JOURNAL OF MATHEMATICAL PSYCHOLOGY 27, 44–85 (1983)

Symmetry, Scale Types, and Generalizations of Classical Physical Measurement

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A review is presented of a number of recent results concerning fundamental measurement structures with a particular emphasis on generalizations of physical measurement. Relational structures are classified in terms of richness and redundancy of their automorphism groups (i.e., in terms of their symmetries). By means of this classification, the possible types of measurement scales are described, and the possibilities for structures with concatenation operations (either associative or nonassociative and either positive or intensive) and for conjoint structures whose automorphisms factor into component automorphisms are explicitly spelled out. Certain conjoint structures with concatenation operations on components are viewed as generalizations of classical physical measurement and some of their most important algebraic and representational properties are explored.

0. INTRODUCTION

The purpose of this paper is to summarize in one place the major results that the authors and M. Cohen have uncovered about the possibilities of generalizing the classical measurement structures of physics. The research is not fully complete in ways that will be indicated, but there is sufficient understanding about the options available to other sciences that it seems useful to us to bring the results together unencumbered by all of the technical detail found in the several research papers.

We begin with a brief summary of what we believe to be the salient features of classical physical measurement, those features that among other things led to the structures of units with which we are all familiar from elementary physics. It is divided into five remarks.

First, the measurement of some one-dimensional attributes, such as length and

This paper is based upon an invited address given by Luce at the 1982 annual meeting of the Society for Mathematical Psychology and upon a talk given by Narens at the 1982 annual meeting of the European Mathematical Psychology Society. The work has been supported by NSF Grant ITS–7924019 to the University of California at Irvine. Address reprint requests to Dr. Louis Narens, Program of Cognitive Science, University of California, Irvine, California 92717.

0022-2496/83 $3.00

mass, rests upon having both a qualitative ordering, denoted $\succsim$, of one object having "more of the attribute" than another, and a qualitative combining operation, denoted $\circ$. The operation is such that if a and b are two objects exhibiting this attribute, then $a \circ b$ is an object in the system that also exhibits the attribute. Classically, these structures are such that they can be represented numerically by a mapping *onto* the positive reals, Re^+, in which $\succsim$ is replaced by $\geqslant$ and $\circ$ by $+$. Moreover, any two such representations onto $\langle \mathrm{Re}^+, \geqslant, + \rangle$ differ only by a positive factor. Representations that differ in this way are said to form a *ratio scale.* The theories of qualitative structures $\langle A, \succsim, \circ \rangle$ having such representations onto $\langle \mathrm{Re}^+, \geqslant, + \rangle$ are generally called *extensive measurement.*

Second, the measurement of some other one-dimensional attributes rests upon factoring the underlying objects into two or more components that each affect the attribute in question. Such qualitative structures for two component attributes are of the form $\langle A \times P, \succsim \rangle$, where A and P are nonempty sets and $\succsim$ is an ordering of $A \times P$. Examples of this are momentum with factors of mass and velocity and kinetic energy with factors of mass and the square of velocity. The representations found in classical physics are mappings ψ_A and ψ_P from A and P, respectively, *onto* Re^+ such that $\psi_A \psi_P$ and $\geqslant$ represents $\succsim$. In other words, the Cartesian product is represented by a numerical product. Any two representations of this type differ by a positive factor and a positive power, i.e., $x \to \alpha x^\gamma$, where $\alpha > 0$ and $\gamma > 0$. These are called *log-interval scales.* An equally satisfactory representation, but not the one conventionally employed, is obtained by taking the logarithm, $\log \psi_A + \log \psi_P$, which is order-preserving and unique up to a positive linear transformation. The set of such representations is called an *interval scale.* Theories of qualitative structures that admit such "additive" representations are collectively known as *additive conjoint measurement.* Note that while the theory concerns the measurement of a single new dimension, $A \times P$, it does so in terms of the components, A and P, so in a sense such "conjoint" measurements are the beginning of something multidimensional.

Third, all measurements of physical attributes can be achieved in one of these two ways, and some can be measured both ways. For example, mass is extensive and also conjoint since it can be affected by two factors: the substance employed and the volume of it used. When two such representations exist, it turns out that they differ by a positive power relation. That is, if φ is the extensive measure of the attribute and $\psi = \psi_A \psi_P$ is the conjoint one (using the multiplicative representation), then for some positive constants β and ρ, $\psi = \beta \varphi^\rho$. The extensive operation may exist on a factor, say A, in which case the extensive measure and the conjoint one on the factor ψ_A are related by a power transformation. This fact implies a distribution property that can be given a qualitative interpretation. Let w, x, y, z be values of ψ_A and u, v those of ψ_P. Suppose they are such that

$$w^\rho u^r = y^\rho v^r \qquad \text{and} \qquad x^\rho u^r = z^\rho v^r.$$

Taking the $1/\rho$ root, adding, and then raising to the ρ power yields the consequence

$$(w + x)^\rho u^r = (y + z)^\rho v^r.$$

This distribution property recast in qualitative structures will play an important role in our development. Note that any ratio scale transformation of the extensive structure appears as that factor raised to the power of the conjoint structure

$$\beta(\alpha\varphi_A)^\rho \psi_P = \alpha^\rho \beta \varphi_A^\rho \psi_P = \alpha^\rho \psi_A \psi_P.$$

Fourth, physics has uncovered a large number of triples of attributes of the sort just mentioned. They are of the form $\langle A \times P, \gtrsim, \circ, \circ_A, \circ_P \rangle$ where at least one of the operations $\circ$, $\circ_A$, and $\circ_P$ on $A \times P$, A, and P, respectively, is assumed to exist. Taken together these triples form a tightly interlocked network of scales that can, in fact, be imbedded in a finite-dimensional, Euclidean, multiplicative vector space. Moreover, a set of extensive scales $\psi_1, \psi_2, \ldots, \psi_N$ can be chosen as the basis of that space, and all other scales are then expressed as products of powers of these base scales.

$$\psi = \alpha \psi_1^{\rho_1} \psi_2^{\rho_2} \cdots \psi_N^{\rho_N}.$$

This fact is reflected in the units of physical scales which can be expressed as products of powers of the basis units, usually chosen to be grams, meters, seconds, etc.

Fifth, physical laws that are formulated in terms of these measures are assumed to exhibit a principle called dimensional invariance, which states a relation among measures is a possible law if it is invariant under any change of units of all the measures generated by changes in the units of the basis scales. This is invoked because these choices of units only reflect arbitrary conventions in the numerical representation of the qualitative information. This postulate was shown by Buckingham (1914) to lead to a very useful representation of all possible physical laws in terms of dimensionless products of powers of the relevant variables. His result is the mathematical key to the powerful method of dimensional analysis.

The topic of this paper is the possibility of useful generalizations of these kinds of structures. This is of interest primarily in nonphysical sciences where the structures of attributes do not closely mimic those of physics. The issue of generalization is, as always, one of relaxing some aspects while retaining others—relaxing those that do not seem essential while retaining those we believe to be essential. In what follows we shall retain four major features.

The first is that we continue to work primarily with ratio and interval scales. These reflect aspects of symmetry in the structures employed, and we shall come to understand very completely why these two types of scales are so important.

The second is that we focus on concepts that exhibit invariance under ratio and interval scale transformations. This is not only natural to do because of the principle of dimensional invariance, but it arises from much more general considerations about the measurement-theoretic concept of meaningfulness that infuse our discussions throughout this paper. (A technical development of results that we have about meaningfulness will not, however, be given in this paper.)

These two features are, in our view, essential to retain, and perhaps they are the only ones that should be. In reality, however, our work has retained two others.

The third feature concerns the qualitative distributive interlock of an operation on one factor of a conjoint structure with that structure. That there are important nondistributive structures is not to be questioned since in relativistic physics the operation of combining ("adding") velocities is not distributive in the additive conjoint structure: distance = velocity × time. However, at this time we do not know how to handle such structures measurement-theoretically, although we are working on the problem.

The fourth feature assumes that the conjoint structures are "solvable" in the sense that given any three of a, b in A, p, q, in P we may find the fourth that fulfils the equivalence $ap \sim bq$ (where we have abbreviated the more usual notation $(a, p) \in A \times P$ by ap). A number of results hold for much weakened versions of solvability, but many of the most important ones assume it and are not valid without it. For a time we believed that weakening the solvabilty assumption was largely a technical matter and that the results would differ little from the unrestrictedly solvable case. We now know this to be false, and in some of our future research we hope to gain a much deeper understanding of the possibilities that arise with weaker forms of solvability.

The paper proceeds as follows: In Section 1 we set up the general mathematical framework of the representation homomorphisms of relational systems by numerical relational systems. In Section 2 we provide a classification of the uniqueness of the representations into different groups of transformations which, when the representations are onto the real numbers, corresponds to aspects of the automorphism groups of the qualitative structures. In Section 3 these theorems are applied to representations of general concatenation structures in order to classify the interesting possibilities. The concept of extensive structures is much generalized in Section 4, and we characterize these monotonic operations that have a ratio scale representation onto Re^+. In Section 5, we present incomplete results about ordered measurement structures with an intensive operation, i.e., an operation $\circ$ such that for all $x \succ y$, $x \succ x \circ y$, $y \circ x \succ y$. Section 6 presents generalizations of conjoint structures, and their relation to the structures of Section 4 is described. In Section 7 we state more precisely what dimensional invariance is. Section 8 discusses the concept of distributivity as a way of interlocking the operations with the conjoint structure, and we see how much it tends to force us toward additive conjoint structures. Because the automorphisms of conjoint structures that play a role in the discussion of physics are all factorizable into separate mappings on the components, we examine more fully and abstractly in Section 9 those conjoint structures that are well endowed with factorizable automorphisms. The final section provides a summary and an indication of what needs to be done next.

1. Representations of Relational Structures

A central concept of measurement is that of a representation of a relational structure. The relational structure is to be thought of as an idealization of an

empirical setting. It consists of a nonempty set of (empirical) objects together with a set of (empirical) relations on those objects. A representation (or scale) for this structure is then an assignment of numbers to the objects of the structure that is consistent with the relations of the structure. The next two definitions precisely capture these concepts.

DEFINITION 1.1. $\mathscr{A} = \langle A, R_0, R_1, ..., R_i, ... \rangle$ is said to be a *relational structure* if and only if A is a nonempty set and $R_0, R_1, ..., R_i, ...$, are finite relations on A. A is called the *domain of discourse* of $\mathscr{A}$ and $R_0, R_1, ..., R_i, ...$, the *relations* of $\mathscr{A}$. Elements of A will often be referred to as "elements of $\mathscr{A}$." $\mathscr{A}$ is said to be a *weakly ordered* (respectively, *totally ordered*) *relational structure* if and only if R_0 is a weak (respectively, a total) ordering.

In Definition 1.1 some or all the relations R_i may be 0-ary relations, that is, may be elements of A. However, each relation R_i must be an n-ary relation for some nonnegative integer n. Some or all of the R_i may be operations since each n-ary operation on A may be considered as a $(n+1)$-ary relation.

DEFINITION 1.2. Let $\mathscr{A} = \langle A, R_0, R_1, ..., R_i, ... \rangle$ and let $\mathscr{N} = \langle N, S_0, S_1, ..., S_i, ... \rangle$ be a numerical relational structure (that is, a relational structure such that $N \subseteq \mathrm{Re}$). φ is said to be a *homomorphism from* $\mathscr{A}$ *into* $\mathscr{N}$ if and only if φ is a function from A into N, for each i and n, R_i is an n-ary relation iff S_i is an n-ary relation, and for each i, if R_i is an n-ary relation and $a_1, ..., a_n$ is in A, then

$$R_i(a_1, ..., a_n) \quad \text{iff} \quad S_i(\varphi(a_1), ..., \varphi(a_n)).$$

φ is said to be a *$\mathscr{N}$-representation* for $\mathscr{A}$ if and only if φ is a homomorphism of $\mathscr{A}$ into $\mathscr{N}$. For the case of weakly ordered or totally ordered $\mathscr{A}$, we will assume by convention that the relation S_0 of the numerical structure is the natural numerical ordering relation $\geqslant$.

Once an appropriate numerical structure $\mathscr{N}$ is chosen, the $\mathscr{N}$-representations for a given empirical situation $\mathscr{A}$ will be considered as the permissible assignments of numbers to the objects of $\mathscr{A}$. This characterization of "permissible assignment" is pervasive throughout the entire measurement literature.

The existence of a numerical representation for a weakly ordered empirical structure imposes certain constraints on that structure. These are captured in

THEOREM 1.1. *Suppose* $\mathscr{A} = \langle A, \succsim, R_1, R_2, ... \rangle$ *is a weakly ordered relational structure. Then the following two statements are equivalent:*

(1) $\mathscr{A}$ *has a* $\mathscr{N}$*-representation for some numerical structure* $\mathscr{N}$*;*

(2) $\langle A, \succsim \rangle$ *has a countable dense subset, i.e., there exists a finite or denumerable subset* B *of* A *such that for each* u, v *in* A, *if* $u \succ v$, *then there exists* y *in* B *such that* $u \succsim y \succsim v$.

The proof of Theorem 1.1 is essentially given in Cantor (1895). A more accessible proof is given in Section 2.1 of Krantz *et al.* (1971).

Frequently we will want to consider cases where the representations of a totally ordered empirical structure are onto a numerical structure with domain of discourse an open interval of reals. The following definition and theorem characterizes such situations:

DEFINITION 1.3. $\langle A, \gtrsim \rangle$ is said to be of *order type* θ if and only if the following four conditions hold:

(i) $\gtrsim$ is a total ordering on the nonempty set A;

(ii) $\langle A, \gtrsim \rangle$ has no maximal or minimal element;

(iii) there exists a denumerable subset B of A such that for all u, v in A, if $u \succ v$, then there exists y in B such that $u \succ y \succ v$;

(iv) $\langle A, \gtrsim \rangle$ is Dedekind complete: each bounded nonempty subset of A has a least upper bound.

A total ordered relational structure $\langle A, \gtrsim, R_1, R_1, \ldots \rangle$ is said to be of *order type* θ if and only if $\langle A, \gtrsim \rangle$ is of order type θ.

(The phrase "order type" is used in set theory to refer to a class of structures isomorphic to a given ordered structure. Those isomorphic to the reals were called type θ by Cantor and others and called type λ by other set theorists.)

THEOREM 1.2. *Suppose* $\mathscr{A}$ *is a totally ordered relational structure. Then the following two statements are equivalent:*

(1) $\mathscr{A}$ *has a representation onto a numerical structure with domain of discourse an open interval of reals.*

(2) $\mathscr{A}$ *is of order type* θ.

Theorem 1.2 was originally proven in Cantor (1895). A proof can also be found in Narens (1983).

2. RATIO, INTERVAL, AND ORDINAL SCALABILITY

DEFINITION 2.1. Suppose $\mathscr{A}$ is a weakly ordered relational structure. $\mathscr{A}$ is said to be *ratio scalable* if and only if there exists a $\mathscr{N}$-representation for $\mathscr{A}$ and the following two conditions hold for all $\mathscr{N}$-representations φ, ψ of $\mathscr{A}$:

(1) for each $r \in \mathrm{Re}^+$, $r\varphi$ is a $\mathscr{N}$-representation;

(2) there exists $r \in \mathrm{Re}^+$ such that $\varphi = r\psi$.

$\mathscr{A}$ is said to be *positively ratio scalable* if and only if conditions (1) and (2) hold and the domain of discourse of $\mathscr{N}$ is a subset of Re^+.

$\mathscr{A}$ is said to be *interval scalable* if and only if there exists a $\mathscr{N}$-representation for $\mathscr{A}$ and the following two conditions for all $\mathscr{N}$-representations φ, ψ of $\mathscr{A}$:

(1′) for each $r \in \mathrm{Re}^+$, $s \in \mathrm{Re}$, $r\varphi + s$ is a $\mathscr{N}$-representation;

(2′) there exist $r \in \mathrm{Re}^+$, $s \in \mathrm{Re}$ such that $\varphi = r\psi + s$.

$\mathscr{A}$ is said to be *ordinal scalable* if and only if there exists a $\mathscr{N}$-representation for $\mathscr{A}$ and the following two conditions hold for all $\mathscr{N}$-representations φ, ψ of $\mathscr{A}$:

(1″) for each strictly increasing f from Re onto Re, $f(\varphi)$ is a $\mathscr{N}$-representation of $\mathscr{A}$;

(2″) there exists a strictly increasing f from Re onto Re such that $\varphi = f(\psi)$.

Note that in the definition of ordinal scalability given, condition (2″) requires that the function f be *onto* Re.

DEFINITION 2.2. Suppose $\mathscr{A} = \langle A, R_0, R_1, \ldots, R_i, \ldots \rangle$ is a relational structure. An *endomorphism* of $\mathscr{A}$ is an homomorphism of $\mathscr{A}$ into itself. Formally, α is an *endomorphism* of $\mathscr{A}$ if and only if α is a function from A into A such that for all relations R_i of $\mathscr{A}$ and all $a_1, \ldots, a_{n_i}$ in A,

$$R_i(a_1, \ldots, a_{n_i}) \quad \text{iff} \quad R_i[\alpha(a_1), \ldots, \alpha(a_{n_i})].$$

An endomorphism of $\mathscr{A}$ that is one-to-one and onto A is called an *automorphism* of $\mathscr{A}$.

Automorphisms are, in fact, isomorphisms of the structure with itself. In other words, for each automorphism the structure exhibits a symmetry that allows itself to be reflected upon itself by that automorphism. The study of symmetry in a structure and the study of its automorphisms are exactly the same topic.

If all $\mathscr{N}$-representations of a totally ordered relational structure are *onto* $\mathscr{N}$, then there is, as the reader can readily verify, a simple relationship between the automorphisms of $\mathscr{A}$ and the $\mathscr{N}$-representations of $\mathscr{A}$, namely, *for all $\mathscr{N}$-representations φ and ψ of $\mathscr{A}$, $\varphi^{-1}\psi$ is an automorphism of $\mathscr{A}$*, and *for all automorphisms α of $\mathscr{A}$ and all $\mathscr{N}$-representations φ of $\mathscr{A}$, $\varphi\alpha$ is a $\mathscr{N}$-representation of $\mathscr{A}$*, i.e., there is a natural one-to-one correspondence between the $\mathscr{N}$-representations and automorphisms of $\mathscr{A}$. For the cases of ratio, interval, and ordinal scalability, this relationship can be exploited for $\mathscr{A}$ of order type θ, as is shown in the next two theorems.

THEOREM 2.1. *Suppose $\mathscr{A}$ is of order type θ and is either positively ratio, interval, or ordinal scalable, and suppose $\mathscr{A}$ has at least one $\mathscr{N}$-representation that is onto $\mathscr{N}$. Then all $\mathscr{N}$-representations of $\mathscr{A}$ are onto $\mathscr{N}$.*

DEFINITION 2.3. Let M, N be nonnegative integers and $\mathscr{A}$ be a totally ordered relational structure. $\mathscr{A}$ is said to satisfy *M-point homogeneity* if and only if for all

$x_1,\ldots,x_M$, $y_1,\ldots,y_M$ in $\mathscr{A}$, if $x_1 \succ x_2 \succ \cdots \succ x_M$ and $y_1 \succ y_2 \succ \cdots \succ y_M$, then there exists an automorphism α of $\mathscr{A}$ such that $\alpha(x_i) = y_i$. $\mathscr{A}$ is said to satisfy *N-point uniqueness* if and only if for all automorphisms β and γ of $\mathscr{A}$, if β and γ agree at N distinct points of $\mathscr{A}$, then $\beta = \gamma$.

Note: If $\mathscr{A}$ is M-point homogeneous and N-point unique, then $M \leqslant N$.

Suppose $\mathscr{A}$ is of order type θ. It is not difficult to show that if $\mathscr{A}$ is positively ratio scalable, then it satisfies 1-point homogeneity and 1-point uniqueness; and if it is interval scalable, then it satisfies 2-point homogeneity and 2-point uniqueness. It is a rather interesting and important fact that the converses of these statements are also true:

THEOREM 2.2. *Suppose $\mathscr{A}$ is of order type θ. Then the following two statements are true:*

(i) *$\mathscr{A}$ is positively ratio scalable if and only if it satisfies 1-point homogeneity and 1-point uniqueness;*

(ii) *$\mathscr{A}$ is interval scalable if and only if it satisfies 2-points homogeneity and 2-point uniqueness.*

Proof. Theorems 1.2 and 1.3 of Narens (1981b). (The proof of part (i) appears in Narens, 1981a.)

Ordinal scalability requires a form of infinite point homogeneity and uniqueness. A characterization of it using such concepts is given in Theorem 1.5 of Narens (1981b).

Ratio, interval, and ordinal scalability are widely used throughout science. The following somewhat surprising theorem perhaps indicates why other forms of scale types may have not arisen:

THEOREM 2.3. *Suppose $\mathscr{A}$ is of order type θ and satisfies N-point homogeneity and N-point uniqueness. Then $N \leqslant 2$.*

Proof. Theorem 1.4 of Narens (1981b).

3. General and Real M-Point Homogeneous Concatenation Structures

This section concerns the specialization of the above concepts to the class of structures in which there is only an ordering and a binary operation. We classify these structures in terms of homogeneity and uniqueness properties and specify some of the possible numerical representations, namely, those onto either Re^+ or Re. The results of this section are taken from Luce and Narens (in preparation).

DEFINITION 3.1. For $A \neq \varnothing$, let $\gtrsim$ be a binary relation on A, and $\circ$ a partial binary operation on A. Then $\mathscr{A} = \langle A, \gtrsim, \circ \rangle$ is a *concatenation structure* if and only if the following four conditions hold for all w, x, y, z in A:

(1) $\gtrsim$ is a total ordering.

(2) There exist u, v in A such that $u \succ v$.

(3) If $x \circ y$ is defined, $x \gtrsim w$, and $y \gtrsim z$, then $w \circ z$ is defined.

(4) (i) If $x \circ z$ and $y \circ z$ are defined, then $x \gtrsim y$ iff $x \circ z \gtrsim y \circ z$,

(ii) If $z \circ x$ and $z \circ y$ are defined, then $x \gtrsim y$ iff $z \circ x \gtrsim z \circ y$.

It is *Archimedian* if and only if there exists n in the set of positive integers, I^+, such that either nx is not defined or $nx \succ y$. (As is usual, $1x = x$ and for $n \in I^+$, nx is defined inductively by $(n+1)x = (nx) \circ x$ when the latter is defined.) The partial operation $\circ$ (and $\mathscr{A}$) is said to be *closed* if and only if $x \circ y$ is defined for all x, y in A. It is *idempotent, weakly positive, or weakly negative* if and only if for all x in A either $x \circ x \sim x$, $x \circ x \succ x$, or $x \circ x \prec x$, respectively.

THEOREM 3.1. *Suppose $\mathscr{A} = \langle A, \gtrsim, \circ \rangle$ is a closed concatenation structure.*

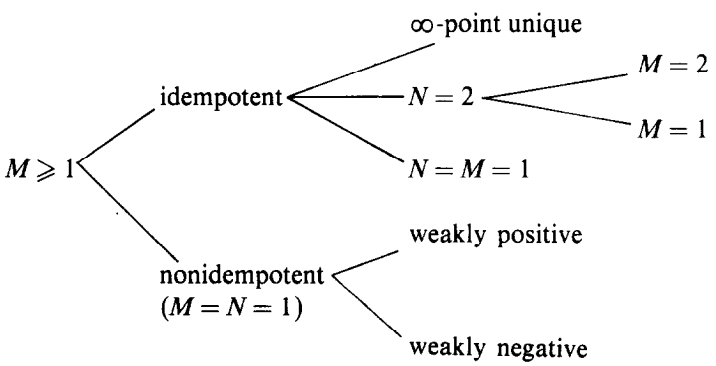
(i) *If $\circ$ is idempotent, then it is intern in the sense that for all x, y in A if $x \succ y$, then $x \succ x \circ y$, $y \circ x \succ y$.*

(ii) *If $\mathscr{A}$ is M-point homogeneous, $M \geqslant 1$, then $\circ$ is either idempotent, weakly positive, or weakly negative.*

(iii) *If $\mathscr{A}$ is 1-point homogeneous and N-point unique, then either $\circ$ is idempotent or $N = 1$.*

(iv) *If $\mathscr{A}$ is N-point unique, $N \geqslant 2$, and $\circ$ is idempotent, then $N = 2$.*

The net effect of this theorem can be summarized in a simple tree diagram that applies to all M-point homogeneous, closed concatenation structures.



Thus, the only cases we need focus on are $(M, N) = (1, 1)$, $(2, 2)$, $(1, 2)$, and (M, ∞). Our next result is a plausible means for ruling out (M, ∞) as a case of interest.

DEFINITION 3.2. Let $\mathscr{A} = \langle A, \gtrsim, \circ \rangle$ be a concatenation structure. For any x, y, in A, a *mix* of x and y is defined inductively as follows: x, y, $x \circ y$, and $y \circ x$ are mixes, and if u and v in A are mixes of x and y, then $u \circ v$ and $v \circ u$ are also mixes.

THEOREM 3.2. *Suppose* $\mathscr{A} = \langle A, \gtrsim, \circ \rangle$ *is an idempotent concatenation structure with the following property: for all* x, y, u, v *in* A, *if* $x \gtrsim u \gtrsim v \gtrsim y$, *then there exists a mix* z *of* x *and* y *such that* $u \succ z \succ v$. *Then* $\mathscr{A}$ *is 2-point unique.*

Let us turn now to those (M, N) concatenation structures that are of order type θ. By Theorem 2.2, we know that the $(1, 1)$ case is ratio scalable and $(2, 2)$ is interval scalable. At present we do not understand the $(1, 2)$ case except that it is nonempty. Theorem 3.3(iii) shows how to construct concatenation structures with groups of transformations of the form $x \to k^n x + s$, where x, s are in Re, $k > 0$ is fixed, and n is an integer (positive, negative, zero). We do not know if all $(1, 2)$ transformation groups are subgroups of the affine group $(2, 2)$ or if something radically different can arise.

Knowledge of these transformation groups is important since it can be used to characterize fully all of the possible real representations. The technique is to write $\circ$ as a function $F : \mathrm{Re} \times \mathrm{Re} \to \mathrm{Re}$ defined by

$$x \circ y = z \qquad \text{iff} \quad F(x, y) = z$$

and then to note that for all automorphisms α, F must satisfy the real functional equation

$$\alpha F(x, y) = F[\alpha(x), \alpha(y)].$$

The following theorem yields *all* possible solutions to this equation for the above three groups.

THEOREM 3.3. *Suppose* $\mathscr{R} = \langle R, \geqslant, \circ \rangle$ *is a real concatenation structure with* $R = \mathrm{Re}$ *or* Re^+ *and let* $\mathscr{G}$ *be its group of automorphisms.*

(i) *If* $\mathscr{G}$ *is the similarity group (multiplication by positive reals), then there exists some* $f : \mathrm{Re}^+ \to \mathrm{Re}^+$ *for which* f *and* f/ι, *where* ι *is the identity map, are strictly monotonic increasing and decreasing, respectively, and for* $x, y \in \mathrm{Re}^+$,

$$x \circ y = yf(x/y).$$

$\circ$ *is idempotent, weakly positive, or weakly negative according as* $f(1) =, >,$ *or* < 1, *respectively.*

(ii) *If* $\mathscr{G}$ *is the affine group (positive linear transformations), then there are constants* a, b *with either* $a, b < 0$, $0 < a, b < 1$, *or* $1 < a, b$ *such that for all* $x, y \in \mathrm{Re}$,

$$x \circ y = ax + (1 - a)y, \qquad x \geqslant y,$$
$$= bx + (1 - b)y, \qquad x < y.$$

(iii) *If for* $k > 0$, $\mathscr{G} = \{k^n x + s \mid x, s \in \mathrm{Re}, n \in I\}$, *then there exist* $g, h : \mathrm{Re} \to \mathrm{Re}$ *that are strictly monotonic increasing and decreasing, respectively, such that for all* $x, y \in \mathrm{Re}$

$$g = h + \iota, \qquad g(kx) = kg(x), \qquad g(0) = h(0) = 0,$$

and

$$x \circ y = g(x - y) + y = h(x - y) + x.$$

This result means that we understand fully all possible representations of concatenation structures (1, 1) and (2, 2) that are onto a real interval and that the only remaining case of interest is the (1, 2) one, whose real transformation groups we have not yet characterized. The next section concerns axiomatizations of qualitative structures that yield (1, 1) and some (0, 1) cases.

4. Positive Concatenation Structures

Specific relational structures that are ratio scalable have been in wide use for at least a century. Helmholtz (1887) and Hölder (1901) used such structures as the theoretical foundations for the measurement of physical attributes. These were later refined by other measurement theorists and became—until recently— the basis of most of measurement theory. These sorts of structures became known in the literature as "extensive structures." They were generalized by Narens and Luce (1976) and Cohen and Narens (1979). These generalizations—called "positive concatenation structures"—are interesting in their own right and play a central role in conjoint measurement.

4.1. *Definition and Uniqueness*

Definition 4.1. For $A \neq \emptyset$, let $\succsim$ be a binary relation on A and let $\circ$ be a partial binary operation on A. Then $\mathscr{A} = \langle A, \succsim, \circ \rangle$ is said to be *a positive concatenation structure* (PCS) if and only if it is an Archimedian concatenation structure (Definition 3.1) for which $\circ$ satisfies the property that for all x, y in A, $x \circ y \succ x, y$, and $\mathscr{A}$ is (*restrictedly*) *solvable* in the sense that for all x, y in A, if $x \succ y$ then there exists u in A such that $y \circ u$ is defined and $x \succ y \circ u$.

Definition 4.1 is the same as the one given in Narens and Luce (1976) except that here $\succsim$ is a total ordering rather than a weak ordering as they assumed. This change is made to simplify certain subsequent definitions; it does not alter the theory in any substantive way.

Definition 4.2. $\mathscr{A}$ is said to be *an extensive structure* if and only if $\mathscr{A}$ is a positive concatenation structure and $\circ$ is associative whenever defined, i.e., for each x, y, z in $\mathscr{A}$, if $(x \circ y) \circ z$ and $x \circ (y \circ z)$ are defined, then $(x \circ y) \circ z = x \circ (y \circ z)$.

Positive concatenation structures have strong uniqueness properties, as the next two theorems show.

Theorem 4.1. *Let $\mathscr{A}$ be a positive concatenation structure. Then $\mathscr{A}$ satisfies* 1-*point uniqueness.*

THEOREM 4.2. *Let* $\mathscr{A} = \langle A, \succsim, \circ \rangle$ *be a positive concatenation structure. Then* (i) *there exists a numerical structure* $\mathscr{N} = \langle R, \geqslant, \odot \rangle$ *with* $R \subseteq \mathrm{Re}^+$ *such that a* $\mathscr{N}$*-representation for* $\mathscr{A}$ *exists; and* (ii) *if* φ *and* ψ *are* $\mathscr{N}$*-representations of* $\mathscr{A}$ *for some* $\mathscr{N}$, $\varphi(A) = \psi(A)$, *and* $\varphi(x) = \psi(x)$ *for some* x *in* A, *then* $\varphi = \psi$.

Theorem 4.1 immediately follows from Lemma 2.2 of Cohen & Narens (1979). Part (i) of Theorem 4.2 follows from Theorem 2.1 of Narens & Luce (1976) and Part (ii) follows from Theorem 4.1 above.

4.2. *Extensive Structures*

DEFINITION 4.3. Suppose $\mathscr{A} = \langle A, \succsim, \circ \rangle$ is an extensive structure. The element a is said to be a *maximal element* of $\mathscr{A}$ if and only if $a \in A$ and for each x in A, $a \succsim x$. It is said to be a *S-maximal element* of $\mathscr{A}$ ("S" is for "solvable") if and only if a is a maximal element of $\mathscr{A}$ and there exists x, y in A such that $a = x \circ y$.

Extensive structures with S-maximal elements occur frequently in science. Perhaps the best known of these are the ones induced by probability structures (Fine, 1971): Associated with the set $\mathscr{E}$ of equivalence classes of equally probable events is a binary relation $\succsim$, and a partial operation $\circ$ such that for each α, β, γ in $\mathscr{E}$, $\alpha \succsim \beta$ if and only if for some $D \in \alpha$ and $E \in \beta$, D is at least as probable as E, and $\alpha \circ \beta = \gamma$ if and only if for some $D \in \alpha$, $E \in \beta$, and $F \in \gamma$, $D \cap E = \varnothing$ and $D \cup E = F$. If the probability structure has a sufficiently rich set of events, then $\langle \mathscr{E}, \succsim, \circ \rangle$ will be an extensive structure. In this case $\langle \mathscr{E}, \succsim, \circ \rangle$ will have a S-maximal element, namely, the equivalence class that contains the sure event X. This equivalence class is clearly maximal, and since by the hypothesized richness of the probability structure $E \cup (X - E) = X$ for some E and $X - E$ of nonzero probabilities, the equivalence class containing X is S-maximal.

The following well-known result shows why extensive structures are of such importance:

THEOREM 4.3. *Let* $\mathscr{A}$ *be an extensive structure such that* $\mathscr{A}$ *has no maximal element or* $\mathscr{A}$ *has a S-maximal element. Let* $\mathscr{N} = \langle \mathrm{Re}^+, \geqslant, + \rangle$. *Then the following two statements are true:*

(1) *There exists a* $\mathscr{N}$*-representation for* $\mathscr{A}$.

(2) *The* $\mathscr{N}$*-representation of* $\mathscr{A}$ *forms a ratio scale.* (*Definition* 2.1.)

4.3. *The Automorphism Group of a Positive Concatenation Structure*

DEFINITION 4.4. Let $\mathscr{A}$ be a relational structure and G be its set of automorphisms. The Greek letter ι denotes the identity automorphism of $\mathscr{A}$, and $*$ denotes the composition of automorphisms.

It is well known that $\langle G, * \rangle$ is a group. For positive concatenation structures, this group is highly constrained, as will shortly be seen.

THEOREM 4.4. *Let $\mathscr{A} = \langle A, \gtrsim, \circ \rangle$ be a positive concatenaion structure and α, β be automorphisms of $\mathscr{A}$. If $\alpha(x) \gtrsim \beta(x)$ for some x in A, then $\alpha(y) \gtrsim \beta(y)$ for each y in A.*

Theorem 4.4 immediately follows from Theorem 2.1 of Cohen & Narens (1979). It justifies the following:

DEFINITION 4.5. Let $\mathscr{A} = \langle A, \gtrsim, \circ \rangle$ be a positive concatenation structure and $\langle G, * \rangle$ be its automorphism group. Define $\gtrsim$ on G as follows: for each α, β in G,

$$\alpha \gtrsim \beta \qquad \text{iff} \qquad \text{for some } x, \quad \alpha(x) \gtrsim \beta(x).$$

THEOREM 4.5. *Let $\mathscr{A} = \langle A, \gtrsim, \circ \rangle$ be a positive concatenation structure and $\langle G, * \rangle$ be its automorphism group. Then $\langle G, \gtrsim, * \rangle$ is an Archimedean, totally ordered, commutative group, i.e.,*

(i) *$\gtrsim$ is a total ordering on G;*

(ii) *$\langle G, * \rangle$ is commutative;*

(iii) *for each α, β, γ in G, $\alpha \gtrsim \beta$ iff $\alpha * \gamma \gtrsim \beta * \gamma$;*

(iv) *for each α, β in G, if $\alpha > \iota$, then for some positive integer n, $\alpha^n > \beta$, where α^k is defined inductively as follows: $\alpha^1 = \alpha$, $\alpha^{k+1} = \alpha^k * \alpha$.*

Proof. Theorem 2.4 and Lemma 2.5 of Cohen & Narens (1979).

Theorem 4.5 allows us conveniently to classify automorphism groups of positive concatenation structures:

DEFINITION 4.6. Let $\mathscr{A} = \langle A, \gtrsim, \circ \rangle$ be a positive concatenation structure and $\mathscr{G} = \langle G, \gtrsim, * \rangle$ its ordered automorphism group. $\mathscr{G}$ is said to be *trivial* if and only if $G = \{\iota\}$. $\alpha \in G$ is said to be *positive* if and only if $\alpha > \iota$. $\mathscr{G}$ is said to be *discrete* if and only if $\mathscr{G}$ has a smallest positive automorphism. $\mathscr{G}$ is said to be *dense* if and only if $\mathscr{G}$ is nontrivial and nondiscrete.

The following are some examples of various kinds of positive concatenation structures:

EXAMPLES 4.1. Let $\circ_1$, $\circ_2$, $\circ_3$ be defined on Re^+ as follows: for each x, y in Re^+,

$$x \circ_1 y = x + y,$$
$$x \circ_2 y = x + y + x^{1/2}y^{1/2},$$
$$x \circ_3 y = x + y + x^{1/4}y^{3/4}.$$

Let $\mathscr{A}_i = \langle \mathrm{Re}^+, \geqslant, \circ_i \rangle$ for $i = 1, 2, 3$. Then $\mathscr{A}_i$ are positive concatenation structures. For each r, s in Re^+, let $\alpha_r(s) = rs$. Then for $i = 1, 2, 3$, and for each r in Re^+, α_r is an automorphism of $\mathscr{A}_i$, and thus $\mathscr{A}_i$ satisfies 1-point homogeneity and therefore has

a dense automorphism group. $\circ_1$ is an associative operation; $\circ_2$ is a commutative and nonassociative operation; and $\circ_3$ is a noncommutative and nonassociative operation. Note that $\mathscr{A}_i$ satisfies 1-point homogeneity and 1-point uniqueness for $i = 1, 2, 3$.

Define $\circ_4$ on Re^+ as follows: for each x, y in Re^+,

$$x \circ_4 y = x + y + (xy)^{1/2}\,[2 + \sin(\tfrac{1}{2}\log[xy])].$$

Let $\mathscr{A}_4 = \langle \mathrm{Re}^+, \geqslant, \circ_4 \rangle$. Then $\mathscr{A}_4$ is a positive concatenation structure. Define for each integer n, β_n on Re^+ by

$$\beta_n(x) = xe^{2\pi n}.$$

Then β_n is an automorphism of $\mathscr{A}_4$ for each integer n, and all automorphism of $\mathscr{A}_4$ are of this form, i.e., $\mathscr{A}_4$ has a discrete automorphism group. Note that $\mathscr{A}_4$ satisfies 0-point homogeneity and 1-point uniqueness.

Define $\circ_5$ on Re^+ as follows: for each x, y in Re^+,

$$x \circ_5 y = x + y + x^2y^2.$$

Let $\mathscr{A}_5 = \langle \mathrm{Re}^+, \geqslant, \circ_5 \rangle$. Then $\mathscr{A}_5$ is a positive concatenation structure that has the identity as its only automorphism. Note that $\mathscr{A}_5$ satisfies 0-point homogeneity and 0-point uniqueness.

Proof. Examples 2.1, 3.1, and 4.2 of Cohen & Narens (1979).

Positive concatenation structures with nontrivial automorphism groups must have closed operations, as the following theorem shows:

THEOREM 4.6. *Suppose $\mathscr{A} = \langle A, \gtrsim, \circ \rangle$ is a positive concatenation structure that has a nontrivial automorphism group. Then the partial operation $\circ$ is in fact an operation.*

Proof. Theorem 2.5 of Cohen & Narens (1979).

4.4. *Fundamental Unit Structures*

DEFINITION 4.7. Let $\mathscr{A} = \langle A, \gtrsim, \circ \rangle$ be a positive concatenation structure. $\mathscr{A}$ is said to be *Dedekind complete* if and only if each nonempty bounded subset S of $\langle A, \gtrsim \rangle$ has a least upper bound in A.

Let $\mathscr{A} = \langle A, \gtrsim, \circ \rangle$ be a Dedekind complete positive concatenation structure. It easily follows that $\langle A, \gtrsim \rangle$ is of order type θ.

DEFINITION 4.8. $\mathscr{A}$ is said to be a *fundamental unit structure* if and only if $\mathscr{A}$ is a Dedekind complete positive concatenation structure that satisfies 1-point homogeneity.

Let $\mathscr{A}$ be a fundamental unit structure. Then since $\mathscr{A}$ is of order type θ and satisfies 1-point homogeneity and 1-point uniqueness (Theorem 4.1), it is ratio

scalable onto some numerical structure $\mathscr{N} = \langle \mathrm{Re}^+, \geqslant, \odot \rangle$. This kind of numerical representing structure has special properties and is given the name "unit representation":

DEFINITION 4.9. $\mathscr{N}$ is said to be a *unit representing structure* if and only if $\mathscr{N}$ is a positive concatenation structure of the form $\langle \mathrm{Re}^+, \geqslant, \odot \rangle$, where $\odot$ is such that there exists a function f from Re^+ onto Re^+ such that for each x, y in Re^+,

$$x \odot y = yf\left(\frac{x}{y}\right).$$

φ is said to be a unit representation for a positive concatenation structure $\mathscr{A}$ if and only if φ is a $\mathscr{N}$-representation for $\mathscr{A}$ for some unit representing structure $\mathscr{N}$.

The following remarks show some of the relationships of unit representations with ratio scalability and fundamental unit structures.

Remark 4.1. $\langle \mathrm{Re}^+, \geqslant, + \rangle$ is a unit representing structure since $x + y = y(1 + (x/y))$.

Remark 4.2. Suppose $\mathscr{N}$ is a unit representing structure. Then $\mathscr{N}$ is a fundamental unit structure and the automorphisms of $\mathscr{N}$ are multiplications by positive reals.

Remark 4.3. Suppose $\mathscr{N}$ is a unit representing structure and $\mathscr{A}$ is a positive concatenation structure that has a $\mathscr{N}$-representation, φ. Then $\{r\varphi \mid r \in \mathrm{Re}^+\}$ is a ratio scaling of $\mathscr{A}$ into $\mathscr{N}$.

Remark 4.4. Suppose $\mathscr{A}$ is a fundamental unit structure, $\mathscr{N} = \langle \mathrm{Re}^+, \geqslant, \odot \rangle$, φ is a $\mathscr{N}$-representation for $\mathscr{A}$ that is onto $\mathscr{N}$, and $\{r\varphi \mid r \in \mathrm{Re}^+\}$ is a ratio scaling of $\mathscr{A}$ onto $\mathscr{N}$. Then $\mathscr{N}$ is a unit representing structure.

THEOREM 4.7. *Suppose $\mathscr{A}$ is a fundamental unit structure. Then there exists a unit representation for $\mathscr{A}$.*

Proof. Theorem 3.2 of Cohen & Narens (1979).

Remarks 4.4 and 4.5 show that there is, for fundamental unit structures, a very strong connection between ratio scalability and unit representations. The following theorem shows how two unit representations are related:

THEOREM 4.8. *Suppose $\mathscr{A}$ is a fundamental unit structure and φ and φ' are unit representations for $\mathscr{N} = \langle \mathrm{Re}^+, \geqslant, \odot \rangle$ and $\mathscr{N}' = \langle \mathrm{Re}^+, \geqslant, \odot' \rangle$ respectively. Then there exist s, t in Re^+ such that*

$$\varphi' = s\varphi^{1/t} \qquad \text{and} \qquad u \odot' v = (u^t \odot v^t)^{1/t}.$$

Proof. Theorem 3.5 of Cohen & Narens (1979).

Let $\mathscr{A} = \langle A, \succsim, \circ \rangle$ be a Dedekind complete, positive concatenation structure. The necessary and sufficient condition given in Definition 4.8 for $\mathscr{A}$ being a fundamental unit structure is that $\mathscr{A}$ satisfies 1-point homogeneity. However, this formulation is not "relational" since it is about the set of automorphisms of $\mathscr{A}$ rather than about relations on A. It is natural to ask if a set of necessary and sufficient conditions for $\mathscr{A}$ being a fundamental unit structure can be given in terms of the relations $\succsim$ and $\circ$. A sufficient condition is that $\circ$ is an associative operation, as is easily seen by applying Theorem 4.3. However, as Example 4.1 shows, this condition is not necessary. A necessary and sufficient condition is given in the following definition and theorem:

DEFINITION 4.10. Suppose $\mathscr{A} = \langle A, \succsim, \circ \rangle$ is a positive concatenation structure and $\circ$ is an operation. For each positive integer n and each x in A, let

$$\alpha_n(x) = nx.$$

α_n is called the *n-copy operator* of $\mathscr{A}$.

THEOREM 4.9. *Let $\mathscr{A} = \langle A, \succsim, \circ \rangle$ be a Dedekind complete, positive concatenation structure. Then the following three statements are equivalent:*

(1) *$\mathscr{A}$ is a fundament unit structure.*

(2) *$\mathscr{A}$ has a dense automorphism group.*

(3) *$\circ$ is an operation and for each positive integer n and each x, y in A, $\alpha_n(x \circ y) = \alpha_n(x) \circ \alpha_n(y)$, where α_n is the n-copy operator of $\mathscr{A}$.*

Proof. Theorems 2.5 and 3.1 of Cohen & Narens (1979).

Note that statement (3) of Theorem 4.9 is tantamount to saying that α_n is an automorphism, since it is easy to prove that α_n preserves the ordering $\succsim$. Thus in some concatenation structures it is possible to define certain specific nontrivial automorphism directly in terms of the partial operation of the structure. However, this is not the case in general: In any interval scalable structure (Definition 2.1), the identity is the only automorphism that is definable in terms of relations of the structure. (See pp. 36–37 of Narens, 1981a, for a proof of this and a discussion of this issue.)

Dedekind completeness is not a necessary condition for a positive concatenation structure to have a unit representation, as the following theorem shows:

THEOREM 4.10. *Suppose $\mathscr{A} = \langle A, \succsim, \circ \rangle$ is a positive concatenation structure that satisfies the following two conditions for all positive integers n and all x, y in A:*

(i) *If either $n(x \circ y)$ or $nx \circ ny$ are defined, then the other is defined and*

$$n(x \circ y) = nx \circ ny;$$

(ii) *there exists z in A such that $nz = x$.*

Then a unit representation for $\mathscr{A}$ exists.

Proof. Theorems 5.3 and 3.2 of Cohen & Narens (1979).

In conjoint measurement, an extension of a positive concatenation structure that has "negative" elements and a "zero" element is sometimes needed. Such an extension is given in the following definition (Luce and Cohen, 1983).

DEFINITION 4.11. Let $A \neq \emptyset$, $\succsim$ be a binary relation on A, $\circ$ be a binary partial operation on A, a_0 be an element of A, and $\mathscr{A} = \langle A, \succsim, \circ \rangle$. Then $\mathscr{A}$ is said to be a *total concatenation structure* if and only if the following six conditions hold:

(1) $\succsim$ is a total ordering.

(2) The restriction of $\mathscr{A}$ to $A^+ = \{x | x \succ a_0\}$ is a positive concatenation structure.

(3) The restriction of $\mathscr{A}$ to $A^- = \{x | a_0 \succ x\}$ is a positive concatenation structure when the converse ordering $\precsim$ is substituted for $\succsim$.

(4) For each x in A, $x \circ a_0 = a_0 \circ x = x$.

(5) For each x, y, z in X,

(i) if $x \circ z$ and $y \circ z$ are defined, then,

$$x \succsim y \quad \text{iff} \quad x \circ z \succsim y \circ z;$$

(ii) if $z \circ x$ and $z \circ y$ are defined, then

$$x \succsim y \quad \text{iff} \quad z \circ x \succsim z \circ y.$$

(6) For $a \in A^+$ and $b \in A^-$, there exist $c, d \in A$ such that $c \circ b$ and $d \circ a$ exist, $c \circ b \succ a$ and $b \succ d \circ a$.

5. INTENSIVE STRUCTURES

In the preceding section, we axiomatized positive concatenation structures, and showed that this rather general subclass of concatenation structures had many important measurement-theoretic characteristics. In particular, for this subclass of structures 1-point homogeneity is equivalent to ratio scalability. Positivity, i.e., $x \circ y \succ x, y$ for all x, y, is a key condition for this subclass of concatenation structures. In this section we will consider some of the measurement-theoretic properties for a subclass of concatenation structures similar to those of the previous section except that positivity fails. In place of positivity, the condition *intern*, which is characterized by $x \succ x \circ y$, $y \circ x \succ y$ for all x, y such that $x \succ y$, will be assumed. The resulting structures are called *intensive*. Despite the fact that intensive structures are the only concatenation structures that are interval scalable, very little has been

written about them. We are aware of only two approaches to them, neither of which produces results as nearly as comprehensive as one would like.

The first approach, which can be found as Theorem 6.10 of Krantz *et al.* (1971), involves imposing a strong condition called *bisymmetry*: for all w, x, y, z in the structure,

$$(w \circ x) \circ (y \circ z) \sim (w \circ y) \circ (x \circ z).$$

With sufficient structural assumptions, it can be shown that $\circ$ induces an additive conjoint structure (see Section 6), from which one is able to derive the existence of an interval scale representation $\langle R, \gtrsim, \bigodot \rangle$, where R is a subset of the real numbers and $\bigodot$ is a binary operation of the following form: for all r, s in R,

$$r \bigodot s = ar + (1-a)s, \qquad 0 < a < 1.$$

Observe that this is the special case with $a = b$ of the possible interval scale representations stated as part (ii) of Theorem 3.3. Of course, bisymmetry is not implied in the general case of $a \neq b$, and so more general constructions are needed for that case as well as for the idempotent ratio and (1, 2) scales.

The second idea is to seek those intensive structures that can be represented within positive concatenation structures, and then to use the numerical representations of the positive concatenation structures as a basis for formation of the representations of the intensive ones. To this end, Narens & Luce (1976) introduced the following concept which turns out to define the relevant mapping:

DEFINITION 5.1. Suppose $\langle A, \gtrsim, * \rangle$ is an intensive structure, $D \subseteq A$, and δ is a function from D into A. The function δ is a *doubling function* if and only if the following five conditions are met for all x, y, x_i in A:

(i) δ is strictly increasing:

(ii) if x is in D and $x \gtrsim y$, then y is in D;

(iii) if $x > y$, then there exists u in D such that $y * u$ is defined and in D and $x > \delta(y * u)$;

(iv) if $x * y$ is defined and in D, then $\delta(x * y) > x, y$;

(v) suppose $\{x_n\}$ is a sequence such that $x_1 \sim x$ and if x_{n-1} is in D, then either $\delta(x_{n-1}) * x$ is defined and equals x_n or x_n is not defined; then for some n either x_n is not defined or $x_n > y$.

First, let us consider the question of the uniqueness of a doubling function. Cohen has established the following result:

THEOREM 5.1. *If an intensive structure has a doubling function, then either it is unique or there is just one other which differs at exactly one point (up to equivalence), and that point is maximal in the structure.*

Next we formulate the close relation of intensive structures to positive concatenation structures, as was shown by Narens & Luce (1976).

THEOREM 5.2. *Suppose* $\gtrsim$ *is an ordering on* $A \neq \varnothing$ *and* $*$ *and* $\circ$ *are binary operations defined for the same pairs of elements and are related by a function* δ *from a subset of* A *into* A *such that for all* x, y *in* A *for which* $x * y$ *is defined then it is in the domain of* δ *and* $x \circ y = \delta(x * y)$. *Then,* $\langle A, \gtrsim, * \rangle$ *is an intensive structure and* δ *is its doubling function if and only if* $\langle A, \gtrsim, \circ \rangle$ *is a positive concatenation structure and* δ^{-1} *is its* half-element function, *i.e., for all* x *in* A *for which* $\delta^{-1}(x) \circ \delta^{-1}(x)$ *is defined it is equal to* x. *Moreover, the two structures have the same group of automorphisms.*

From the representation of a positive concatenation structure (Theorem 4.2) and this theorem, a representation of the following form can be constructed: There exists a binary operation $\odot$ on Re^+ with half-element function h (i.e., a function h such that $x = h(x) \odot h(x)$ for all x in A) and a mapping φ from A into Re^+ such that for all x, y in A

$$x \gtrsim y \quad \text{iff} \quad \varphi(x) \geqslant \varphi(y),$$
$$\text{if } x * y \text{ is defined, } \varphi(x * y) = h[\varphi(x) \odot \varphi(y)],$$
$$\text{and if } x \text{ is in the domain of } \delta, \ \varphi(x) = h\varphi\delta(x).$$

When we restrict attention to (1, 1) structures onto Re^+, which have unit representations, then the following result (Luce & Narens, unpublished) characterizes the existence of doubling functions.

THEOREM 5.3. *Suppose* $\langle \mathrm{Re}^+, \geqslant, \odot \rangle$ *is a real intensive structure with a unit representation* g, *i.e.,*

$$x \odot y = yg(x/y).$$

Then the necessary and sufficient conditions for it to have a doubling function are the existence of a constant $k > 0$ *such that*

(1) $\lim_{x \to \infty} x/g(x) = k$,

(2) $\lim_{x \to 0} 1/g(x) \leqslant k$,

(3) *if* $g_n(k)$ *is defined by* $g_1(k) = 1$ *and* $g_n(k) = g[kg_{n-1}(k)]$ *then* $\lim_{n \to \infty} g_n(k) = \infty$.

In this case, the doubling function is kx *and the resulting positive concatention structure has the unit representation* $f = kg$.

Since by Theorem 5.2 the automorphism groups are necessarily the same, Theorem 5.3 covers all of the homogeneous intensive structures that have doubling functions and are onto Re^+. We suspect that this is only a small fraction of the possible intensive structures.

6. Solvable Conjoint Structures

6.1. *Background and Definition*

Perhaps the most natural reason for developing a theory of numerical representations for orderings on Cartesian products of empirical objects is the fact that they abound in physics. Such Cartesian products are called "conjoint structures" in the measurement literature. Any simple physical law involving three dimensions, for example, $E = mv^2/2$, can be looked at as a conjoint structure, with, in the example, the ordering of mass-velocity pairs by kinetic energy. But historically the motive for developing such a theory came mostly from economics and psychology. The major economic stimulus was consideration of utility functions for commodity bundles consisting of several components. Psychologists throughout this century commonly have reported data in terms of tradeoffs between two independent factors that leave constant some psychological variable. Three examples of this are (i) the amount and delay of reward that maintains a constant response rate, (ii) the combinations of frequency and intensity of pure tones that exhibit the same loudness, and (iii) the combinations of intensity and duration of light flashes that result in equal detectability. The question naturally arose as to whether these equal-attribute contours could in some fashion be used to establish a numerical scale of that attribute.

As additive forms of conjoint theories began to be developed, two important facts began to be recognized. The first, due to Holman (1971) and later generalized by Narens & Luce (1976), is that it is possible under suitable solvability conditions to induce on a single component of the conjoint structure an operation the encodes all of the information about the structure, thereby reducing the study of such structures to that of certain operations (see Section 6.3). The second is that this induced operation is totally distinct from the usual physical operations that are often found on one or another component of a conjoint structure. Moreover, in classical physics at least, this "induced" operation interlocks with the usual physical operations on the components so that there is a representation of these structures as products of powers of the additive scales for the physical extensive structures of the components. This and generalizations of it are treated in Section 8.

The precise definition of a conjoint structure that we shall use is:

Definition 6.1. Suppose $\succsim$ is a binary relation on $A \times P$, a_0 is in A, and p_0 is in P. $\langle A \times P, \succsim, a_0, p_0 \rangle$ is said to be a *conjoint structure that is A-solvable relative to* $a_0 p_0$ if and only if for all a, b in A and p, q in P:

(1) Weak ordering: $\succsim$ is transitive and connected.

(2) Independence: $ap \succsim bp$ iff $aq \succsim bq$; $ap \succsim aq$ iff $bp \succsim bq$.

(3) Solvability: There exist $\xi(a, p)$ in A and $\pi(a)$ in P such that:

$$\xi(a, p)\, p_0 \sim ap$$

$$a_0 \pi(a) \sim ap_0.$$

(4) Density: If $ap_0 \succ bp_0$, then there exist p, q in P such that

$$ap_0 \succ bp \succ bp_0 \qquad \text{and} \qquad ap_0 \succ aq \succ bp_0.$$

(5) Archimedean: For all a in A and all positive integers n, define a *standard sequence* na inductively by: $1a = a$, $na = \xi[(n-1)a, \pi(a)]$. Then every strictly bounded standard sequence is finite.

A symmetric definition of being *P-solvable relative to* $a_0 p_0$ can be given; the only required changes are in axioms (4) and (5).

The axioms of Definition 6.1 are of three types. Axioms (1) and (2) are necessary conditions for the existence of a numerical representation of the following form: There exist real functions φ_A and φ_P on A and P, respectively, and a monotonic binary operation $\odot$ on the reals such that for all $a, b \in A$ and $p, q \in P$

$$ap \gtrsim bq \qquad \text{iff} \quad \varphi_A(a) \odot \varphi_P(p) \geqslant \varphi_A(b) \odot \varphi_P(q).$$

Observe that axiom (2) renders well defined the following induced orderings on the components A and P: for $a, b \in A$,

$$a \gtrsim_A b \qquad \text{iff} \qquad \text{for all} \quad p \text{ in } P, \; ap \gtrsim bp,$$

and for all $p, q \in P$,

$$p \gtrsim_P q \qquad \text{iff} \qquad \text{for all} \quad a \text{ in } A, ap \gtrsim aq.$$

The last axiom, the Archimedean one, or something equally unempirical, is needed if the representation is to be into the reals. Much of interest in a real representation can, however, be obtained without the Archimedean axiom by considering representations into richer algebraic structures such as the nonstandard reals. Axioms (3) and (4) are structural in nature, i.e., they restrict the scope of the theory. In some of the first developments (Debreu, 1960; Luce & Tukey, 1964) some version of *unrestricted solvability* was assumed, namely, that every equation $ap \sim bq$ can be solved for the fourth variable when the other three are given. For psychological applications this assumption was unrealistic, and later (Luce, 1966; Krantz *et al.*, 1971, Chap. 6) a much weaker condition was shown to be adequate for additive representations. The present version (taken from Luce & Cohen, 1983) is still different but is just what is needed in order to induce a closed binary operation on the first component. Somewhat weaker conditions will suffice for partial operations, but they have not yet been worked out except for the case of structures with a minimal element (Narens & Luce, 1976). The density axiom, which is innocuous in most physical and psychological examples, is needed to establish some of the desired properties of the induced operations.

6.2. *Conjoint Structures as Relational Structures*

Consider $\langle A \times P, \gtrsim, a_0, p_0 \rangle$, where $\gtrsim$ is a binary relation on $A \times P$ and $a_0 \in A$ and $p_0 \in P$. Let $C = A \times P$, $e = a_0 p_0$, and $\mathscr{C} = \langle C, \gtrsim, e \rangle$. If we consider $\mathscr{C}$ as a relational

structure, then we have no way of talking about C being a Cartesian product, and thus cannot define concepts such as independence and solvability in terms of the primitives of $\mathscr{C}$. However, we can capture these concepts in a relational way by considering a different structure: Let $X = A \cup P$, and for each a, p, b, q in $A \cup P$, let R be the 4-ary relation on x such that

$$ap \gtrsim bq \qquad \text{iff} \quad R(a, p, b, q).$$

Let $\mathscr{X} = \langle X, R, a_0, p_0 \rangle$. Then $\mathscr{X}$ is a relational structure and the concepts of independence and solvability are definable in terms of its primitives. Let α be an automorphism of $\mathscr{X}$. Then for each $a, b \in A$ and $p, q \in P$,

$$\begin{aligned} ap \gtrsim bq \qquad &\text{iff} \quad R(a, p, b, q) \\ &\text{iff} \quad R[\alpha(a), \alpha(p), \alpha(b), \alpha(q)] \\ &\text{iff} \quad \alpha(a)\,\alpha(p) \gtrsim \alpha(b)\,\alpha(q). \end{aligned}$$

From the invariance of R under α, it must be the case that $\alpha(a) \in A$ and $\alpha(p) \in P$. Let α_A be the restriction of α to A and α_P be the restriction of α to P. In this way, we can write each automorphism of α of $\mathscr{X}$ as an automorphism γ of the structure $\mathscr{C}$, where γ is defined as follows: for each $ap \in C$,

$$\gamma(ap) = \alpha_A(a)\,\alpha_P(p).$$

Such a γ is called *factorizable* since in the structure $\langle A \times P, \gtrsim \rangle$ it can be written as a "product" of a transformation on A with a transformation on P. Not all automorphisms of $\mathscr{C}$ need be factorizable. Indeed, if $A = P = \mathrm{Re}$, $\gtrsim$ is defined by

$$ap \gtrsim bq \qquad \text{iff} \quad a^2 + p^2 \geqslant b^2 + q^2,$$

and $e = 00$. Thus, $ap \gtrsim bq$ if and only if in the Cartesian plane the distance of (a, p) from the origin $(0, 0)$ is at least as great as the distance of (b, q) from the origin. Compositions of rotations and dilations about the origin are automorphisms of $\gtrsim$ and it is easy to show that not all of these are factorizable.

In summary, orderings $\gtrsim$ on a Cartesian product can be formulated in terms of a relational structure, but in doing so certain types of automorphisms of $\gtrsim$ are omitted. In the following sections, which rely heavily on concepts involving automorphisms, we will not use such a relational structure formulation of $\gtrsim$ so that we can handle the theory in its full generality.

6.3. *Relation to Total Concatenation Structures*

As was noted above, one of the major insights into conjoint structures, first realized for additive ones and later recognized as equally useful for any with adequate solvability conditions, was that they can be mapped into operations on either the components or on the structure itself. The formal definitions in terms of the π and ξ notation on Definition 6.1 are:

DEFINITION 6.2. Suppose $\mathscr{C} = \langle A \times P, \succsim, a_0, p_0 \rangle$ is a conjoint structure that is A-solvable relative to $a_0 p_0$. Define operations $*_A$, $*_P$, and $*$ relative to $a_0 p_0$ induced on A, P, and $A \times P$ respectively, as follows: for all $a, b \in A$, $p, q \in P$:

(i) $a *_A b = \xi[a, \pi(b)]$.

(ii) $p *_P q = \pi\xi[\xi(a_0, p), q]$

(iii) $ap * bq = \xi(a, p)\, \pi\xi(b, q) = \xi(a, p) *_A \xi(b, q), p_0$.

It is convenient to have names for the induced structures. We use $\mathscr{S}$ as the generic symbol and subscript it by the name of the defining set: A, P, $A \times P$, or

$$A^+ = \{a \mid a \in A \text{ and } a \succ_P p_0\}, \qquad A^- = \{a \mid a \in A \text{ and } a \prec_A a_0\},$$

$$P^+ = \{p \mid p \in P \text{ and } p \succ_P p_0\}, \qquad P^- = \{p \mid P \in A \text{ and } p \prec_P p_0\}.$$

It is implicit that the appropriate ordering and operation are restricted to the defining set.

The first theorem establishes that the notions of positive and total concatenation structures describe what is induced by a solvable conjoint structure.

THEOREM 6.1. *Suppose* $\mathscr{C} = \langle A \times P, \succsim, a_0, p_0 \rangle$, $a_0 \in A$, $p_0 \in P$, *is a conjoint structure that is A-solvable relative to* $a_0 p_0$.

(1) *If* $a_0 p_0$ *is minimal with respect to* $\succsim$, *then* $\mathscr{S}_{A^+}$, $\mathscr{S}_{P^+}$, *and* $\mathscr{S}_{A^+ \times P^+}$ *are closed positive concatenation structures.*

(2) *If* $a_0 p_0$ *is neither minimal nor maximal, then* $\mathscr{S}_A$, $\mathscr{S}_P$, and $\mathscr{S}_{A \times P}$ *are total concatenation structures.*

(This is Theorem 2 of Luce & Cohen, 1983, and it generalizes Theorem 4.1 of Narens & Luce, 1976.)

One can, of course, ask if this is the best possible result under the circumstances or are there further constraints on the induced operation that we have failed to capture. The next result (Theorem 3 of Luce & Cohen) shows we cannot do better.

THEOREM 6.2. *Suppose* $\mathscr{A} = \langle A, \succsim, \circ, a_0 \rangle$ *is a closed total concatenation structure. Then for* $\mathscr{A}'$ *isomorphic to* $\mathscr{A}$, *there exists a conjoint structure* $\mathscr{C} = \langle A \times A', \succsim'', a_0, a_0' \rangle$ *that is A-solvable relative to* $a_0 a_0$ *and for which* $\mathscr{S}_A$ *is isomorphic to* $\mathscr{A}$.

Observe that these results both involved closed operations, whereas the general concept of a total concatenation structure entails partial operations. Presumably, there is some weakening of the solvability conditions that will be just adequate to yield the constraints of axiom (3) of Definition 3.1, but so far this has not been developed, except for structures with a minimal element (Narens & Luce, 1976).

If we combine the results of Theorems 4.2 and 6.1, we obtain the following representation of a general solvable conjoint structure:

THEOREM 6.3. *Suppose* $\mathscr{C} = \langle A \times P, \gtrsim, a_0, p_0 \rangle$ *is a conjoint structure that is A-solvable relative to* $a_0 p_0$. *Then there exists real functions* φ_A *on A and* φ_P *on P and a binary numerical operation* $\odot$ *such that*

(i) $\varphi_A(a_0) = 0$, $\varphi_P(p_0) = 0$,

(ii) *for real* x, $x \odot 0 = 0 \odot x \doteq x$,

(iii) *for all* $a, b \in A$, $p, q \in P$,

$$ap \gtrsim bq \quad \text{iff} \quad \varphi_A(a) \odot \varphi_P(p) \geqslant \varphi_A(b) \odot \varphi_P(q).$$

6.4. *The Thomsen Condition*

The earliest representation that was studied, and by far the best understood, is the additive one, where the numerical operation $\odot$ is $+$. It was early recognized that a key qualitative property underlying additivity of the representation is the so-called Thomsen condition which first appeared in the theory of webs (Blaschke & Bol, 1938).

DEFINITION 6.3. A conjoint structure $\langle A \times P, \gtrsim \rangle$ satisfies the *Thomsen condition* if and only if for all $a, b, f \in A$ and $b, q, x \in P$,

$$\text{if} \quad ax \sim fq \quad \text{and} \quad fp \sim bx, \quad \text{then} \quad ap \sim bq.$$

(In their treatment of the additive case, Luce & Tukey, 1964, used a stronger version of this property in which $\sim$ is everywhere replaced by $\gtrsim$.) Luce & Cohen (1983) have proved:

THEOREM 6.4. *Suppose* $\mathscr{C} = \langle A \times P, \gtrsim, a_0 p_0 \rangle$ *is a conjoint structure that is A-solvable relative to* $a_0 p_0$. *The structure satisfies the Thomsen condition if and only if the induced operation* $*_A$ *is associative.*

COROLLARY. *The induced operation* $*_A$ *is also commutative.*

Under somewhat different solvability conditions, Holman (1971) showed that the Thomsen condition implies associativity. Before turning to the representation that results from this theorem, we turn to an alternative characterization of the Thomsen condition that arises from comparing the induced structures arising from different reference points.

DEFINITION 6.4. A conjoint structure has *invariant induced operations* if and only if for every $a_0 p_0$, $a_0' p_0'$ for which it is solvable, the respective induced operations $*_A$ and $*_A'$ satisfy the following condition for every $a, b, c, d, \in A$,

$$a *_A b \gtrsim c *_A d \quad \text{iff} \quad a *_A' b \gtrsim c *_A' d.$$

Luce & Cohen (1983, Theorem 6) proved:

THEOREM 6.5. *Suppose* $\mathscr{C} = \langle A \times P, \succsim \rangle$ *is a (conjoint) structure that satisfies axioms* (1), (2), *and* (5) *of Definition* 6.1 *and, in addition, is unrestrictedly solvable. Then* $\mathscr{C}$ *satisfies the Thomsen condition if and only if it has invariant induced operations.*

It will follow from Theorems 9.2 and 9.3 that when the Thomsen condition holds, any two induced structures are isomorphic.

6.5. *Additivity and Factorizable Automorphisms*

The representation that follows from the Thomsen condition is, by now, classical:

THEOREM 6.6. *Suppose that* $\mathscr{C}$ *is an A-solvable conjoint structure for which the Thomsen condition holds. Then in the representation of Theorem* 6.3 *the operation* $\odot$ *can be chosen to be* $+$.

There are other versions of this result using somewhat different solvability conditions; see Chapter 6 of Krantz *et al.* (1971).

The question of the uniqueness of this additive representation can be formulated in terms of automorphisms, just as was the case for fundamental unit structures.

THEOREM 6.7. *Suppose* $\succsim_+$ *is the relation on* $\mathrm{Re} \times \mathrm{Re}$ *defined by: for all* $x, y, u, v \in \mathrm{Re}$,

$$xy \succsim_+ uv \qquad \textit{iff} \quad x + y \geqslant u + v.$$

A function $\alpha : \mathrm{Re} \times \mathrm{Re} \to \mathrm{Re}$ *is an order automorphism of* $\succsim_+$ *if there are real constants* $r > 0$, s_1, s_2 *such that for all* $x, y \in \mathrm{Re}$,

$$\alpha(x, y) = (rx + s_1, ry + s_2).$$

This is a classical result. We say that such automorphisms are *factorizable* into correlated (since r is common to both) transformations on each component separately. Because the property of factorizability appears to play a crucial role in the classical theory of dimensional analysis (see Sects. 7 and 8), we explore it more fully in Section 9 by considering classes of conjoint structures that are richly endowed with factorizable automorphisms.

6.6. *The Set of Translations*

A somewhat different perspective on the concepts we have been examining can be obtained by studying the family of translations corresponding to the induced operation $*_A$:

DEFINITION 6.5. Suppose $\mathscr{C} = \langle A \times P, \succsim, a_0 p_0 \rangle$ is a conjoint structure that is A-solvable relative to $a_0 p_0$, and let $*_A$ be the operation given in Definition 5.2. For each a in A, the transformation

$$\theta_a(x) = x *_A a, \qquad x \quad \text{in } A,$$

is called a *translation*, and the set of all translations is denoted $\mathscr{T}$.

THEOREM 6.8. *Suppose $\mathscr{C}$ is a conjoint structure that is A-solvable relative to $a_0 p_0$ and $\mathscr{E}$ is the set of its translations. Then the following five statements are true:*

(1) *$\mathscr{E}$ satisfies 1-point uniqueness.*

(2) *$\mathscr{E}$ satisfies 1-point homogeneity if and only if $\mathscr{C}$ is unrestrictedly solvable in the sense that if a, b in A and p in P are given, then there exists q in P such that $ap \sim bq$. In this case, $\mathscr{E}$ includes inverse translations.*

(3) *$\mathscr{E}$ is closed under function composition if and only if $*_A$ is associative.*

(4) *$\mathscr{E}$ is commutative if and only if $\mathscr{C}$ satisfies the Thomsen condition, in which case $*_A$ is associative and commutative.*

(5) *If $\mathscr{C}$ is unrestrictedly solvable and Archimedean and $*_A$ is associative, then $\mathscr{C}$ satisfies the Thomsen condition and $\mathscr{E}$ is an Archimedean ordered group.*

Theorem 6.8 is partially proved in Narens (1981a), more fully in Luce & Cohen (1983), and completely in Luce & Narens (unpublished). A further result about $\mathscr{E}$ is given in Theorem 8.3.

7. DIMENSIONAL ANALYSIS IN PHYSICS

7.1. *Extensive and Conjoint Measures of the Same Things*

Classical physical measures are of two types: extensive and additive conjoint (the latter usually written in terms of a multiplicative rather than an additive representation). For example, length, mass, and time are modeled as extensive structures, whereas momentum and density arise out of conjoint structures. To be specific, consider density. Let A denote a set of homogeneous liquids and P a set of containers. Then $ap \in A \times P$ is interpreted as the amount of liquid a needed to fill container p, and $\succsim$ is the ordering of $A \times P$ obtained by comparing the masses of the two liquids poured from the two containes into the pans of an equal-arm pan balance. In this situation, the axioms of additive conjoint measurement are met to a high degree of accuracy, and so we know by Theorem 6.6 that there is a representation of the form $\psi_A \psi_P$, where both scales are positive, that preserves the mass ordering.

Since mass is extensively measurable, it has a representation m that is additive over concatenation of objects. Thus, there must be an increasing function f so that $\psi_A \psi_P = f(m)$. Focussing on the components, ψ_A is a measure associated with the liquids that is induced by the mass ordering and ψ_P is one associated with the volumes of the containers. Once again, we know that volume is an extensive quantity with a measure V that is additive over concatenations of volume. So there is an increasing function f_P such that $\psi_P = f_P(V)$, and so we can write the representation as

$$m = f^{-1}[\psi_A f_P(V)].$$

But physics tells us more, namely, that the functions f and f_P can, within the uniqueness of the conjoint representation, both be selected to be the identity function so that

$$m = \psi_A V.$$

In this case, ψ_A is abbreviated ρ and called the density measure.

More generally, one of the major discoveries of classical physics was that whenever there is a triple of variables that have a conjoint structure and some are also extensively measurable, then each pair of dual measures, one conjoint and one extensive, are related by a power function. So, for example, if a quantity has a multiplicative conjoint representation $\psi_A \psi_P$ and each component has an extensive (additive) representation φ_A and φ_P, then there are (rational) numbers r and s and a (real) number c such that

$$\psi_A \psi_P = c\varphi_A^r \varphi_P^s.$$

If the attribute with the representation $\psi_A \psi_P$ is itself extensive with an additive representation φ, then there is a (rational) number t and a real number c such that

$$\psi_A \psi_P = c\varphi^t.$$

The combination of these facts of physics ultimately results in our system of units, where each physical unit can be written as the product of powers of units from a fixed, small set of basis units.

Thus, one problem to be solved by a qualitative theory for physical measurement is the laws on a conjoint structure $\langle A \times P, \gtrsim \rangle$, which has one or more of the empirical operations $\circ$, $\circ_A$, $\circ_P$ on, respectively, $A \times P$, A, and P, that are adequate to lead to the following representation: There are real functions φ, φ_A, and φ_P that transform one or more of $\circ$, $\circ_A$, and $\circ_P$ into addition and result in a conjoint representation of the form

$$\varphi = \varphi_A^r \varphi_P^s.$$

One solution to this problem was suggested in Chapter 10 of Krantz *et al.* (1971), but a better one was formulated in Narens & Luce (1976) and subsequently improved. It is summarized in Section 8.

7.2. *Structure of Physical Quantities*

As was suggested above in the remark about the structure of units in physics, there is a substantial extension from triples of variables to a structure of all physical quantities that is represented as a multiplicative vector space. In particular, a basis of m extensive quantities, with additive measures $\varphi_1, \ldots, \varphi_m$, can be selected such that any other measure of the space can be written as

$$\psi = c\varphi_1^{\rho_1}\varphi_2^{\rho_2} \cdots \varphi_m^{\rho_m}, \tag{7.1}$$

where the $\rho_1,...,\rho_m$ are rational numbers and c is a real number. Such a numerical structure is postulated explicitly in most books on dimensional analysis. Perhaps the most explicit and simple version of what is assumed is due to Whitney (1968) (for a discussion, see also Sect. 10.2.1 of Krantz *et al.*, 1971). We do not describe here explicitly how one can impose axioms on the underlying qualitative structures that are sufficient to ensure that triples of the sort discussed both in Section 7.1 and again in Section 8 lead to the representation just mentioned. One such attempt can be found in Section 10.9 of Krantz *et al.* (1971) and a better version will appear in Vol. II of that work.

7.3. *Physical Laws and Dimensional Invariance*

Within the framework of interlocked physical dimensions just discussed, a physical system is assumed to be described as collections of coexisting values on some, but not all, of the possible physical measures. A description of all the possible combinations of these variables that can be observed for the system is said to be the *law* describing that system. Such a mathematical relation, for that is what it is, is usually characterized as follows. Let $x_1,...,x_n$ denote values on the n physical attributes that characterize the system. Then for some function F from $\mathrm{Re}^n \to \mathrm{Re}$, the n-tuple $(x_1,...,x_n)$ is a realizable configuration of the system if and only if

$$F(x_1,...,x_n) = 0. \tag{7.2}$$

Having said this, the question immediately arises about the uniqueness of F as we vary the representations used for the variables. Each of the physical measures is defined, at best, on a ratio scale, and so the information we have about the structure can be recoded by scale changes of the form

$$(x_1,...,x_n) \to (r_1 x_1,...,r_n x_n), \tag{7.3}$$

where $r_1,...,r_n$ are positive reals. However, in general, not all of the changes of scale represented by Eq. (7.3) are possible because the scales are not all independent. This is reflected in Eq. (7.1) where we have said there is a basis of m scales and all the remaining ones can be expressed as products of powers of these. So, once the scales of the basis are fixed, then those of all of the other scales are also fixed by relations of the form

$$r = r_1^{\rho_1} \rho_2^{\rho_2} \cdots r_m^{\rho_m}. \tag{7.4}$$

So transformations of the form of Eq. (7.3) but subject to constraints given by Eq. (7.4) are the ones we need to consider. These are called *similarities*.

The condition physicists have imposed on laws of the type given in Eq. (7.2) is that they must be invariant under similarly transformations, i.e., if $(\alpha_1,...,\alpha_n)$ is a similarity transformation, then

$$F(x_1,...,x_n) = 0 \qquad \text{iff} \quad F(\alpha_1 x_1,...,\alpha_n x_n) = 0.$$

This property is known as *dimensional invariance*.

One question that has troubled all who have written about dimensional analysis is just why one should assume dimensional invariance (Birkhoff, 1950; Bridgeman, 1931; Causey, 1969; Luce, 1959, 1978; Sedov, 1959). On the one hand, it seems most plausible; but on the other hand, it has proved elusive to justify formally. One can provide an account of it in terms of a concept of meaningfulness cast in terms of invariance under automorphisms of qualitative structures (Narens, 1981a).

Once dimensional invariance is accepted, use may be made of the famous Π-Theorem of Buckingham (1914) which characterizes more fully the nature of such a law. We do not state the result formally (see Theorem 10.4 of Krantz *et al.*, 1971) but do attempt to convey its meaning. Suppose F is defined over n variables in a minimal structure of physical quantities having a basis of m quantities. The assertion is, then, that there are $n-m$ independent products of powers of the n variables, $\Pi_1,\ldots,\Pi_{n-m}$, that are each dimensionless, and so do not undergo any change in value under a similarity, and that the law F can be replaced by a function of just these $n-m$ dimensionless quantities. In some applications there is a single dependent variable, say x_1, of interest and it appears in just one of the dimensionless quantities, say the first one. In that case one can usually rewrite the law as

$$\Pi_1 = G(\Pi_2,\ldots,\Pi_{n-m}).$$

Further, this can be solved for the dependent variable x_1 and, taking into account that the Π's are products of powers of the several variables of the system, it must be of the form

$$x_1 = x_2^{\rho_2}\cdots x_n^{\rho_n} G(\Pi_2,\ldots,\Pi_{n-m}),$$

where some of the ρ's may be 0. This provides a great deal of information about the system, especially if, as sometimes is the case, $\Pi_2,\ldots,\Pi_{n-m}$ are parameters of the system that remain constant and so can be estimated experimentally.

7.4. *Physically Similar Systems and Dimensional Constants*

All we have said about dimensionally invariant laws presupposed that we know the relevant variables of the problem. If we do not, we can be seriously misled. Sometimes it is quite a subtle matter to know what the variables are. If, for example, the physics of the process is understood to the point where a full set of dynamic equations have been developed, as is true in hydrodynamics and electromagnetism, then the relevant variables and parameters are those that appear in the equations. Usually some of the parameters are not the least bit obvious. A simple example is illustrative.

Consider the law—Hooke's—that describes the linear behavior of springs. If we let l denote the amount of deformation of the spring and F the force required to achieve it, we find F and l are related by a law of the form

$$F - Cl = 0,$$

where C is characteristic of the spring involved. Change the spring and C changes; it is called the *spring constant.* Now it is easy to see that this law is dimensionally

invariant if and only if C has dimensions of mass/time^2. This follows from the fact that if mass, length, and time form a basis, and we introduce the similarity (r_m, r_l, r_t), then F is transformed by $r_m r_l / r_t^2$ and l by r_l. So, if we believe that laws must be dimensionally invariant, we are forced to conclude that C is transformed by r_m / r_t^2. In this case, there is just one dimensionless quantity $\Pi_1 = F/Cl$.

This is typical of physics. In addition to quantities that may vary within the system, parameters are associated with the system and are called dimensional constants. One can think of such situations as a family of systems, as in the spring example, and a mapping from the family into a dimension that in some sense serves to identify the different members of the family. And any laws about the family will involve these constants as well as the variables. For a detailed development of this idea, see Causey (1969), Sect. 10.10.2 of Krantz *et al.* (1971), and Luce (1971, 1978).

8. Distributive Structures

As we have just discussed, a basic and very important class of measurement structures is formed by adding positive concatenation operations to conjoint structures. These structures are called "distributive" and appear ubiquitously throughout physical, psychological, and economic settings. The published literature concerning the abstract nature of these structures is contained in five papers: Narens (1976), Narens & Luce (1976), Luce (1978), Narens (1981a) and Luce & Cohen (1983). Since the focus of each of these papers is on other topics, the uses of distributive structures in them are mostly technical or illustrative, and a really clear and systematic development is not given in any of them. (Luce & Cohen 1983 comes closest to such a development.) Nevertheless, taken in sum, an impressive number of theorems and concepts have evolved. Unfortunately, the best and sharpest results are not directly stated as theorems; however, they are usually easily obtainable by combining together the proofs given for various theorems. In this section we will state a few of the most useful and important results about distributive structures and indicate how some of these may be generalized.

Definition 8.1. $\langle A \times P, \succsim \rangle$ is said to be an *unrestrictedly solvable conjoint structure* if and only if $\langle A \times P, \succsim \rangle$ satisfies the conditions of weak ordering and independence given in Definition 6.1 and the following:

Unrestricted solvability: for x, y in X and each p, q in P, there exists z in X and r in P such that $xp \sim yr$ and $xp \sim zq$.

Note that Definition 8.1 does not assume the density and Archimedean conditions for a solvable conjoint structure (Definition 6.1).

Definition 8.2. $\langle A \times P, \succsim, \circ \rangle$ is said to be an *A-distributive structure* if and only if the following four conditions hold:

(1) $\langle A \times P, \succsim \rangle$ is an unrestrictedly solvable conjoint structure;

(2) $\langle A, \succsim_A \rangle$ is Dedekind complete and totally ordered;

(3) $\langle A, \succsim_A, \circ \rangle$ is a positive concatenation structure with a closed operation (Definition 3.1);

(4) *A-distributive*: for each xp, yp, uq, vq in $A \times P$, if $xp \sim uq$ and $yp \sim vq$, then $(x \circ y)\, p \sim (u \circ v)\, q$.

The theory of distributive structures can easily be developed without condition (2) of Definition 8.2; it is assumed here to simplify notation and the statement of theorems.

The following theorem presents what is probably the most important result about distributive structures:

THEOREM 8.1. *Suppose* $\langle A \times P, \succsim, \circ \rangle$ *is an A-distributive structure and* $\mathscr{A} = \langle A, \succsim_A, \circ \rangle$. *Then the following two statements are true:*

(1) $\mathscr{A}$ *is a fundamental unit structure.*

(2) *For each ratio scale* $\mathscr{F}$ *of* $\mathscr{A}$ *and each* φ *in* $\mathscr{F}$, *there exists* $\psi : P \to \mathrm{Re}^+$ *such that* $\langle \varphi, \psi \rangle$ *is a multiplicative representation for* $\langle A \times P, \succsim \rangle$, *i.e., such that for each ap, bq in* $A \times P$,

$$ap \succsim bq \qquad \textit{iff} \quad \varphi(a)\, \psi(p) \geqslant \varphi(b)\, \psi(q).$$

Proof. Statement (1) follows by the proof of statement iv of the corollary to Theorem 10 of Luce & Cohen (1983). Statement 2 follows from statement 1 and Theorem 4.1 of Narens (1981b).

Notice that if $\langle A \times P, \succsim \rangle$ is an unrestrictedly solvable conjoint structure, $\mathscr{A} = \langle A, \succsim_A, \circ \rangle$, and statements (1) and (2) of Theorem 8.1 hold, then it easily follows that $\langle A \times P, \succsim, \circ \rangle$ is a distributive structure, so that an appropriately formulated converse of Theorem 8.1 is true.

Also notice that the conclusion of Theorem 8.1 implies the Thomsen condition holds for $\langle A \times P, \succsim \rangle$.

We do not know if a result comparable to Theorem 8.1 holds when $\langle A, \succsim_A, \circ \rangle$ is an intensive structure or when the assumption of unrestricted solvability is weakened. We do, however, have an example that shows both changes void the conclusion. Consider $\langle (\mathrm{Re}^+ \cup \{0\}) \times \mathrm{Re}^+, \succsim \rangle$, where for all $x, y \in \mathrm{Re}^+ \cup \{0\}$ and $u, v \in \mathrm{Re}^+$,

$$(x, u) \succsim (y, v) \qquad \text{iff} \quad xu + u^2 \geqslant yv + v^2.$$

This fails to be unrestrictedly solvable since $(x, u) \sim (y, v)$ holds if $x = (yv + v^2 - u^2)/u$, which is not always $\geqslant 0$. We establish later (from the corollary to Theorem 9.7) that this conjoint structure is not transformable into a multiplicative representation. Let the operation $\circ$ on the first component be

$$x \circ y = \frac{x + y}{2},$$

which is easily seen to be intensive, closed, and distributive. Then Theorem 8.1 is not true for an intensive operation and a conjoint structure that is not unrestrictedly solvable.

The next theorem establishes a crucial link between distributivity and a qualitative analog of dimensional invariance.

THEOREM 8.2. *Suppose* $\langle A \times P, \succsim \rangle$ *is an unrestrictedly solvable conjoint structure and* $\mathscr{A} = \langle A, \succsim_A, \circ \rangle$ *is a fundamental unit structure. Then the following two statements are equivalent:*

(1) $\circ$ *satisfies distributivity* (*Definition* 8.2).

(2) *For all* ap, bq *in* $A \times P$ *and all automorphisms* α *of* $\mathscr{A}$, $ap \sim bq$ *iff* $\alpha(a)\, p \sim \alpha(b)\, q$.

Proof. Statement (1) implies statement (2) by Remark 4.2 and Theorem 8.1. Statement (2) implies statement (1) is provided in Theorem 10 of Luce & Cohen (1983).

The next result, proved in Luce & Cohen (1983), establishes a connection between the translations of $*_A$, the automorphisms of $\mathscr{A}$, and distributivity:

THEOREM 8.3. *Suppose* $\mathscr{C} = \langle A \times P, \succsim \rangle$ *is an unrestrictedly solvable conjoint structure,* $*_A$ *is one of its induced operations,* $\mathscr{F}$ *is the set of translations of* $*_A$ (*see Theorem* 6.8), *and* $\mathscr{A} = \langle A, \succsim_A, \circ \rangle$ *is a positive concatenation structure with the automorphism group* $\mathscr{G}$. *Then* $\mathscr{F} = \mathscr{G}$ *if and only if* $\circ$ *is* A*-distributive.*

Note that by Theorem 6.8, if $\mathscr{C}$ is Archimedean and $\circ$ is distributive, then it follows from Theorem 8.3 that $\mathscr{F}$ is a group, $*_A$ is associative, and the Thomsen condition holds. Thus, $\mathscr{C}$ has an additive representation.

A-distributive structures have an operation $\circ_A$ on the A-component. An analogous definition of P-distributive structures can be given in the obvious way. Such structures have an operation $\circ_P$ on the P-component. The following definition extends the concept of distributivity to structures of the form $\langle A \times P, \succsim, \circ \rangle$, where $\circ$ is an operation on $A \times P$:

DEFINITION 8.3. $\langle A \times P, \succsim, \circ \rangle$ is said to be an $A \times P$-distributive structure if and only if the following four conditions hold:

(1) $\langle A \times P, \succsim \rangle$ is an unrestrictedly solvable conjoint structure;

(2) $\langle A, \succsim_A \rangle$ and $\langle P, \succsim_P \rangle$ are Dedekind complete, totally ordered sets;

(3) $\langle A \times P, \succsim, \circ \rangle$ is a positive concatenation structure;

(4) $A \times P$*-distributivity:* for each a, b, c in A and each p, q in P, $(ap) \circ (bp) \sim cp$ iff $(aq) \circ (bq) \sim cq$.

The following theorem establishes the relationships between $A \times P$-distributivity and A- and P-distributivity.

THEOREM 8.4. *Suppose* $\langle A \times P, \succsim, \circ \rangle$ *is an* $A \times P$*-distributive structure. Then the following two statements are true:*

(1) *For each* a, b *in* A *and each* p, q, r *in* P, $(ap) \circ (aq) \sim ar$ *iff* $(bp) \circ (bq) \sim br$.

(2) *Define* $\circ_A$ *and* $\circ_P$ *on* A *and* P, *respectively by: for each* a, b, c *in* A:

$$a \circ_A b = c \quad \text{iff} \quad \text{for some } p \text{ in } P, \quad (ap) \circ (bp) \sim cp,$$

and for each q, r, s *in* P,

$$q \circ_P r = s \quad \text{iff} \quad \text{for some } d \text{ in } A, \quad (dq) \circ (dr) \sim ds.$$

Then $\langle A \times P, \succsim, \circ_A \rangle$ *and* $\langle A \times P, \succsim, \circ_P \rangle$ *are respectively* A*-distributive and* P*-distributive structures.*

Proof. See Section 5 of Narens & Luce (1976).

In Classical Physics, it is commonplace to find structures of the form $\mathscr{C} = \langle A \times P, \succsim, \circ_A, \circ_P \rangle$, where $\circ_A$ and $\circ_P$ are, respectively, A- and P-distributive. They often occur when $\mathscr{A} = \langle A, \succsim_A, \circ_A \rangle$ and $\mathscr{P} = \langle P, \succsim_P, \circ_P \rangle$ are extensive structures of fundamental physical qualities such as length and mass. The fundamental qualities are measured by giving additive representations φ and ψ to $\mathscr{A}$ and $\mathscr{P}$, respectively. By the appropriate use of Theorem 8.1 and the uniqueness result for additive conjoint structures, Theorem 6.7, it follows that $\langle A \times P, \succsim \rangle$ has a multiplicative representation of the following form: there exist r, s in Re^+ such that for each ap, bq in $A \times P$,

$$aq \succsim bq \quad \text{if} \quad \varphi(a)^r \psi(p)^s \geqslant \varphi(b)^r \psi(q)^s. \tag{8.1}$$

In the above representation, r and s are not uniquely determined although the ratio r/s is. Qualitative axioms can be stated in terms of the structure $\mathscr{C}$ that specify any particular ratio r/s (see Chap. 10, Krantz *et al.*, 1971).

Not all structures in physics are appropriately distributive. For example, there exists the structure $\langle V \times T, \succsim, \circ_V \rangle$, where V is the set of positive velocities in a given direction that are less than the velocity of light, c, T is the set of times, and $\succsim$ is the ordering of distance, i.e., $v_1 t_1 \succsim v_2 t_2$ stands for $v_1 t_1$ (the distance that a particle with veocity v_1 displaces in time t_1) is at least as great as $v_2 t_2$. Then in relativistic physics, $\langle V, \succsim_V, \circ_V \rangle$ is an extensive structure, $\langle V \times T, \succsim \rangle$ is an unrestrictedly solvable conjoint structure (which in fact is an additive conjoint structure), but $\circ_V$ does not satisfy V-distributivity. This can all be easily checked by considering the representations used in relativistic physics,

$$v_1 t_1 \succsim v_2 t_2 \quad \text{iff} \quad \varphi(v_1)\, \psi(t_1) \geqslant \varphi(v_2)\, \psi(t_2)$$

and

$$\varphi(v_1 \circ_V v_2) = \frac{\varphi(v_1) + \varphi(v_2)}{1 - (\varphi(v_1)\, \varphi(v_2)/c^2)}$$

for all $v_1 t_1$ and $v_2 t_2$ in $V \times T$.

9. Conjoint Structures with Factorizable Automorphisms

In Theorem 6.7 we noted that the automorphisms of an additive representation of a conjoint structure can be thought of as factorizable into correlated linear transformations on each of the components, and considerable use of that fact is made in dimensional analysis. In addition, we noted in Section 6.2 that if a conjoint structure is recast as a relational structure, the automorphisms of that relational structure correspond exactly to the factorizable automorphisms of the conjoint structure. The question posed in this section is whether the notion of a conjoint structure with a rich set of factorizable automorphisms is an interesting generalization of the concept of an additive conjoint structure. It is clear that the methods of dimensional analysis are generalizable without revision to this class of structures. The discussion is divided into four major parts. The first establishes some rather general properties of structures with factorizable automorphisms. The second introduces suitable concepts of homogeneity and uniqueness of the group of factorizable automorphisms, and this leads to a partial classification of such structures. The third considers structures that have numerical representations with a certain amount of smoothness (in the sense of derivatives existing). The fourth considers the case in which the structure has an intrinsic zero, in the sense of an element that transforms into itself under all factorizable automorphisms. In this case, it is shown that representations exist that are very closely related to those of fundamental unit structures. In the other cases that have been investigated (Section 9.5) the additive structure or a minor variant arise.

9.1. *General Properties*

We begin with the obvious formal definition of factorizable automorphisms.

Definition 9.1. Suppose $\mathscr{C}$ is a conjoint structure. An order automorphism α of $\mathscr{C}$ is *factorizable* iff there exist 1:1 functions $\theta : A \to A$ and $\eta : P \to P$ such that for all $a \in A$, $p \in P$, $\alpha(a, p) = \theta(a)\,\eta(p)$.

$\mathscr{F}$ will denote the set of all factorizable automorphisms, $\mathscr{F}_A$ will denote those transformations that arise on A, i.e.,

$$\mathscr{F}_A = \{\theta \mid \theta : A \xrightarrow{\text{onto}} A \text{ and there exists } \eta : P \xrightarrow{\text{onto}} P \text{ such that } \langle \theta, \eta \rangle \in \mathscr{F}\}$$

$\mathscr{F}_P$ will denote those that arise on P, i.e.,

$$\mathscr{F}_P = \{\eta \mid \eta : P \xrightarrow{\text{onto}} P \text{ and there exists } \theta : A \xrightarrow{\text{onto}} A \text{ such that } \langle \theta, \eta \rangle \in F\}.$$

Note $\mathscr{F} \subseteq \mathscr{F}_A \times \mathscr{F}_P$ but in general, $\mathscr{F} \neq \mathscr{F}_A \times \mathscr{F}_P$. In fact, if $\mathscr{F}$ is nontrivial, equality is almost never the case.

It follows readily that under function composition all three of $\mathscr{F}$, $\mathscr{F}_A$, and $\mathscr{F}_P$ are groups.

Recall from our discussion of fundamental unit structures that the n-copy operator is an automorphism (Theorem 4.9). This gives rise to the following result (Theorem 7 of Luce & Cohen, 1973), which can be used to show the existence of nonadditive conjoint structures with factorizable automorphisms.

THEOREM 9.1. *Suppose* $\mathscr{C} = \langle A \times P, \succsim \rangle$ *is an A-solvable conjoint structure relative to* $a_0 p_0 \in A \times P$. *Then the mapping* α_n, *which is defined for each integer n by* $\alpha_n(a, p) = (na, np)$, *is a factorizable automorphisms of* $\mathscr{C}$ *if and only if the n-copy operator of* $*_A$ *induced by* $\mathscr{C}$ *on A is an automorphism of* $\mathscr{J}_A = \langle A, \succsim_A, *_A \rangle$.

The next result, while not terribly difficult to prove, is absolutely central to all work on structures with factorizable automorphisms because it establishes very strong interlocks between the two factors of the automorphism and certain of the induced positive concatenation structures.

THEOREM 9.2. *Suppose that conjoint structure* $\mathscr{C} = \langle A \times P, \succsim \rangle$ *is A-solvable relative to both* $a_0 p_0$ *and* $a_0' p_0'$, θ *is a function from A onto A with* $\theta(a_0) \sim_A a_0'$, *and* η *is a function from P onto P with* $\eta(p_0) \sim_P p_0'$. *Let* π *and* π' *denote the solutions relative to* $a_0 p_0$ *and* $a_0' p_0'$, *respectively of Definition* 6.1. *Then* $\langle \theta, \eta \rangle$ *is a factorizable automorphism of* $\mathscr{C}$ *if and only if* $\eta = \pi' \theta \pi^{-1}$ *and* θ *is an isomorphism from* $\mathscr{J}_A = \langle A, \succsim_A, *_A, a_0 \rangle$ *onto* $\mathscr{J}_A' = \langle A, \succsim_A, *_A', a_0' \rangle$.

COROLLARY. *Let* ι_A *and* ι_P *denote the identity maps of A and P, respectively. If* $\langle \theta, \iota_P \rangle$ (*respectively*, $\langle \iota_A, \eta \rangle$) *is a factorizable automorphism, then for some* $c \in A$ (*respectively*, $r \in P$) $\theta \sim_A c *_A \iota_A$ (*respectively*, $\eta \sim_P \iota_P *_P r$).

The condition that the two induced structures are isomorphic together with the existence of a factorizable automorphism forces enormous regularity on the conjoint structure. However, something more is needed to get the Thomsen condition, as the next result shows:

THEOREM 9.3. *Suppose that* $\mathscr{C}$ *is a conjoint structure that is A-solvable relative to* $a_0 p_0$ *and that for every a in A, p in P there exists b in A such that* $bp \sim ap_0$. *Then the following two statements are true:*

(1) $\mathscr{C}$ *satisfies the Thomsen condition if and only if for every c in A, r in P,* $\langle c *_A i_A, \iota_P *_P r \rangle$ *is an automorphism of* $\mathscr{C}$.

(2) *Suppose* θ *is a one-to-one function from A onto A and* η *is a one-to-one function from P onto P. Under the conditions of part* 1, $\langle \theta, \eta \rangle$ *is an automorphism of* $\mathscr{C}$ *if and only if for some automorphism* θ^* *of* $\mathscr{J}_A$,

$$\theta = \theta(a_0) *_A \theta^* \qquad \text{and} \qquad \eta = \pi \theta^* \pi^{-1} *_P \eta(p_0).$$

9.2. *Assumptions about the Factorizable Automorphisms*

In Section 1 it was shown that assumptions about automorphisms—specifically M-point homogeneity and N-point uniqueness—limit in significant and interesting ways

the possible representations of a relational structure. Within the context of conjoint structures with factorizable automorphisms, it is clear that some modifications of these concepts are needed since it is not only properties of the automorphisms of the conjoint ordering that matter but also those of the components.

Recall that when placed in relational form, the conjoint structure becomes $\langle A \cup P, R \rangle$, where for all a, b in A and p, q in P,

$$R(a, b, p, q) \qquad \text{iff} \quad ap \gtrsim bq.$$

An automorphism α of this structure has the property

$$R(a, b, p, q) \qquad \text{iff} \quad R[\alpha(a), \alpha(b), \alpha(p), \alpha(q)].$$

It is, thus, clear that α maps A into A and P into P; it is not usually a map between A and P. Thus, in general the relational structure is necessarily 0-point homogeneous. However, we can ask about the behavior of those groups that are obtained by restricting the automorphisms to A and to P separately, and to consider levels of homogeneity of these relational substructures. When translated back into the Cartesian formulation we arrive at the following notions of homogeneity and uniqueness.

DEFINITION 9.2. Suppose $\mathscr{C} = \langle A \times P, \gtrsim \rangle$ is a conjoint structure. Its factorizable automorphisms satisfy *component M-point homogeneity* if and only if for every $a_1, \ldots, a_M$, $b_1, \ldots, b_M \in A$, $p_1, \ldots, p_M$, $q_1, \ldots, q_M \in P$ such that for all $i = 1, \ldots, M-1$, $a_{i+1} \succ_A a_i$, $b_{i+1} \succ_A b_i$, $p_{i+1} \succ_P p_i$, and $q_{i+1} \succ_P q_i$, there are factorizable automorphisms $\langle \theta, \eta \rangle$ and $\langle \theta', \eta' \rangle$ such that for $i = 1, \ldots, M$,

$$\theta(a_i) = b_i \qquad \text{and} \qquad \eta(p_M) = q_M,$$

$$\theta'(a_M) = b_M \qquad \text{and} \qquad \eta'(p_i) = q_i.$$

If the value of M differs on the two components, say M and M', respectively, we speak of (M, M')-component homogeneity.

They satisfy *component N-point uniqueness* iff for all $a_i, \ldots, a_N$, $p_i, \ldots, p_N$ with $a_{i+1} p_{i+1} \succ a_i p_i$, $i = 1, \ldots, N-1$, and factorizable automorphisms $\langle \theta, \eta \rangle$, $\langle \theta', \eta' \rangle$ with $\theta(a_i) = \theta'(a_i)$, $\eta(p_i) = \eta'(p_i)$, $i = 1, \ldots, N$, if $a_{i+1} \succ_A a_i$, $i = 1, \ldots, N-1$, then $\theta = \theta'$, or if $p_{i+1} \succ_P p_i$, $i = 1, \ldots, N-1$, then $\eta = \eta'$.

Note that if the factorizable automorphisms satisfy component M-point homogeneity, they satisfy M-point homogeneity in the ordered structure $\mathscr{C}$; whereas if they satisfy N-point uniqueness in the $\mathscr{C}$, they satisfy component N-point uniqueness.

An important distinction among conjoint structures with factorizable automorphisms is whether or not there is a fixed point. We formulate this as:

DEFINITION 9.3. Suppose $\mathscr{C} = \langle A \times P, \gtrsim \rangle$ is a conjoint structure with a nontrivial set $\mathscr{F}$ of factorizable automorphisms and $a_0 \in A$, $p_0 \in P$. The point $a_0 p_0$ is an *intrinsic zero* of $\mathscr{C}$ if and only if for every $\langle \theta, \eta \rangle \in \mathscr{F}$, $\theta(a_0) = a_0$ and $\eta(p_0) = p_0$.

9.3. *Assumptions about a Real Representation*

In the theorems to be reported, we begin with the assumption that the conjoint structure has a real representation, and attempt to characterize it is terms of properties of its automorphisms. The nature of the representation is captured in the next definition.

DEFINITION 9.4. A function $F: \mathrm{Re} \times \mathrm{Re} \to \mathrm{Re}$ is a C^n *conjoint representation* of the conjoint structure $\langle \mathrm{Re} \times \mathrm{Re}, \gtrsim \rangle$ where, for $x, y, u, v \in \mathrm{Re}$,

$$xy \gtrsim uv \qquad \text{iff} \quad F(x, y) \geqslant F(u, v),$$

provided:

(i) $F(x, \cdot)$ and $F(\cdot, y)$ map Re onto Re.

(ii) F is strictly increasing and continuous in each variable.

(iii) F is C^n, i.e., it is continuously differentiable of order n.

Such a representation is additive if and only if there exist strictly increasing functions f_1, f_2, and f from Re onto itself such that

$$F(x, y) = f[f_1(x) + f_2(y)].$$

We say 0 is an identity if and only if for all $x \in \mathrm{Re}$,

$$F(x, 0) = F(0, x) = x.$$

In terms of such a numerical representation, a factorizable automorphism is a strictly increasing function $\alpha_{\theta,\eta}$ from Re onto Re together with one-to-one functions θ, η from Re onto Re such that for all $x, y \in \mathrm{Re}$,

$$F[\theta(x), \eta(y)] = \alpha_{\theta,\eta}[F(x, y)].$$

LEMMA 9.1. *Suppose F is C^0 and f_1, f_2, and f are strictly monotonic increasing functions from* Re *onto* Re. *Then $F^* = f[F(f_1, f_2)]$ is a C^0 conjoint representation and the two groups of factorizable automorphisms are isomorphic.*

9.4. *The Case of an Intrinsic Zero*

THEOREM 9.4. *Suppose $\mathscr{C}$ has a C^0 real representation, and it is unrestrictedly solvable. Let $\mathscr{F}$ be its group of factorizable automorphisms. If $\mathscr{C}$ has an intrinsic zero, then $\mathscr{F}$ satisfies component* 1*-point uniqueness. If, in addition, $\mathscr{F}$ on $\langle A^+ \times P^+, \gtrsim \rangle$ satisfies component* 1*-point homogeneity and* 1*-point uniqueness, then there exists functions ϕ_A from A onto* Re *and ϕ_P from P onto* Re *such that $\phi_A(a_0) = 0$ and $\phi_P(p_0) = 0$, and a function F from* Re $\times$ Re *onto* Re *that is strictly increasing in both variables such that $F(\phi_A, \phi_P)$ is a representation of $\mathscr{C}$ and F is of the following form: There exist strictly increasing functions f_+ from* Re *onto* $[1, \infty)$ *and f_- from*

Re *onto* $(-\infty, -1]$, *where* f_+ *and* $-f_-$ *satisfy the properties of a unit representation* (*Definition* 4.9) *and*

$$F(x, y) = |y| f_{\operatorname{sign} y}(x/|y|), \qquad y \neq 0,$$
$$= x, \qquad y = 0.$$

This, which is Theorem 13 of Luce & Cohen (1983), is clearly a generalization of the usual representation of dimensional analysis. However, we are not aware of any empirical interpretations where such an intrinsic zero plays an important role.

9.5. *The Case of No Intrinsic Zero*

THEOREM 9.5. *Suppose F is a conjoint representation for which* 0 *is the identity. Each of the following conditions is sufficient for F to be additive.*

(1) *F is* C^4, $F_x \neq 0$, $F_y \neq 0$, *and* $\mathscr{F}$ *satisfies component* 1*-point homogeneity.*

(2) *F is* C^3, $F_x \neq 0$, $F_y \neq 0$, *and* $\mathscr{F}$ *satisfies component* (1, 2)- *or* (2, 1)*-homogeneity.*

(3) *F is* C^0 *and unrestrictedly solvable,* $\mathscr{F}$ *satisfies* 2*-point uniqueness and component* 2*-point homogeneity, and* $\mathscr{F}_A$ *on* $\langle A, \succsim_A \rangle$ *and* $\mathscr{F}_P$ *on* $\langle P, \succsim_P \rangle$ *both satisfy* 2*-point uniqueness.*

The three parts of this result are, respectively, Theorems 17, 18, and 19 of Luce & Cohen (1983). It appears that there is some tradeoff between smoothness assumptions and the degree of homogeneity imposed. One suspects that other sufficient conditions entailing C^1 or C^2 smoothness can be found.

The proofs of (1) and (2) use a result that is closely related to one of Scheffé (1965). Recall that he proved

THEOREM 9.6. *Suppose F is a* C^2 *conjoint representation with* $F_x F_y \neq 0$. *Then F is additive if and only if* $F_{xy}/F_x F_y$ *is a function of F.*

Luce & Cohen (1983, Theorem 16) show

THEOREM 9.7. *Suppose F is a* C^1 *conjoint representation. Let*

$$\Psi(x, y) = \log[F_x(x, y)/F_y(x, y)].$$

Then F is additive if and only if there exist functions ψ_1 *and* ψ_2 *such that* e^{ψ_1} *and* $e^{-\psi_2}$ *are integrable functions on any bounded domain and*

$$\Psi(x, y) = \psi_1(x) + \psi_2(y).$$

COROLLARY. *Suppose F is* C^3 *and* $F_x F_y \neq 0$. *Then F is additive if and only if* $\Psi_{xy} \equiv 0$.

The theorem has the advantage over Scheffé's that only C^1 is assumed, but the disadvantage that sometimes it may not be obvious whether Ψ is the sum of functions on the two factors. At the expense of going to C^3, the corollary reduces the question to showing that the second partial of Ψ is or is not identically zero, which may be very much easier to do than to show that F_{xy}/F_xF_y is or is not a function of F.

As an example, consider the one mentioned following Theorem 8.1:

$$F(x, y) = xy + y^2, \qquad x \geqslant 0, \quad y > 0.$$

Then,

$$F_x = y, \qquad F_y = x + 2y, \qquad F_{xy} = 1$$

So the Sheffé criterion is whether $1/(xy + 2y^2)$ is a function of F. The criterion of Theorem 9.7 is whether

$$\Psi(x, y) = \log \frac{y}{x + 2y} = -\log \left(\frac{x}{y} + 2\right)$$

is additive. The criterion of the corollary is whether

$$\psi = \frac{2y}{(x + 2y)^2}$$

is identically 0, which it obviously is not.

10. Conclusions and Open Problems

The basic strategy of the work reported has been to lay bare what we believe to be the key ideas of classical measurement, particularly those embodied in the structure of physical quantities and dimensional invariance. In this paper, the focus has been primarily upon generalizing these structures, and only implicitly have we considered issues involved in generalizing the corresponding meaningfulness concepts related to dimensional invariance. We have worked with four important interconnected measurement-theoretic concepts: (1) conjoint structures, (2) concatenation structures (which often appear on a component of a conjoint structure), (3) distributive operations (which give a powerful interlock between a conjoint structure and a component concatenation structure), and (4) the automorphism groups of all of the above structures and operations, which necessarily have strong interconnections and describe the underlying symmetries.

We have studied three important, major ideas about the automorphism groups of measurement structures: (1) the richness of the group relative to the structure, which is captured by the notion of M-point homogeneity; (2) the redundancy of the group relative to the structure, which is captured by the notion of N-point uniqueness; and (3) the nature of the interlock among the automorphism groups that arise in conjoint

measurement structures, and in particular, the idea of factorizable automorphisms. These three concepts provide a classification scheme that is very useful for understanding the measurement possibilities: (1, 1) structures that map onto Re^+ are equivalent to the concept of being ratio scalable onto Re^+; (2, 2) structures that map onto Re are equivalent to the concept of being interval scalable onto Re; and (M, M) structures, $M > 2$, that map onto Re or Re^+ cannot exist. For concatenation structures that are M-point homogeneous, $M \geqslant 1$, only the (1, 1), (2, 2), (1, 2), and (M, ∞) cases can arise, and the (M, ∞) case is ruled out once a very reasonable density condition is imposed. At present, the nature of the (1, 2) group has not been worked out although examples of measurement structures with such groups have been given. Except for some of the (1, 1) cases, all of the concatenation structures are intensive. The nonintensive (1, 1) structures are well understood at this time, but very few results exist about the other, intensive, concatenation structures.

Once an automorphism group of a measurement structure has been characterized, it is possible to set up a numerical functional equation whose solutions yield the possible numerical representations of the measurement. Such functional equations have been set up and solved in general for (1, 1) and (2, 2) structures and for the particular example we know of a (1, 2) structure. Similarly for conjoint structures with factorizable automorphisms, analogous concepts of homogeneity and uniqueness allow us to derive the possible numerical representations by solving appropriate numerical functional equations as before. Thus in the conjoint case we understand well the analog of the (1, 1) and (2, 2) cases, but are in the dark about the analog of the (1, 2) case.

Once the possible kinds of scale types of representations are known, we can ask for qualitative conditions that specify measurement structures with those scale types. This is, of course, the kind of measurement theory that has dominated the field until recently. In particular, classical extensive measurement can be generalized by replacing associativity with a much more general qualitative condition that is equivalent to 1-point homogeneity, and the resulting positive concatenation structure has representations that form a ratio scale. Intensive bisymmetric structures have interval scale representations, as do additive conjoint structures. These, and their variants, and ordinal scalable structure are basically the ordered measurement structures that have been axiomatized up until the writing of this paper. There remains much more to be done since we have axiomatizations for only a small fraction of intensive structures of types (1, 1), (2, 2), and (1, 2) and for the corresponding analogous conjoint structures.

There are vast gaps in our knowledge about the measurement possibilities for the following kinds of structures: (1) those that are neither of the concatenation nor conjoint type; (2) structures that are 0-point homogeneous and thus lack any substantial degree of symmetry; and (3) conjoint structures that are rich in automorphisms but not in factorizable ones.

REFERENCES

BIRKHOFF, G. *Hydrodynamics. A study in logic, fact and similitude.* Princeton, N. J.: Princeton Univ. Press, 1950, 1960.

BLASCHKE, W., & BOL, G. *Geometrie der Gewebe.* Berlin: Springer, 1938.

BRIDGMAN, P. *Dimensional analysis.* New Haven: Yale Univ. Press, 1922, 1931.

BUCKINGHAM, E. On physically similar systems: illustrations of the use of dimensional equations. *Physical Review*, 1941, **4**, 345–376.

CANTOR, G. Beiträge zur Begründung der transfiniten Mengenlehre. *Mathematische Annalen*, 1895, **46**, 481–512.

CAUSEY, R. L. Derived measurement, dimensions and dimensional analysis. *Philosophy of Science*, 1969, **36**, 252–270.

COHEN, M., & NARENS, L. Fundamental unit structures: A theory of ratio scalability. *Journal of Mathematical Psychology*, 1979, **20**, 193–232.

DEBREU, G. Topological methods in cardinal utility theory. In K. J. Arrow, S. Karlin, & P. Suppes (Eds.), *Mathematical methods in the social sciences, 1959.* Stanford: Standford Univ. Press, 1960. Pp. 16–26.

FINE, T. *Theories of probability.* New York: Academic Press, 1971.

HELMHOLTZ, H. V. Zählen und Messen erkenntnis-theoretische betrachet. *Philosophische Aufsätze Eduard Zeller gewidmet.* Leipzig, 1887. Reprinted in *Gessammelte Abhandl.*, Vol. 3, 1895. Pp. 356–391. English translation by C. L. Bryan, *Counting and measuring.* Princeton, N.J.: van Nostrand, 1930.

HÖLDER, O. Die Axiome der Quantitat und die Lehre vom Mass. *Ver. Verh. Kgl. Sachisis. Ges. Wiss. Leipzig, Math.-Phys. Classe.* 1901, **53**, 1–64.

HOLMAN, E. W. A note on conjoint measurement with restricted sovability. *Journal of Mathematical Psychology*, 1971, **8**, 489–494.

KRANTZ, D. H., LUCE, R. D., SUPPES, P., & TVERSKY, A. *Foundations of measurement* (Vol. I). New York: Academic Press, 1971. (Vol. II), in preparation.

LUCE, R. D. On the possible psychophysical laws. *Psychological Review*, 1959, **66**, 81–95.

LUCE, R. D. Two extensions of conjoint measurement. *Journal of Mathematical Psychology*, 1966, **3**, 348–370.

LUCE, R. D. Similar systems and dimensionally invariant laws. *Philosophy of Science*, 1971, **38**, 157–169.

LUCE, R. D. Dimensionally invariant numerical laws correspond to meaningful qualitative relations. *Philosophy of Science*, 1978, **45**, 1–16.

LUCE, R. D., & COHEN, M. Factorizable automorphisms in solvable conjoint structures, I. *Journal of Pure and Applied Algebra*, 1983, **27**, 225–261.

LUCE, R. D., & NARENS, L. Classification of concatenation structures according to scale type. In preparation.

LUCE, R. D., & TUKEY, J. W. Simultaneous conjoint measurement: A new type of fundamental measurement. *Journal of Mathematical Psychology*, 1964, **1**, 1–27.

NARENS, L. Utility-Uncertainty trade-off Structures. *Journal of Mathematical Psychology*, 1976, **13**, 296–322.

NARENS, L. On the scales of measurement. *Journal of Mathematical Psychology*, 1981, **24**, 249–275. (a)

NARENS, L. A general theory of ratio-scalability with remarks about the measurement-theoretic concept of meaningfulness. *Theory and Decision*, 1981, **13**, 1–70. (b)

NARENS, L. *Abstract measurement theory.* Cambridge, Mass.: MIT Press, 1983, in preparation.

NARENS, L., & LUCE, R. D. The algebra of measurement. *Journal of Pure and Applied Algebra*, 1976, **8**, 197–233.

SCHEFFÉ, H. *The analysis of variance.* New York: Wiley, 1959.

SEDOV, L. I. *Similarity and dimensional methods in mechanics.* Moscow, 1943, 1956. English translation of 1956 edition by M. Holt & M. Friedman. New York: Academic Press, 1959.

WHITNEY, H. The mathematics of physical quantities. I. Mathematical models for measurement; II. Quantity structures and dimensional analysis. *American Mathematical Monthly*, 1968, **75**, 115–138; 227–256.

RECEIVED: February 10, 1983