-point unique. Moreover, a is a fixed point, so the subgroup of dilations at a are almost 1-point homogeneous and 1-point unique. Therefore, by Theorem 5.3, the structure is of type (2, 2). By Theorem 3.11 it has a log dual linear representation (Eq. (3.5)). We next establish $c = d$, which proves that the structure is bisymmetric. It is easy to verify that for $r > 0$,

$$\begin{aligned} \theta_n(r) &= r(r/A)^{(n-1)(1-c)/(1-d)} && \text{if } r \geqslant A, \\ &= r(r/A)^{(n-1)(1-d)/(1-c)} && \text{if } r < A. \end{aligned}$$

Consider any r, s such that $r > A > r \otimes s > s$. It is routine to show that θ_n is an automorphism, as assumed, iff $c = d$.

(3) is equivalent to (5). By Theorem 6.2 (5), $\mathscr{D}(a) = \mathscr{T}(*_a)$, and by Theorem 6.3 this set is a subgroup of the automorphisms iff condition (3) holds. Q.E.D.

Observe that for a real concatenation structure with an operation that is continuous in each variable, the conditions of the theorem are fulfilled, which settles affirmatively the conjecture on p. 299 of Aczél (1966) that for a continuous, monotonic operation right autodistributivity implies bisymmetry.

As we mentioned at the end of Theorem 4.4, we cannot strengthen the conclusion of that result from $*' = *''$ to $\succsim' = \succsim''$, as the following argument shows: Suppose it were so, and consider the case of a bisymmetric structure satisfying the hypotheses of Theorem 6.4. In the first part of the proof, it would follow that $h = \iota$, and so $H = A\iota$, whence Eq. (6.1) becomes $f(XYA) = Yf(X/Y)$, and so setting $X = 1/YA$ and solving we find $f(X) = X^{1/2}$. But this is absurd since there are bisymmetric structures with $c \neq \frac{1}{2}$.

Given Theorem 4.4, the following question seems natural. Suppose $\langle X, \succsim, \circ\rangle$ and $\langle X, \succsim, \circ'\rangle$ are both concatenation structures that are closed, idempotent, dense, solvable, and Archimedean. Then, under what conditions is each operation distributive relative to the other in the following sense: if $x \circ p = u \circ q$ and $y \circ p = v \circ q$, then $(x \circ' y) \circ p = (u \circ' v) \circ q$, and the corresponding statement with $\circ$ and $\circ'$ interchanged? A sufficient condition is dual bisymmetry (Definition 6.1 (6)) since from the hypotheses and the monotonicity of $\circ'$ we have

$$(x \circ p) \circ' (y \circ p) = (u \circ q) \circ' (v \circ q),$$

whence by dual bisymmetry,

$$(x \circ' y) \circ (p \circ' p) = (u \circ' v) \circ (q \circ' q).$$

By the idempotence of $\circ'$, it follows that $\circ'$ distributes across $\circ$. The other half is similar. The converse, if true, may be difficult to prove since it implies the first three parts of Theorem 6.4.

7. Application to Utility of Gambles

7.1. *Regular Mixture Spaces*

The problem of choosing between gambles is often cast in the framework of a mixture space in which the typical element has three components: an event A, sometimes assumed to be fully characterized by its probability $p = \Pr(A)$, and two outcomes x and y, where x is received if A occurs and y is received if A fails to occur. The outcomes x and y are assumed to be elements of the mixture space and so can be gambles themselves or pure outcomes, such as amounts of money. We may think of such a gamble as an "experiment" in much the same sense as statisticians use that term. The symbols (x, A, y) and $(x, A; y, \sim A)$, where $\sim A$ signifies the complement of A, are often used to denote such gambles, the latter notation having the advantage that it generalizes in a natural way to gambles with more than two outcomes. When only the probabilities matter, one writes (x, p, y) or $(x, p; y, 1-p)$ or even xpy. If $x = (u, B, v)$, then $(x, A, y) = ((u, B, v), A, y)$ is interpreted to mean that outcome u is received when both A and B occur, v when both A and $\sim B$, and y when neither A nor B. Lurking behind this notation for mixes of mixes is a question of independence. Our intention is to think of a mix of mixes as two statistically independent gambles in which the gamble based on A is carried out first and then following that, as an independent experiment, the one based on B is carried out. We do not think of A and B as subevents of a single experiment! In particular, in the gamble $((x, A, y), A, z)$ the second occurrence of A is to be thought of as an event that is statistically independent from the first occurence of A. Should ambiguity seem imminent, one may always write A and A' or A_1 and A_2.

In order to reduce the possibility of inadvertently misinterpreting the notation, we find it helpful to think of the process of forming gambles based on an event A as an operation, $\circ_A$. So, in this paper we shall write $x \circ_A y$ instead of the more usual notation (x, A, y). Thus, the structure we consider is based upon the following primitives: a nonempty set $\mathscr{E}$ of events (the events can be thought of as sets themselves), a nonempty set X composed of pure outcomes and all gambles that can be generated inductively by the binary operations $\circ_A$, A in $\mathscr{E}$, operating on pure outcomes to generate gambles and on gambles to generate mixes of gambles, etc., and a binary relation $\succsim$ on X.

Definition 7.1. Let X, $\mathscr{E}$, $\succsim$, and $\circ_A$, A in $\mathscr{E}$, be as above. Then the relational structure

$$\mathscr{M} = \langle X, \succsim, \circ_A \rangle_{A \in \mathscr{E}} \tag{7.1}$$

is said to be a *mixture space* iff $\succsim$ is a weak ordering. The mixture space $\mathscr{M}$ is said to be *regular* iff each $\mathscr{X}_A = \langle X, \succsim, \circ_A \rangle$ is a weakly ordered concatenation structure that is closed, idempotent, dense, solvable, and Dedekind complete. The mixture space $\mathscr{M}$ is said to be *closed* iff there is a function $\chi\colon \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ such that for every x, y in X,

$$(x \circ_A y) \circ_B y \sim x \circ_{\chi(A,B)} y. \tag{7.2}$$

Observe that "solvability" in Definition 7.1 means that with the event A fixed and three of the four outcomes x, y, u, v given, the fourth outcome can be found so as to satisfy $x \circ_A y \sim u \circ_A v$. Also observe that $\mathscr{E}$ is assumed to be a nonempty set of events and not necessarily an algebra of events. In fact, if $\mathscr{M}$ is regular one would not want to include either X or $\varnothing$ in $\mathscr{E}$ since $\circ_A$ is intern for each A in $\mathscr{E}$. We have adopted this approach so that definitions, statements of theorems, and proofs will be less complicated since the special cases of $A = X$ and $A = \varnothing$ do not have to be considered. It does not constitute a loss of generality.

The weak ordering assumption is common in utility theory, although recently Fishburn (1982, 1983) has questioned it and has developed a weakening of expected utility theory that avoids it. Closure under the operation $\circ_A$ is a widely accepted idealization which implies that each experiment can be repeated, independently, an arbitrary number of times. Idempotence is trivial given the interpretation of $x \circ_A x$. Monotonicity within utility theory is highly controversial and is believed by many to be rejected by a substantial body of empirical data. There is no doubt that the data reject something embodied in the traditional expected utility theory, but as we argue in detail in Sections 7.4 and 7.5 it does not appear to be the version of monotonicity displayed by the concatenation operation $\circ_A$ that is the problem. Density, solvability, and Archimedean are all plausible and have been little questioned in discussions of the theory. Closure of $\mathscr{M}$ simply means that the space of events is sufficiently rich that two successive independent events can be replaced by a single event.

7.2. *The Dual Bilinear Utility Representation*

If we just look at the ordered set of gambles $G = \langle X, \succsim \rangle$, a homomorphism of G into $\langle \mathrm{Re}, \geqslant \rangle$ is said to be a *utility function.* Note that we have shifted from Re^+ to Re in deference to the fact that psychologists and economists, unlike physicists, prefer to work with interval scales on Re rather than with log interval scales on Re^+. A *utility scale* for G is a nonempty set $\mathscr{U}$ of utility functions. The concepts of M-point homogeneity and N-point uniqueness are unchanged, and we are interested in conditions that lead to the (2, 2) scale type of the mixture structure $\mathscr{M}$ given in Eq. (7.1) so that we can show the existence of a utility scale for G that is an interval scale.

The general type of utility model that has been considered is one in which the utility of a gamble $U(x \circ_A y)$ can be expressed as some function of the utilities of its components, $U(x)$ and $U(y)$, and some weighting function of the event A that, in general, depends also on x and y. This very general model takes the following form: If $\mathscr{U}$ is a utility scale for G, then there exists a collection of functions $\langle F_U, W_U \rangle_{U \in \mathscr{U}}$, where F_U: $\mathrm{Re}^3 \to \mathrm{Re}$ and W_U: $\mathrm{Re}^2 \times \mathscr{E} \to \mathrm{Re}$, such that for all x, y in X, A in $\mathscr{E}$, and U in $\mathscr{U}$,

$$U(x \circ_A y) = F_U(U(x), U(y), W_U[U(x), U(y), A]). \tag{7.3}$$

This is far too general to say much about. The usual rational model for utility of gambling, which is known as the *subjective expected utility* (SEU) model, makes a number of very strong assumptions about Eq. (7.3). (For a general survey of models of this type see Fishburn (1981).) Specifically, $\mathscr{U}$ is taken to be an interval scale; the weights are assumed not to depend on U or x or y, but just on A (and thus may be written as just $W(A)$), and this weighting function is assumed to be a (finitely) additive probability function on $\mathscr{E}$, which is typically assumed to be a Boolean algebra of events; and finally the combining function F also does not depend upon U and in fact has the bisymmetric form

$$F(r, s, p) = rp + s(1 - p). \tag{7.4}$$

In accounting for human behavior, the general model of Eq. (7.3) is simply too general and the SEU one has turned out to be a bit too restrictive (see below). So our aim is to find something in between. We shall accept the following feature of the rational model, namely, that neither the weighting function W nor the combining function F shall be subscripted by the utility function. So, Eq. (7.3) takes the form

$$U(x \circ_A y) = F(U(x), U(y), W[U(x), U(y), A]). \tag{7.5}$$

LEMMA 7.1. *Suppose $\mathscr{M}$ is a mixture space* (*Definition* 7.1) *such that for each A in $\mathscr{E}$, $\mathscr{X}_A$ is 2-point unique, that $\langle X, \succsim \rangle$ has a utility scale $\mathscr{U}$ that is 2-point homogeneous in the sense that for all x, y in X and r, s in* Re, *if $x \succ y$ and $r > s$, then for some V in $\mathscr{U}$, $V(x) = r$ and $V(y) = s$, and suppose each element of $\mathscr{U}$ is onto* Re *and there exist functions F and W such that Eq.* (7.5) *holds for all U in $\mathscr{U}$. Then for each A, B in $\mathscr{E}$, the automorphism groups of $\mathscr{X}_A$ and $\mathscr{X}_B$ are identical and are* (2, 2).

Proof. For A in $\mathscr{E}$, we show $\mathscr{X}_A$ is 2-point homogeneous which with the hypothesis that it is 2-point unique proves it is of scale type (2, 2). Suppose x, y, u, v in X are such that $x \succ y$ and $u \succ v$ and V is in $\mathscr{U}$. Since by the assumption that $\mathscr{U}$ is 2-point homogeneous, there exists U in $\mathscr{U}$ such that $U(x) = V(u)$ and $U(y) = V(v)$. Since, by hypothesis, U and V are onto Re and order preserving, $\alpha = U^{-1}V$ is a function from X into X. In fact it is onto, since for any z in X, $\alpha[V^{-1}U(z)] = z$. By the fact that V and U^{-1} are order preserving, so is α. And by choice, $\alpha(u) = x$ and $\alpha(v) = y$. Since X is the domain of $\mathscr{X}_A$, to show that α is an automorphism of $\mathscr{X}_A$ it suffices to show that for all w, z in X, $\alpha(w \circ_A z) = \alpha(w) \circ_A \alpha(z)$. By Eq. (7.5) and the fact that $U\alpha = UU^{-1}V = V$,

$$\begin{aligned} U[\alpha(w) \circ_A \alpha(z)] &= F(U\alpha(w), U\alpha(z), W[U\alpha(w), U\alpha(z), A]) \\ &= F(V(w), V(z), W[V(w), V(z), A]) \\ &= V(w \circ_A z), \end{aligned}$$

and applying U^{-1} yields the result. Thus, $\{U^{-1}V \mid U, V \text{ in } \mathscr{U}\}$ is a set of automorphisms of $\mathscr{X}_A$ for every A in $\mathscr{E}$, and therefore, since $\mathscr{X}_A$ is 2-point unique, $\mathscr{X}_A$ must be of type (2, 2). Q.E.D.

THEOREM 7.1. *Suppose $\mathscr{M}$ is a regular mixture space with a family of utility scales that are onto* Re, *are 2-point homogeneous, and are such that Eq.* (7.5) *holds for all A in $\mathscr{E}$. Then there exist an interval scale $\mathscr{U}$ and functions S^+ and S^- from $\mathscr{E}$ into* (0, 1) *such that for all U in $\mathscr{U}$,*

(i) *U is order preserving, and*

(ii) *for all x, y in X and A in $\mathscr{E}$,*

$$\begin{aligned} U(x \circ_A y) &= U(x)\,S^+(A) + U(y)[1 - S^+(A)] && \text{if } x \succ y, \\ &= U(x) && \text{if } x \sim y, \qquad (7.6) \\ &= U(x)\,S^-(A) + U(y)[1 - S^-(A)] && \text{if } x \prec y. \end{aligned}$$

Proof. By Theorem 5.1, each $\mathscr{X}_A$ of a regular mixture space is 2-point unique, and so by Lemma 7.1 they have a common group of (2, 2) automorphisms. Thus, by Theorem 3.12 and Eq. (3.5) following it, each can be represented as in Eq. (7.6) since the constants c and d depend upon A. (Note that Theorem 3.12 and Eq. (3.5) are for totally ordered concatenation structures, but they easily extend to weakly ordered ones.) Q.E.D.

We refer to $\langle \mathscr{U}, S^+, S^- \rangle$ that satisfies Eq. (7.6) as the *dual bilinear utility representation* of $\mathscr{M}$. In formulating this result, we have introduced the symbol S for the weights that depend upon both A and the ordering of x and y so as to suggest the subjective nature of the weights. Note that nothing has been said about how the two weighting functions S^+ and S^- relate to one another nor have we suggested that they behave like probabilities. If the two weighting functions are identical, then we speak of Eq. (7.6) as the *bisymmetric utility representation*, and if in addition $\mathscr{E}$ is a Boolean algebra of events, Eq. (7.6) is extended in the obvious way to include events X and $\varnothing$, and the common weighting function is a finitely additive probability measure on it, then we call Eq. (7.6) the (*rational*) SEU *model for* $\mathscr{M}$.

We turn now to various restrictions that further constrain the model. After that we will explore how the model relates to various empirical facts that invalidate the rational SEU model as one for human decision making.

7.3. Various Additional Restrictions

All of the following properties are plausible in varying degrees, and they impose additional restrictions on the form of the dual bilinear representation.

DEFINITION 7.2. In all of the following statements $\mathscr{M}$ is a mixture space and A, B, C are in $\mathscr{E}$ and w, x, y, z, u, v are in X.

(1) *Commutativity* holds in $\mathscr{M}$ iff $\mathscr{M}$ is closed and

$$\chi(A, B) = \chi(B, A),$$

which is true iff

$$(x \circ_A y) \circ_B y \sim (x \circ_B y) \circ_A y.$$

(2) *Complementation* holds in $\mathscr{M}$ iff $\mathscr{E}$ is a set of subsets that is closed under complementation with respect to X and

$$x \circ_A y \sim y \circ_{\sim A} x.$$

(3) *Self distribution* holds in $\mathscr{M}$ iff

$$x \circ_A w \sim u \circ_A z \qquad \text{and} \qquad y \circ_A w \sim v \circ_A z$$

imply

$$(x \circ_A y) \circ_A w \sim (u \circ_A v) \circ_A z.$$

(4) *Right autodistributivity* holds in $\mathscr{M}$ iff

$$(x \circ_A y) \circ_A z \sim (x \circ_A z) \circ_A (y \circ_A z).$$

(5) *Bisymmetry* holds in $\mathscr{M}$ iff

$$(x \circ_A y) \circ_A (u \circ_A v) \sim (x \circ_A u) \circ_A (y \circ_A v).$$

(6) *Outcome independence* holds in $\mathscr{M}$ iff for some x, y in X, $x \circ_A y \sim x \circ_B y$, then for all u, v in X for which $u \succsim v$ iff $x \succsim y$ it follows that $u \circ_A v \sim u \circ_B v$.

(7) *Monotonicity of events* holds in $\mathscr{M}$ iff for all A, B, C in $\mathscr{E}$ such that for $A \cap C = B \cap C = \varnothing$ and $A \cup C$, $B \cup C$ are in $\mathscr{E}$, then

$$x \circ_A y \succsim x \circ_B y \qquad \text{iff} \quad x \circ_{A \cup C} y \succsim x \circ_{B \cup C} y.$$

(8) *Consistency of preferences with event inclusion holds in* $\mathscr{M}$ iff when $A \supseteq B$, then

$$x \succsim y \qquad \text{iff} \quad x \circ_A y \succsim x \circ_B y.$$

In each case, if one thinks through what the two sides of the condition mean in terms of the events that can occur and the outcomes associated with them, they are highly plausible.

THEOREM 7.2. *Suppose $\mathscr{M}$ is a closed, regular, Dedekind complete mixture space with a dual bilinear utility representation. Then the following statements are true:*

(1) *Commutativity holds in $\mathscr{M}$.*

(2) *Complementation holds in $\mathscr{M}$ iff $\mathscr{E}$ is closed under complementation and for every A in $\mathscr{E}$,*

$$S^{+}(A)+S^{-}(\bar{A})=1. \tag{7.7}$$

(3) *The following are equivalent:*

(i) *self-distribution holds in $\mathscr{M}$,*

(ii) *right autodistributivity holds in $\mathscr{M}$,*

(iii) *bisymmetry holds in $\mathscr{M}$,*

(iv) $S^{+}=S^{-}$.

(4) *Outcome independence holds in $\mathscr{M}$.*

(5) *Monotonicity of events holds in $\mathscr{M}$ iff for all A, B, C in $\mathscr{E}$ such that $A\cap C=B\cap C=\varnothing$ and $A\cup C$, $B\cup C$ in $\mathscr{E}$, then for $i=+,-$,*

$$S^{i}(A)\geqslant S^{i}(B) \text{ iff } S^{i}(A\cup C)\geqslant S^{i}(B\cup C). \tag{7.8}$$

(6) *Consistency of preferences with event inclusion holds in $\mathscr{M}$ iff for $i=+,-$,*

$$A\supseteq B \text{ implies } S^{i}(A)\geqslant S^{i}(B). \tag{7.9}$$

Proof. (1) Applying the dual bilinear representation, Eq. (7.6), to both sides of $(x\circ_A y)\circ_B y\sim(x\circ_B y)\circ_A y$ and using the closure of $\mathscr{M}$ yields $\chi(A, B)=\chi(B, A)$.

(2) Select $x\succ y$, and observe that

$$\begin{aligned}
x\circ_A y\sim y\circ_{\bar{A}} x \quad &\text{iff } U(x\circ_A y)=U(y\circ_{\bar{A}} x)\\
&\text{iff } U(x)\,S^{+}(A)+U(y)[1-S^{+}(A)]\\
&\qquad =U(y)\,S^{-}(\bar{A})+U(x)[1-S^{-}(\bar{A})]\\
&\text{iff } [U(x)-U(y)][S^{+}(A)+S^{-}(\bar{A})-1]=0,
\end{aligned}$$

from which the assertion follows since $U(x)-U(y)>0$.

(3) The equivalence of (i), (ii), and (iii) was established in Theorem 6.4. To show that (iv) is equivalent to (iii), it suffices to apply Eq. (7.6) to the bisymmetry condition, much as in (2).

(4) If $x \sim y$, the result is trivial. If $x \succ y$, then applying Eq. (7.6) to $x \circ_A y \sim x \circ_B y$ we see that $S^+(A) = S^+(B)$, and so $u \circ_A v \sim u \circ_B v$ holds for all u, v with $u \succ v$. The case $x \prec y$ is similar.

(5) Observe that from Eq. (7.6),

$$\begin{aligned} U(x \circ_A y) - U(x \circ_B y) &= [U(x) - U(y)]\,[S^+(A) - S^+(B)] && \text{if } x \succsim y, \\ &= [U(x) - U(y)]\,[S^-(A) - S^-(B)] && \text{if } x \prec y, \end{aligned}$$

and

$$\begin{aligned} U(x \circ_{A \cup C} y) - U(x \circ_{B \cup C} y) &= [U(x) - U(y)]\,[S^+(A \cup C) - S^+(B \cup C)] && \text{if } x \succsim y, \\ &= [U(x) - U(y)]\,[S^-(A \cup C) - S^-(B \cup C)] && \text{if } x \prec y. \end{aligned}$$

The equivalence of monotonicity of events and Eq. (7.8) follow immediately from the two equations above.

(6) Equation (7.9) follows directly by applying Eq. (7.6) to the definition of event consistency. Q.E.D.

Before turning to applications of the dual bilinear model, it is perhaps worth mentioning the possibility of a utility model that is either a ratio scale or of type (1, 2). If we suppose in Theorem 7.2 that we are dealing with a general unit representation (now switching from Re to Re^+ and setting $\varphi = \exp U$) of the form

$$\varphi(x \circ_A y) = \varphi(y)\, f_A[\varphi(x)/\varphi(y)], \tag{7.10}$$

then using the same line of argument one can establish conditions on the f_A's equivalent to the various ones stated. In particular, commutativity in $\mathscr{M}$ is equivalent to

$$f_A f_B = f_B f_A, \tag{7.11}$$

and complementation in $\mathscr{M}$ is equivalent to

$$f_{\sim A}(r) = r f_A(1/r), \qquad r > 0. \tag{7.12}$$

Combining Eqs. (7.11) and (7.12) yields the functional equation

$$f_A[r f_A(1/r)] = f_A(r)\, f_A[1/f_A(r)], \qquad r > 0. \tag{7.13}$$

It is not difficult to verify that the log interval version of the dual bilinear utility model satisfies Eq. (7.13), and T. Alper (personal communication) has exhibited some other strictly increasing solutions. It would be interesting to find all of its solutions because commutativity and complementation are extremely weak conditions of rationality that are probably satisfied by most people.

7.4. *The Allais and Ellsberg Paradoxes*

A typical version of the Allais paradox (Allais, 1953; see also Allais and Hagen, 1979) is of the following form. Suppose A, B, C are events, and A and B are statistically independent, and $P(C) = P(A) P(B) > 0$, where P is an objective probability with all of its usual properties. Let x and y be outcomes with $x \succ y$, and let 0 denote the outcome of receiving nothing. One often finds humain beings making choices in which

$$x \succ y \circ_A 0 \quad \text{and} \quad x \circ_B 0 \prec y \circ_C 0.$$

Assuming the bisymmetric model with $S = P$, we see that these statements are equivalent to:

$$U(x) - U(0) > [U(y) - U(0)]\, P(A)$$

and

$$\begin{aligned} [U(x) - U(0)]\, P(B) &< [U(y) - U(0)]\, P(C) \\ &= [U(y) - U(0)]\, P(A)\, P(B). \end{aligned}$$

Dividing the latter by $P(B) > 0$ reveals an inconsistency. The failure of the model is clear. This failure is easily remedied by assuming a subjective model in which independence does not hold. So we turn to the more taxing Ellsberg paradox.

The Ellsberg paradox (Ellsberg, 1961; also see Fishburn, 1983) is simply that for a substantial fraction of people and for some choices of x, y in X and A, B, and C in $\mathscr{E}$, with $A \cap C = B \cap C = \varnothing$ and $A \cup C$ and $B \cup C$ in $\mathscr{E}$, the property of monotonicity of events in $\mathscr{M}$ (Definition 7.2 (7)) is violated. As we have seen in Theorem 7.2, this property is equivalent in the dual bilinear utility model to Eq. (7.8). In the special case of SEU, where $S = S^+ = S^-$ is a probability measure, the monotonicity property is met because

$$\begin{aligned} S(A \cup C) - S(B \cup C) &= S(A) + S(C) - S(B) - S(C) \\ &= S(A) - S(B). \end{aligned}$$

Of course, if we admit weights that are not probability measures, which is consistent with an interval scale representation, the property of monotonicity of events is not forced and so the dual bilinear utility model can accommodate the Ellsberg paradox. It is important to recognize that the event monotonicity violated in Ellsberg's example is completely distinct from the monotonicity assumed for the concatenation operations of the mixture space. This distinction has not always been carefully maintained.

7.5. *Prospect Theory as a Special Case of Dual Bilinearity*

In a widely cited paper, Kahneman and Tversky (1979) have summarized a number of empirical studies for which classical expected utility and, in some cases, SEU have been shown to be inadequate. Further, to encompass these data, they proposed a variant on the rational SEU model that they called "prospect theory." We review briefly the phenomena cited, evaluate them in the light of the dual bilinear utility model, and then show that prospect theory, where it is defned, is a special case of dual bilinearity. Moreover, for some of the phenomena which Kahneman and Tversky interpret as imposing special constraints on the behavior of the utility function above and below the 0 outcome, we will show that they are equally well accounted for by the dual bilinear utility model, which branches in a relative rather than an absolute fashion.

The phenomena discussed by Kahneman and Tversky were grouped according to the following titles:

"Certainty, probability, and possibility." The examples in this category (numbers 1–8 in their paper) were all like the Allais paradox in that they are inconsistent with the strongest version of expected utility, but they are easily accommodated by a subjective version of the theory, including classical SEU.

"Reflection effect." They pointed out that for monetary outcomes x and y, with $x > y$, and appropriate events A and B, a substantial proportion of subjects both select $y \circ_B 0$ over $x \circ_A 0$ and $-x \circ_A 0$ over $-y \circ_B 0$. They remarked that "... the reflection effect implies that risk aversion in the positive domain is accompanied by risk seeking in the negative domain." With the dual bilinear model (Eq. (7.6)) no such implication follows since for $x, y > 0$

$$\frac{U(x \circ_A 0) - U(0)}{U(y \circ_B 0) - U(0)} = \frac{[U(x) - U(0)]\, S^+(A)}{[U(y) - U(0)]\, S^+(B)}$$

and

$$\frac{U(-x \circ_A 0) - U(0)}{U(-y \circ_B 0) - U(0)} = \frac{[U(-x) - U(0)]\, S^-(A)}{[U(-y) - U(0)]\, S^-(B)}.$$

The fact that one ratio is larger than 1 while the other is less than 1 can arise as much from the fact that S^+ is not identical to S^- as from special properties of the utility function. Of course, Kahneman and Tversky were assuming the SEU model in which $S^+ = S^-$, and for that case their conclusion was correct.

"Probabilistic insurance." Without going into the details, they presented an expected utility argument showing that many people's response to a version of probabilistic insurance is inconsistent with the usual assumption that utility is a concave function of money ($U'' < 0$). They argued that this too has special implications for the form of the utility function, namely, that it should be concave on one side of zero and convex on the other. This conclusion does not follow if one uses a subjective model in which $S(A) + S(\tilde{A}) = 1$ is violated.

"Isolation effect." Kahneman and Tversky noted that some gambling choices can be converted to equivalent ones by use of monotonicity arguments. They discussed in detail an example of the following type. Suppose $x > y$. The choice between $x \circ_A 0$ and $y \circ_B 0$ is equivalent, by monotonicity of the operations, to a choice between $x \circ_C 0$ and y provided that $(x \circ_C 0) \circ_B 0 \sim x \circ_A 0$, i.e., $\chi(C, B) = A$. In their example, the events were assumed to have objective probabilities satisfying the definition of independence, i.e., $P(A) = P(B)\,P(C)$. At a theoretical level, using the dual bilinear utility model, the equivalence of the gambles is easily shown to be equivalent to: $S^i(A) = S^i(B)\,S^i(C)$, $i = +, -$, which can be thought of as subjective independence of the events. The fact that subjects are often inconsistent in these choices suggests that objective independence of events does not necessarily force their subjective independence.

We turn now to prospect theory. A prospect is taken to be a monetary gamble of the form

$$z = (x, p; y, q; 0, 1 - p - q),$$

which means that one obtain x with probability p, y with probability q, and nothing with probability $1 - p - q$. We identify this prospect with the mixture

$$z = (x \circ_A y) \circ_B 0,$$

where A and B are independent realizations of events with objective probabilities chosen so that

$$P(A) = p/(p + q) \qquad \text{and} \qquad P(B) = p + q.$$

A prospect is called *regular* iff either $x > 0 > y$ or $x < 0 < y$. The theory has three components. First, the value function for a regular prospect is of the form

$$V(z) = V(x)\,\pi(p) + V(y)\,\pi(q), \tag{7.14}$$

where $V(0) = 0$, $\pi(0) = 0$, and $\pi(1) = 1$. For nonregular prospects with $p + q = 1$ and either $x > y > 0$ or $x < y < 0$,

$$V(z) = V(y) + [V(x) - V(y)]\,\pi(p). \tag{7.15}$$

Third, they impose assumptions on the form of V dictated by the data mentioned above. We do not go into these here since within the framework of the dual bilinear utility model the data do not necessarily imply anything about the form of the utility function.

Consider how Eqs. (7.14) and (7.15) relate to (7.6). For a regular prospect $z = (x \circ_A y) \circ_B 0$, we obtain from Eq. (7.6) for U with $U(0) = 0$,

$$U(z) = U(x)\,\pi_1(A, B) + U(y)\,\pi_2(A, B), \tag{7.16}$$

where

$$\begin{aligned}
\pi_1(A, B) &= S^+(A)\,S^+(B) && \text{if } x \succ y,\ x \circ_A y \succ 0,\\
&= S^+(A)\,S^-(B) && \text{if } x \succ y,\ x \circ_A y \prec 0,\\
&= S^-(A)\,S^+(B) && \text{if } x \prec y,\ x \circ_A y \succ 0,\\
&= S^-(A)\,S^-(B) && \text{if } x \prec y,\ x \circ_A y \prec 0,
\end{aligned} \tag{7.17a}$$

$$\begin{aligned}\pi_2(A, B) &= [1 - S^+(A)]\, S^+(B) && \text{if } x \succ y, x \circ_A y \succ 0\\ &= [1 - S^+(A)]\, S^-(B) && \text{if } x \succ y, x \circ_A y \prec 0\\ &= [1 - S^-(A)]\, S^+(B) && \text{if } x \prec y, x \circ_A y \succ 0\\ &= [1 - S^-(A)]\, S^-(B) && \text{if } x \prec y, x \circ_A y \prec 0.\end{aligned} \tag{7.17b}$$

If $S^+ = S^-$, then π_1 and π_2 are independent of the relations among the outcomes and Eq. (7.16) reduces to exactly the same form as Eq. (7.14).

For nonregular prospects with $p + q = 1$, the event B must be the universal one Ω and if we add the plausible assumption that $x \circ_\Omega y \sim x$, the prospect reduces to $x \circ_C y$, where $C = \chi(A, \Omega)$, and so by Eq. (7.6) we have

$$U(z) = U(y) + [U(x) - U(y)]\, \pi(C), \tag{7.18}$$

where

$$\begin{aligned}\pi(C) &= S^+(C) && \text{if } x > y > 0,\\ &= S^-(C) && \text{if } x < y < 0.\end{aligned} \tag{7.19}$$

Equation (7.18) has the same form as Eq. (7.15) when $S^+ = S^-$.

There are two important differences in the theories. First, the dual bilinear utility theory applies to more gambles than does prospect theory, which is restricted to regular prospects and nonregular ones with $p + q = 1$. Second, the same π-function appears in both Eqs. (7.14) and (7.15) whereas that is not necessarily the case in Eqs. (7.16) and (7.18). Assume $S^+ = S^-$. Suppose the events are such that in their objective probabilities $P(C) = P(A)\, P(B)$. Since $\pi_1(A, B) = S(A)\, S(B)$ and $\pi(C) = S(C)$, we see that the same π-function arises iff $S(C) = S(A)\, S(B)$. This is not a property one would anticipate unless for some constant $\beta > 0$, $S = P^\beta$. Were this to hold, then the dual bilinear utility model would fail to account for the isolation effect (see discussion above).

In summary, the dual bilinear model of Eq. (7.6), which encompasses prospect theory where that theory is explicit, appears to be quite flexible, and we believe that it deserves close scrutiny as a generalization of SEU. Assuming this is correct, it is highly desirable to devise an axiomatization of it comparable to those that have been given for SEU (Krantz *et al.*, 1971, Chap. 8; Savage, 1954). At this time we do not have any.

7.6. *Ratio Scale Utility Representations*

By analogy to what we have just carried out for the interval scale case, we work out the ratio scale possibilities for a regular mixture space. The primary change is to replace the restrictive Eq. (7.6) of the dual bilinear representation by the general ratio expression:

$$U(x \circ_A y) = U(y)\, f_A[U(x)/U(y)]. \tag{7.20}$$

As before, the question is to understand the limitations that are imosed by the several conditions formulated in Definition 7.2. The following result, which says in essence that under plausible smoothness conditions on f_A there are no ratio scale representations beyond those of the interval scale case, is due in major part to M. A. Cohen. We proved a version of parts (1) and (2), and after correcting an error, Cohen solved the functional equation of part (2).

Without difficulty, we can extend the concept of a regular mixture space to situations where $\mathscr{E}$ includes the sure event Ω. Throughout this section we will suppose that this has been done.

THEOREM 7.3. *Suppose that $\mathscr{M}$ is a regular mixture space that can be represented by a family of ratio scales satisfying Eq. (7.20) for all A in $\mathscr{E}$; that there exists Ω in $\mathscr{E}$ such that for all x, y in X, $x \circ_\Omega y \sim x$; that the function f_A satisfies restricted solvability in terms of the events; and that, for all A, B in $\mathscr{E}$ such that f_A and f_B, $B \neq \Omega$, are distinct, the sequence $\{x_n\}$ defined by $x_1 = x$ and $x_{n+1} = f_B^{-1} f_A(x_n)$ is either increasing and unbounded or decreasing with g.l.b. 0.*

(1) *If commutativity and outcome independence hold in $\mathscr{M}$, then there exists a strictly increasing function h from* Re$^+$ *to* Re *and two functions S^+ and S^- from $\mathscr{E}$ onto $[0, 1]$ such that for all x in* Re$^+$ *and A in $\mathscr{E}$,*

(i) $h(1) = 0$ *and* $S^i(\Omega) = 1$, $i = +, -$,

(ii)
$$\begin{aligned} f(x) &= h^{-1}[h(x)\, S^+(A)] && \text{if } x > 1, \\ &= 1 && \text{if } x = 1, \\ &= h^{-1}[h(x)\, S^-(A)] && \text{if } x < 1. \end{aligned} \tag{7.21}$$

(2) *Suppose, in addition, complementation holds in $\mathscr{M}$, and the functions θ^i, $i = +, -$, are defined on $[0, 1]$ by $\theta^i[S^i(\,\tilde{}A)] = S^i(A)$, where A is in $\mathscr{E}$. Then for $x > 1$ and α in $[0, 1]$, h satisfies the functional equation*

$$\begin{aligned} h\{xh^{-1}[h(1/x)\,\alpha]\} &= h(x)\,\theta^+(\alpha) && \text{if } x > 1, \\ &= 1 && \text{if } x = 1, \\ &= h(x)\,\theta^-(\alpha) && \text{if } x < 1. \end{aligned} \tag{7.22}$$

(3) (Cohen). *If, in addition, h satisfies the following conditions:*

(i) *h is continuously differentiable on* Re$^+$ *except possibly at $x = 1$, and*

(ii) *there exist positive constants μ and δ such that the limits* $\lim_{x \downarrow 1} d[h(x)^\mu]/dx$ *and* $\lim_{x \uparrow 1} d[h(x)^\delta]/dx$ *exist and are nonzero,*

then the structure is dual bilinear.

Proof. For x, y in $(1, \infty)$ and A, B in $\mathscr{E}$, define $\succsim'$ by

$$(x, A) \succsim' (y, B) \qquad \text{iff} \quad f_A(x) \geqslant f_B(y).$$

We show that $\langle (1, \infty) \times \mathscr{E}, \succsim' \rangle$ is an additive conjoint structure (Definition 6.7, Krantz *et al.*, 1971). The relation $\succsim'$ satisfies independence because, for each A, f_A is strictly monotonic, and for each $x > 1$, monotonicity holds over $\mathscr{E}$ by outcome independence. To show double cancellation, suppose $(x, B) \succsim' (y, C)$ and $(y, A) \succsim' (z, B)$, i.e., $f_B(x) \geqslant f_C(y)$ and $f_A(y) \geqslant f_B(z)$. Apply f_A to the first inequality and f_C to the second and use the commutativity assumption,

$$f_B f_A(x) = f_A f_B(x) \geqslant f_A f_C(y) = f_C f_A(y) \geqslant f_C f_B(z) = f_B f_C(z).$$

So, by the monotonicity of f_B, $(x, A) \succsim' (z, C)$. Restricted solvability holds because f_A is onto (Theorem 5.1) and by the assumption that it satisfies restricted solvability for events. Each component is essential by the strict monotonicity of f and by the existence of events other than Ω. A nontrivial standard sequence $\{x_n\}$ corresponds to some $A, B, B \neq \Omega$, in $\mathscr{E}$ such that f_A and f_B are distinct and $(x_n, A) \sim' (x_{n+1}, B)$, which in turn corresponds to $x_{n+1} = f_B^{-1} f_A(x_n)$. Thus, by assumption, the Archimedean property holds. So by the representation theorem for such structures (Theorem 6.2 of Krantz *et al.*, 1971), there exist functions h, S^+, and k such that

$$k[f_A(x)] = h(x)\, S^+(A).$$

Since for all x, y in X, $x \circ_\Omega y \sim x$, we have

$$U(y)\, f_\Omega[U(x)/U(y)] = U(x),$$

whence

$$k(z) = k[f_\Omega(z)] = h(z)\, S^+(\Omega).$$

If we choose S^+ so that $S^+(\Omega) = 1$, which by the uniqueness of additive conjoint structures is possible, then $k \equiv h$. Setting $z = 1$ and noting that, for each A in $\mathscr{E}$, $f_A(1) = 1$, we see that,

$$h(1) = h[f_A(1)] = h(1)\, S^+(A),$$

and so $h(1) = 0$.

In a completely analogue fashion, we define an additive conjoint structure for x, y in $(0, 1)$, which permits us to extend h onto that interval and introduces S^- on $\mathscr{E}$. Thus, Eq. (7.21) holds.

Note that because f_A is onto and solvability holds for the events, the functions S^i map onto a continuum, which by choice of Ω is $[0, 1]$.

(2) Define θ^+ as in the statement of the theorem. Observe, for $U(x)$, $U(y) > 1$,

$$x \circ_A y \sim y \circ_{\bar{A}} x \quad \text{iff} \quad U(y)\, f_A[U(x)/U(y)] = U(x)\, f_{\bar{A}}[U(y)/U(x)]$$

$$\text{iff} \quad \text{for all } z > 0,$$

$$h^{-1}[h(z)\, S^+(A)] = z h^{-1}[h(1/z)\, S^+(\bar{A})].$$

Setting $S^+(\tilde{\ }A) = \alpha$, we see that h must satisfy the upper part of Eq. (7.22). The proof is similar for the bottom part.

(3) Define the function H on Re^+ by

$$\begin{aligned} H(x) &= h^\mu(x) && \text{if} \quad x > 1, \\ &= 0 && \text{if} \quad x = 1, \\ &= h^\delta(x) && \text{if} \quad x < 1. \end{aligned}$$

Observe that by part (2), H must satisfy the functional equation:

$$\begin{aligned} H\{xH^{-1}[H(1/x)\,\alpha^\mu]\}/H(x) &= \theta^+(\alpha)^\delta && \text{if} \quad x > 1, \\ H\{xH^{-1}[H(1/x)\,\alpha^\delta]\}/H(x) &= \theta^-(\alpha)^\mu && \text{if} \quad x < 1. \end{aligned} \tag{7.23}$$

Note that if H satisfies Eq. (7.23) and $r, s > 0$, then

$$\begin{aligned} H_{r,s}(x) &= rH(x) && \text{if} \quad x > 1, \\ &= 0 && \text{if} \quad x = 1, \\ &= sH(x) && \text{if} \quad x < 1, \end{aligned}$$

also satisfies Eq. (7.23). Therefore, by an appropriate choice of r and s, there is no loss of generality in assuming that H has equal right and left derivatives at $x = 1$ and that they equal 1. Hence, H is continuously differentiable everywhere.

In Eq. (7.23), take the limits as $x \to 1$, which we may do by using l'Hospital's rule, and it yields

$$\begin{aligned} \theta^+(\alpha)^\delta &= 1 - \alpha^\mu, \\ \theta^-(\alpha)^\mu &= 1 - \alpha^\delta. \end{aligned}$$

Substituting this into Eq. (7.23) yields

$$\begin{aligned} H(x) &= H\{xH^{-1}[H(1/x)\,\alpha^\mu]\}/(1 - \alpha^\mu) && \text{if} \quad x > 1, \\ &= 0 && \text{if} \quad x = 1, \\ &= H\{xH^{-1}[H(1/x)\,\alpha^\delta]\}/(1 - \alpha^\delta) && \text{if} \quad x < 1. \end{aligned}$$

Now, for $x = 1$, take the limit as $\alpha \to 1$, again using l'Hospital's rule, yielding

$$H(x) = -xH(1/x)/H'(1/x), \qquad x \neq 1.$$

Set $y = 1/x$, solve for H'/H,

$$H'(y)/H(y) = -1/yH(1/y), \qquad y \neq 1. \tag{7.24}$$

Since H is continuously differentiable, we may take the derivative of Eq. (7.24) to obtain

$$\left[\frac{H''(y)}{H'(y)}-\frac{H'(y)}{H(y)}\right]\frac{H'(y)}{H(y)}=\frac{H(1/y)+yH'(1/y)(-1/y^2)}{y^2H(1/y)^2}$$

$$=\frac{1}{yH(1/y)}\left[\frac{1}{y}-\frac{H'(1/y)}{y^2H(1/y)}\right]$$

$$=\frac{-H'(y)}{yH(y)}\left[1+1/H(y)\right].$$

Dividing out H'/H and setting $H(y)=G(\log y)$, we see that G satisfies the differential equation

$$\frac{G''}{G'}-\frac{G'}{G}=-\frac{1}{G}.$$

It is easy to verify that this is equivalent to

$$\frac{d}{dy}\log[G'(y)-1]-\frac{d}{dy}\log G(y)=0,$$

i.e.,

$$\frac{G'(y)-1}{G(y)}=\beta,$$

whence

$$G(y)=\sigma e^{\beta y}-1/\beta.$$

Since $H=G\log$, H is continuous at 1, and $H(1)=0$, we see that $\sigma=1/\beta$, whence

$$\begin{aligned} h(y)&=[(y^\beta-1)/\beta]^{1/\delta} && \text{if } y>1,\\ &=0 && \text{if } y=1,\\ &=[(y^\beta-1)/\beta]^{1/\mu} && \text{if } y<1. \end{aligned}$$

Substituting into the unit expression for $\circ_A$,

$$x\circ_A y = yf_A(x/y)$$

$$=\begin{cases} y(h^{-1}[h(x/y)\,S^+(A)] & \text{if } x>y,\\ y & \text{if } x=y,\\ y(h^{-1}[h(x/y)\,S^-(A)]) & \text{if } x<y,\end{cases}$$

$$=\begin{cases} \{x^\beta S^+(A)^\mu + y^\beta[1-S^+(A)^\mu]\}^{1/\beta} & \text{if } x>y,\\ y & \text{if } x=y,\\ \{x^\beta S^-(A)^\delta + y^\beta[1-S^-(A)^\delta]\}^{1/\beta} & \text{if } x<y.\end{cases}$$

Thus, in the transformed variable x^β, we see that the structure has a dual bilinear representation and so is, necessarily, (2, 2). Q.E.D.

The upshot of this is quite clear. Suppose that a regular mixture space is an adequate model of gambling; that the decision maker satisfies the following four conditions: restricted solvability for events, outcome independence, commutativity, and complementation, each of which seems plausible; and that there is a ratio scaling of utility that produces a smooth representation. Then the only possible representation is the interval scale, dual bilinear one. Clearly, this representation warrants further study.

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