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SCALES AND MEANINGFULNESS OF QUANTITATIVE LAWS*

1. INTRODUCTION

There is a view of quantitative science which goes roughly as follows. In a first stage, data are collected. Next, these data are summarized and organized along the lines of a mathematical theory, which provides a temporary explanation. Such an explanation never fits the data perfectly. The discrepancies between theory and data suggest alternative theories, and further experiments. Science pursues its course towards an increasingly reliable description and explanation of the world.

Such a view, even though certainly largely correct, only gives a simplified picture of the quantitative approach to science. In particular, there is a slow but steady recognition of the odd fact that the language itself which we use in our quantitative description of the world, conditions in a subtle way the image that we obtain. To establish with some hope of accuracy the relative importance in this image of our particular quantitative language, and of the data that it purports to explain, raises problems of considerable difficulty.

Our paper aims at providing a specific contribution to that enterprise. It will be shown, through a detailed discussion of an important, exemplary case, that if we observe certain natural and plausible conventions regarding the interplay between changes of scale of the relevant empirical variables and the invariance of the (unknown) empirical law relating them, then the possible forms of the law are extremely limited. The precise formulation of this result suggests new ways of approaching empirical data and of formulating and testing mathematical models. In spirit at least, this program can be seen as bearing some resemblance to one enunciated by Luce in his influential 1959 paper, "On the Possible Psychophysical Laws."

We shall begin by the observation that, in quoting quantitative empirical laws, scientists frequently neglect to specify the various scales entering in the equations. Illustrations of this widespread habit abound in most scientific fields. For concreteness, a couple of examples are given below, chosen for their simplicity and diversity.

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EXAMPLES. 1. (Electromagnetism.)

The force in a homogeneous isotropic medium of infinite extent between two point charges is proportional to the product of their magnitude, divided by the square of the distance between them. (Coulomb's Law, quoted from the *American Institute of Physics Handbook*, 1957).

2. (Visual Perception.)

A flash of light of short duration, presented to the eye in any condition of adaptation, provides a given effect (e.g., a brightness match against a standard) that can be achieved by the reciprocal manipulation of luminance and duration of the flash. This statement means that the given effect can be produced by a dim light that acts for a relatively long time, or by an intense flash that acts for a short time. Stated mathematically, $L \times t = C$, where L is the light intensity, t is the duration of flash, and C is a constant. This relationship is sometimes known, for human vision, as Bloch's Law (1885), or, because of its applicability to many other photochemical systems, as the Bunsen–Roscoe Law. (Quoted from Graham 1965, p. 77).

In addition to considerably simplifying the life of the student in quantitative science, such practice fortunately also makes good sense, in these and many other similar examples. Indeed, the mathematical forms of the quoted laws are unaffected by transformations of the scales entering in the equations formalizing the laws, provided that such transformations are "admissible." (We shall be more specific in a moment.) Let us illustrate this remark in the case of Example 1. Each of the numerical assignments for the three variables involved in Coulomb's law, namely, force, magnitude of charge and distance is a so-called *ratio-scale*, that is, these assignments are only defined up to multiplications by positive constants. Such multiplications play here the role of the "admissible" transformations. Let us formalize Coulomb's Law by the expression

$$(1) \qquad F_{1,2} = c\,\frac{Q_1 Q_2}{d_{1,2}^2}$$

in which: $F_{1,2}$ is the force acting on two point charges p_1, p_2; Q_1, Q_2, are the respective magnitudes of the charges; and c is the constant of proportionality. Let α_F, α_Q and α_d be three positive constants. Then, obviously, Equation 1 holds if and only if

$$(2) \qquad \alpha_F F_{1,2} = \frac{\alpha_d^2 \alpha_F C}{\alpha_Q^2} \times \frac{(\alpha_Q Q_1)(\alpha_Q Q_2)}{\alpha_d d_{1,2}^2}$$

Actually, Equation 2 is simply a restatement of Coulomb's law with the new assignments, and a different constant of proportionality. Clearly, specific properties of the mathematical form of Coloumb's law were essential in reaching our conclusions. Similar arguments could obviously be used in the case of Example 2.

The reader should notice what will be shown to be an important difference between the two examples: in Coulomb's Law the numbers assigned to two of the variables, the charges, are manipulated by the same admissible transformations. Such a situation does not arise in Bloch's Law. We will see later that whether quantities entering into a law are manipulated by the same or by different admissible transformations will have a critical impact on the form of the law.

To sum up, our discussion shows that the following three concepts are intimately interrelated:

(i) the admissible transformations of the variables (scales) entering into an equation describing some empirical law;

(ii) the mathematical expression of this equation;

(iii) the invariance of this equation under the admissible transformations of the variables.

The approach taken in this paper is to assume that we are dealing with an empirical situation that is governed by an empirical law of which we know a little of its mathematical form and a little of its invariance properties, but a lot about the structure of the admissible transformations of its variables, and use this information to greatly delimit the possible equations that express this law.

Invariance is an ill-understood but important and useful concept in mathematics and physics. In the philosophy of science it plays a critical role in the justification of certain routine scientific practices. In the theory of measurement, a particular form of invariance is often called "meaningfulness". Our notion of this concept is germane to the one in measurement, but differs from it in a critical manner, as we shall see. Both of these notions of 'meaningfulness" are very close to the notion of invariance in classical geometry, particularly in the sense of Klein in his famous Erlanger Program. Invariance also plays an important role in physics, and in particular the methods of dimensional analysis of physics can be viewed as a form of measurement-theoretic meaningfulness (Luce 1978). Examples throughout the paper will illustrate the relationship between our use of this concept and these others.

A basic concept in our developments is that of a *measurement scale*. This and related notions are defined in the next section.

2. SCALES, CONJUGATE SCALE FAMILIES

DEFINITION 1. We write Re for the set of real numbers, and Re_+ for the set of positive real numbers. Let I be any nonempty, real open interval. A strictly increasing continuous function mapping I onto I will be called a *scale on I*. The identity scale $x \to x$ on I will be denoted by ι_I. A set $\mathcal{K}$ of scales on I containing ι_I will be called a *scale family with domain I*, or more briefly, a *scale family on I*. Any subset of a scale family $\mathcal{K}$ containing the identity scale is a *scale subfamily of $\mathcal{K}$*. A scale family $\mathcal{K}$ on I satisfies *n-point homogeneity* iff for all pairs of sequences (x_i), (y_i), $1 \leq i \leq n$, in I, such that $x_i < x_{i+1}$ and $y_i < y_{i+1}$ for $1 \leq i \leq n-1$, there exists $k \in \mathcal{K}$ such that $k(x_i) = y_i$, $1 \leq i \leq n$. When $\mathcal{K}$ satisfies 1-point homogeneity, we shall simply say that $\mathcal{K}$ is *homogeneous*. A scale family $\mathcal{K}$ on I satisfies *n-point uniqueness*, where n is a positive integer, iff for any k, $k^* \in \mathcal{K}$ we have $k = k^*$ whenever $k(x_i) = k^*(x_i)$, $1 \leq i \leq n$, for some sequence of distinct points $x_1, x_2, \ldots, x_n \in I$. Clearly, if a scale family $\mathcal{K}$ satisfies n-point uniqueness, then any scale subfamily of $\mathcal{K}$ satisfies p-point uniqueness, for all $p \geq n$. On the other hand, a scale subfamily of an homogeneous scale family is not necessarily homogeneous. A scale family $\mathcal{K}$ is called a *scale group* iff $\mathcal{K}$ is a group for the operation of composition of functions. A scale family $\mathcal{K}$ is *commutative* iff for any f, $k \in \mathcal{K}$, we have $f \circ k = k \circ f$. (We write $\circ$ for the composition of functions. Notice that we do not require that $f \circ k$, $k \circ f \in \mathcal{K}$.)

REMARKS. (1) For certain technical and philosophical reasons, it is convenient to restrict considerations to scales as mappings onto their own domain. Some rationale for this choice are discussed in Narens (1980).

(2) It will be shown that the homogeneity and uniqueness conditions constitute important classification principles for scale families. In particular, the most frequently used scales (see below) fall naturally in one or the other of the categories of these classifications.

DEFINITION 2. A scale family K is called a

ratio scale family,
interval scale family, iff, respectively,
log interval scale family,

$\mathcal{K} = \{k | k(x) = \lambda_k x$, for some $\lambda_k \in \mathrm{Re}_+$, and all $x \in \mathrm{Re}_+\}$,

$\mathcal{K} = \{k | k(x) = \lambda_k x + \gamma_k$, for some $\lambda_k \in \mathrm{Re}_+$, $\gamma_k \in \mathrm{Re}$, and all $x \in \mathrm{Re}\}$,

$\mathcal{K} = \{k | k(x) = \gamma_k x^{\lambda_k}$, for some γ_k, $\lambda_k \in \mathrm{Re}_+$ and all $x \in \mathrm{Re}_+\}$.

REMARKS. (1) Notice that a ratio scale family satisfies one point uniqueness and one point homogeneity. Interval and log interval scale families satisfy 2-point uniqueness and 2-point homogeneity.

(2) Any interval scale family has a ratio scale subfamily.

Quite often the data, at an early stage of experimental research, are not coded in terms of ratio scale families or interval scale families. Effort is then exerted to recode the data in terms of such scales. The critical condition for a successful recoding is that the initial scaling shares certain basic structural properties with the intended final scaling. The next definition gives the general form of such recodings.

DEFINITION 3. Two scale families $\mathcal{K}$, $\mathcal{H}$ are called *conjugate* iff there exists a strictly increasing, continuous function u mapping the domain of $\mathcal{K}$ onto the domain of $\mathcal{H}$ such that

$$\mathcal{H} = \{h | h = u \circ k \circ u^{-1}, \text{ for some } k \in \mathcal{K}\}.$$

In such a case, we shall say that $\mathcal{H}$ is the *u-conjugate* of $\mathcal{K}$, and we shall write

$$\mathcal{H} = u\mathcal{K}u^{-1}.$$

Clearly, $\mathcal{K}$ is then the u^{-1}-conjugate of $\mathcal{H}$. Conjugation is thus a symmetric relation. It is obviously reflexive and it is also transitive since if $\mathcal{F}$ is the v-conjugate of $\mathcal{H}$, then

$$\begin{aligned}\mathcal{F} &= v\mathcal{H}v^{-1}\\ &= v(u\mathcal{K}u^{-1})v^{-1}\\ &= (v \circ u)\mathcal{K}(u^{-1} \circ v^{-1})\\ &= (v \circ u)\mathcal{K}(v \circ u)^{-1};\end{aligned}$$

that is, $\mathcal{F}$ is the $(v \circ u)$-conjugate of $\mathcal{K}$. We conclude that conjugation is an equivalence relation. A scale family is a *quasiratio* (respectively, *interval) scale family* iff it is conjugate to a ratio (respectively, interval) scale family. Notice in passing that if a scale family $\mathscr{F}$ is u-conjugate to some ratio scale family $\mathscr{H}$, then for any constants λ, θ such that $\lambda\theta > 0$, $\mathscr{F}$ is (λu^{θ})-conjugate to $\mathscr{H}$.

EXAMPLES. 3. Any log interval scale family $\mathscr{K}$ is conjugate to some interval scale family $\mathscr{H}$. Actually, $\mathscr{H}$ is the log-conjugate of $\mathscr{K}$ since

$$\log(\gamma_k e^{\lambda_k s}) = \log \gamma_k + \lambda_k s,$$

for γ_k, $\lambda_k \in \mathrm{Re}_+$ and $s \in \mathrm{Re}$. More generally, a scale family $\mathscr{F}$ is u-conjugate to an interval scale family iff it is e^u-conjugate to a log interval scale family.

4. Define

$$\mathscr{K} = \{k \mid k(x) = [(x-1)^3 + \beta_k]^{1/3} + 1, \text{ for some } \beta_k \text{ and all } x \text{ in Re}\}.$$

Then $\mathscr{K}$ is an homogeneous, commutative, scale family satisfying one-point uniqueness. It is also a quasi ratio scale family. Indeed, with

$$u(x) = e^{(x-1)^3}, \qquad \lambda_k = e^{\beta_k},$$

we have

$$u\mathscr{K}u^{-1} = \{f \mid f: x \to \lambda_f x, \text{ for some } \lambda_f \text{ and all } x \text{ in } \mathrm{Re}_+\}.$$

5. Notice that a quasi ratio scale family is commutative. More generally, many important properties of scale families are preserved under conjugation. In particular:

THEOREM 1. In scale families, the properties of n-point homogeneity, commutativity, and n-point uniqueness are preserved under conjugation.

Proof. Let I, J be the domains of two scale families $\mathscr{H}$, $\mathscr{K}$ respectively, and suppose that $\mathscr{K}$ is the u-conjugate of $\mathscr{H}$. Thus $u\mathscr{H}u^{-1} = \mathscr{K}$.

(i) Assume that $\mathscr{H}$ satisfies n-point homogenity. Let (x_i), (y_i), $1 \leq i \leq n$ be two finite sequences in J such that both $x_i < x_{i+1}$, $y_i < y_{i+1}$ for $1 \leq i \leq n-1$. Then for $1 \leq i \leq n$, successively, $u^{-1}(x_i)$, $u^{-1}(y_i) \in I$, $h[u^{-1}(x_i)] = u^{-1}(y_i)$ for some $h \in \mathscr{H}$; this yields $u\{h[u^{-1}(x_i)]\} = y_i$, and with $k = u \circ h \circ u^{-1}$, we have $k(x_i) = y_i$ with $k \in \mathscr{K}$.

(ii) Assume that $\mathscr{H}$ is commutative, and take k, $k^* \in \mathscr{K}$. Then for some h, $h^* \in \mathscr{H}$,

$$
\begin{aligned}
k \circ k^* &= (u \circ h \circ u^{-1}) \circ (u \circ h^* \circ u^{-1}) \\
&= u \circ h \circ h^* \circ u^{-1} \\
&= u \circ h^* \circ h \circ u^{-1} \\
&= k^* \circ k.
\end{aligned}
$$

(iii) Assume that $\mathscr{H}$ satisfies n-point uniqueness, and that $k(x_i) = k^*(x_i)$, $1 \leqslant i \leqslant n$ for some sequence $x_1, x_2, \ldots, x_n$ of distinct points of J and some k, $k^* \in \mathscr{K}$. There are $h, h^* \in \mathscr{H}$ such that $k = u \circ h \circ u^{-1}$, $k^* = u \circ h^* \circ u^{-1}$. This entails

$$u\{h[u^{-1}(x_i)]\} = u\{h^*[u^{-1}(x_i)]\}, \qquad 1 \leqslant i \leqslant n,$$

yielding $h = h^*$ by the strict monotonicity of u and the n-point uniqueness of $\mathscr{H}$, and hence, $k = k^*$.

Q.E.D.

For example, any scale family conjugate to a ratio scale family satisfies one-point uniqueness and commutativity. The following result (Narens 1981) captures the structure of quasi ratio scale families and quasi interval scale families in terms of their homogeneity and uniqueness properties.

THEOREM 2. Let $\mathscr{H}$ be a scale group satisfying N-point homogeneity and N-point uniqueness. Then $N \leqslant 2$. Moreover: if $N = 1$, then $\mathscr{H}$ is a quasi ratio scale family; if $N = 2$, then $\mathscr{H}$ is a quasi interval scale family.

Thus in particular, in the case $N = 1$, $\mathscr{H}$ is commutative (cf. Blaschke and Bol, 1938; Aczel, Belousof, and Hosszu, 1960; Levine, 1970). The impossibility of the existence of a case $N > 2$ in this Theorem may be part of the reason why so few scale families have arisen in science.

In this paper, various conditions and results of Krantz et al. (1971) will be used. This reference will be abbreviated as F.M. I. The second, forthcoming volume of this work will be referred to as F.M. II.

3. MEANINGFUL, ISOTONE FAMILIES OF NUMERICAL CODES

We shall consider an empirical situation in which the data collected have been coded numerically in terms of two input quantities

$$(a, x),$$

which are respectively evaluated in terms of two (input) scales

$$f \in \mathscr{F},\ g \in \mathscr{G},$$

yielding an output quantity P, a function of (a, x). However, in the course of determining P, the input scales f, g are used, and to allow for this dependence we will use the explicit notation

$$M_{f,g}(a, x),$$

to specify the output P. Thus $M_{f,g}$ is a real valued function of two real variables. In the sequel such functions will be referred to as *numerical codes*. Even though such detailed notation may appear at first needlessly heavy, our development will show that it is fully justified, in fact, unavoidable. Clearly, the empirical situation is compatible with many numerical codes. To the extent that these numerical codes bear a "strong resemblance" to one another, one is tempted to describe the situation as "lawful" as is often done in practice. But what is the meaning of "strong resemblance"? The next and following definitions try to capture some important aspects of this slippery concept.

DEFINITION 4. Let $\mathscr{F}$, $\mathscr{G}$ be two scale families on A, X respectively, let R be a subset of $\mathscr{F} \times \mathscr{G}$, such that $(\iota_A, \iota_X) \in R$. For any $(f, g) \in R$, let $M_{f,g}$ be a real valued, continuous function defined on $A \times X$, strictly increasing in the first argument, and strictly monotonic in the second argument. Then

$$\mathscr{M} = \{M_{f,g} \mid (f, g) \in R\}$$

is a *family of numerical codes*. (Some remarks on the role of R will be made shortly.) Each $M_{f,g} \in \mathscr{M}$ will be referred to as a *numerical code*. If $\mathscr{F}$, $\mathscr{G}$ are homogeneous, we shall say by extension that $\mathscr{M}$ is homogeneous. Since R is technically a binary relation from $\mathscr{F}$ to $\mathscr{G}$, we shall often abbreviate $(f, g) \in R$ as fRg. Notice that, by definition

$$M_{\iota_a, \iota_X} \in \mathscr{M}.$$

For simplicity, we shall adopt the abbreviation

$$M = M_{\iota_A, \iota_X}.$$

We shall occasionally refer to M as the *initial code*. We are now in a position to formulate a very general invariance property for families of numerical codes.

DEFINITION 5. A family $\mathcal{M} = \{M_{f,g} | fRg\}$ of numerical codes is *(one-to-one) meaningful* iff whenever fRg, f^*Rg^*, then

$$M_{f,g}[f(a), g(x)] = M_{f,g}[f(b), g(y)]$$

(3) iff

$$M_{f^*,g^*}[f^*(a), g^*(x)] = M_{f^*,g^*}[f^*(b), g^*(y)]$$

for all points a, b in the domain of f, f^*, and all points x, y in the domain of g, g^*. For convenience, the specification "one-to-one" will be omitted in the sequel. The family $\mathcal{M}$ is called *order-meaningful* iff the two equations in the equivalence (3) are replaced by identical inequalities (say, $\leq$). By abuse of language, we shall sometimes say that a particular numerical code, or a particular numerical law, is *(order-)meaningful*, to signify that the corresponding family $\mathcal{M}$ of numerical codes, with a domain made clear by the context, is (order-) meaningful. A similar convention will be used freely throughout this paper for other properties of a family $\mathcal{M}$ of numerical codes.

REMARKS. (1) The relation R allows for a suitable generality in our definitions. In Bloch's Law, we have $R = \mathcal{F} \times \mathcal{G}$: This law can be formulated for any choice of two scales, measuring light intensity and duration. In the case of Coulomb's Law, we have $\mathcal{F} = \mathcal{G}$ and R is the identity function of $\mathcal{F}$: the magnitudes of the two charges are measured using the same scale. (Remember that the distance between the two points is assumed to remain constant.)

(2) We stress that, in one important respect, our definition of "meaningfulness" differs from that most frequently encountered in the measurement literature. A key feature of this definition is that it applies to a family of relations (the family of numerical codes), rather than to a single relation. We shall go back to this point in our discussion section.

(3) Some reflection will probably convince that the concept of meaningfulness, as defined here, represents a rather minimal (yet essential) requirement for a family of numerical codes to be worthy of consideration for scientific purposes. This may not be obvious. The following remarks may help the reader's examination of this notion.

Any numerical code $M_{f,g}$ is a translation, depending on the chosen scales f,g, of a collection of empirical facts. To be specific, to the Cartesian product $A \times X$ corresponds a set $A^\circ \times X^\circ$ of empirical situations (inputs). Thus, $(\alpha, \xi) \in (A^\circ \times X^\circ)$ is an empirical situation characterized by two aspects that are in general non-numerical. There is also an empirical set E° (of outputs), and a mapping

$$(\alpha, \xi) \to \rho(\alpha, \xi)$$

of $A^\circ \times X^\circ$ onto E°. The notation $\rho(\alpha, \xi)$ symbolizes the output in E^0 generated by the input (α, ξ). We assume that some initial scaling has taken place, involving two real valued mappings

$$\alpha \to \alpha, \quad \xi \to \xi'$$

respectively of A° onto A and X° onto X.

From an empirical viewpoint, the critical information is contained in the function ρ. This information should be preserved by any numerical translation. In particular, the fact that two inputs (α, ξ), (β, ζ) generate the same output should be preserved no matter which numerical code $M_{f,g}$ is chosen. Symbolically,

$$\begin{aligned} &\rho(\alpha, \xi) = \rho(\beta, \zeta) \\ (4) \qquad &\text{iff} \\ &M_{f,g}[f(\alpha'), g(\xi')] = M_{f,g}[f(\beta'), g(\zeta')]. \end{aligned}$$

Clearly, this leads to the definition of meaningfulness of families of numerical codes adopted here. The definition of order-meaningfulness arises when there is a natural ordering on the set E° of outputs.

(4) Meaningfulness and order-meaningfulness of families of numerical codes are conditions which are both weak and natural to postulate. Meaningfulness concepts in general (usually referred to by other names such as "invariance") have a long history in mathematics and science. These concepts are concerned with the invariance of sets and relations under admissible classes of transformations. The following example from geometry illustrates this concept and its relationships to Definitions 4 and 5.

Let Π be the results of an initial scaling of the plane, i.e., the points of Π are input quantities (x, y) where x and y are real numbers. Different geometries on Π can be specified in many ways. For example, the Euclidean geometry is obtained by defining appropriate geometric entities in terms of the Euclidean distance function,

$$D[(x_1, y_1), (x_2, y_2)] = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2},$$

It follows that $\langle \Pi, D \rangle$ can be considered as a formulation of the

Euclidean geometry on Π. It can be shown that the set of transformations on Π that leave D invariant are exactly those that are generated by the translations, rotations, and reflections of Π. Call this set of transformations the "Euclidean transformations (on Π)". It turns out that Euclidean concepts, and therefore also the Euclidean geometry on Π, can be specified directly in terms of these transformations: a concept about Π is said to be "meaningful" for Euclidean geometry if and only if it is invariant under Euclidean transformations; thus line and circle are meaningful Euclidean concepts since Euclidean transformations map the class of lines onto itself and the class of circles onto itself. In addition, D itself is meaningful, and it can be shown that all meaningful distance functions (called technically, *metrics*) on Π are of the form rD where r is a positive real. Another geometry on Π, and one more analogous to the situation described in this paper, is obtained by considering transformations on Π resulting from positive linear transformations of coordinates of points of Π, i.e., transformations of the form

$$(5) \qquad (x, y) \to (rx + s, ty + u) \qquad r, s, t, u \in \mathrm{Re}, r > 0, t > 0.$$

Let $\mathcal{F}$ be the set of transformations defined in (5). Suppose that $\mathcal{G} = \mathcal{F}$ and $R = \mathcal{F} \times \mathcal{F}$. Then we are in a setting covered by Definitions 4 and 5. Consider the set S of ellipses with an axis in the direction of the abscissa. In this context, S is meaningful since the appropriate transformations are mappings of S onto itself. (Notice however that, unlike the Euclidean case, the concept of circle is not meaningful.) In the spirit of Definitions 4 and 5, consider an empirical situation where the relationship between the two variables of interest has been initially scaled so that it can be expressed in Π by $x^2 + y^2 = 1$, the equation of the unit circle. (That this equation expresses a relation between x and y rather than a function, as in Definitions 4 and 5, is unimportant for this discussion.) We will describe this equation by $E_{\iota,\iota}$ where (ι, ι) stands for, depending upon how one looks at it, the identity transformation or the result of the initial scaling of the relationship between the empirical variables. The notation $E_{f,g}$ will stand for the transformation of $E_{\iota,\iota}$ by $(f, g) \in \mathcal{F} \times \mathcal{F}$. It is easy to verify that $E_{f,g}$ is

$$(6) \qquad \frac{(x-a)^2}{r^2} + \frac{(y-b)^2}{s^2} = 1$$

for some $a, b, r, s \in \mathrm{Re}$, i.e., $E_{f,g}$ is the equation of an ellipse with an axis in the direction of the abscissa. The set of equations $\{E_{f,g} | (f, g) \in \mathcal{F} \times \mathcal{F}\}$ captures numerically the original relationship of the variables, and this set corresponds in a natural way to the set of ellipses. In practice, this set of equations is said to express the "numerical law" that connects the original two empirical variables.

The following simple consequence of Definition 5 will be useful in the sequel.

THEOREM 3. A family $\mathcal{M} = \{M_{f,g} | fRg\}$ of numerical codes is meaningful (respectively, order-meaningful) iff for all $(f, g) \in R$, there exists a one-to-one (respectively, strictly increasing, continuous) function $H_{f,g}$ mapping the range of the initial code M onto the range of $M_{f,g}$, such that

(7) $$H_{f,g}[M(a, x)] = M_{f,g}[f(a), g(x)]$$

whenever both members are defined.

Indeed, if $\mathcal{M}$ is meaningful, it suffices to define $H_{f,g}$ by (7): the equivalence (6), with f^* and g^* as the two identity scales, ensures that $H_{f,g}$ is a well defined function and has the required properties. In the case of order-meaningfulness, the function $H_{f,g}$ is strictly increasing by definition, which implies by the continuity of the numerical codes M and $M_{f,g'}$ that $H_{f,g}$ is also continuous. The converse is clear.

However obvious, the following consequence of this Theorem deserves explicit mention.

COROLLARY. If a family $\mathcal{M} = \{M_{f,g} | fRg\}$ of numerical codes is order-meaningful and the range of each element of $\mathcal{M}$ is the same nonempty open interval, then there exists a scale family $\mathcal{H}$ and a function $(f, g) \to H_{f,g}$ from R onto $\mathcal{H}$, such that Equation (7) holds.

The generalization of Definitions 4, 5 to functions of more than two variables is certainly clear to the reader. It turns out then that practically all numerical laws of importance result from considering families of numerical codes which are meaningful in this generalized sense. It is not difficult, however, to manufacture examples of hypothetical "laws" which are not meaningful.

EXAMPLE 6. (Psychophysical choice). Let x, y be numbers representing the light intensities of two visual stimuli. We assume that the numerical assignments x, y involve a ratio scale family. Let $P(x, y)$ be the probability that x is judged as brighter than y by a subject.

Suppose that

$$(8) \qquad P(x, y) = F[(1+x)/(1+y)]$$

where F is assumed to be a strictly increasing, continuous function. A critical piece of information is missing in the notation of the above equation: the particular scale used is not indicated. Thus, we rewrite (8) more explicitly as

$$P_\lambda(x, y) = F_\lambda[(1+x)/(1+y)]$$

where $\lambda > 0$ symbolizes the scale in the sense that $x = \lambda x'$, $y = \lambda y'$ with x', y' representing the initial scaling of the intensities. The family $\{P_\lambda\}$ of numerical codes is easily shown not to be meaningful: It cannot be the case that

$$P_\lambda(\lambda x, \lambda y) = P_\lambda(\lambda z, \lambda w)$$

iff

$$P_{\lambda^*}(\lambda^* x, \lambda^* y) = P_{\lambda^*}(\lambda^* z, \lambda^* w)$$

for all $x, y, z, w, \lambda, \lambda^* \in (0, \infty)$.

The concepts introduced in the following definition will play a central role in subsequent developments.

DEFINITION 6. Let $\mathscr{F}$, $\mathscr{G}$ be two scale families on A, X respectively; let $\mathscr{M} = \{M_{f,g} \mid fRg\}$ be a family of numerical codes, with $R \subset \mathscr{F} \times \mathscr{G}$. A numerical code $M_{f,g} \in \mathscr{M}$ is called *dimensionally invariant* iff whenever $f^* R g^*$, then

$$M_{f,g}[f^*(a), g^*(x)] \leq M_{f,g}[f^*(b), g^*(y)]$$

iff

$$M_{f,g}(a, x) \leq M_{f,g}(b, y)$$

for all $a, b \in A$ and $x, y \in X$. The family $\mathscr{M}$ is called *dimensionally invariant* iff all its numerical codes are dimensionally invariant. This definition generalizes the classical notion used in dimensional analysis (cf. Causey, 1969, or F.M.I.). Notice that a numerical code $M_{f,g} \in \mathscr{M}$ is dimensionally invariant iff for all $f^* R g^*$, there exists a strictly increasing, continuous function $Q_{f,g;f^*,g^*}$ such that

$$M_{f,g}[f^*(a),\, g^*(x)] = Q_{f,g;\, f^*,g^*}[M_{f,g}(a,\, x)]$$

whenever both members of this equation are defined.

We shall say that $\mathcal{M}$ is *isotonically generated*, or more simply, *isotone*, iff there exists a real valued function M^*, defined on $A \times X$ such that whenever fRg, then

$$(9) \qquad M_{f,g} = m_{f,g} \circ M^*$$

for some strictly increasing, continuous function $m_{f,g}$ mapping the range of M^* onto the range of $M_{f,g}$. Notice that there is no loss of generality in assuming that $M^* = M$. Indeed, we have by definition of isotonicity

$$M = m_{\iota_A, \iota_X} \circ M^*$$

which yields for any $(f,\, g) \in R$,

$$M_{f,g} = m_{f,g} \circ m^{-1}_{\iota_A, \iota_X} \circ M,$$

and the function $m_{f,g} \circ m^{-1}_{\iota_A, \iota_X}$ is strictly increasing, continuous, and maps the range of M onto the range of $M_{f,g}$. In other terms, if $\mathcal{M}$ is isotone, then any numerical code $M_{f,g}$ can be obtained by some strictly increasing, continuous transformation of M.

A comparison of (7) and (9) (with $M^* = M$), together with the definition of dimensional invariance may suggest that the conditions of order-meaningfulness, dimensional invariance, and isotonicity are related. Actually, these three conditions are pairwise independent.

In particular, it is readily checked that Example 6, in which isotonicity holds, fails to satisfy both order-meaningfulness and dimensional invariance. The two Examples below establish the independence for the remaining cases.

EXAMPLE 7. (Psychophysical choice revisited.) With the binary probabilities $P_\lambda(x,\, y)$ having the same meaning as in Example 6, suppose that

$$(10) \qquad P_\lambda(x\, y) = F_\lambda[(\lambda K + x)/(\lambda K + y)]$$

for all $x,\, y,\, \lambda > 0$, where $K > 0$ is a constant and F_λ is a strictly increasing, continuous function. Then (10) is order-meaningful, since for any $x,\, y,\, z,\, w,\, \lambda > 0$, we have

$$P_\lambda(\lambda x,\, \lambda y) \leq P_\lambda(\lambda z,\, \lambda w)$$

iff

$$F_\lambda[(\lambda K + \lambda x)/(\lambda K + \lambda y)] \leqslant F_\lambda[(\lambda K + \lambda z)/(\lambda K + \lambda w)]$$

iff

$$(K + x)/(K + y) \leqslant (K + z)/(K + w).$$

This last expression is independent of λ. The reader can verify that the family of numerical codes defined by (10) is neither isotone, nor dimensionally invariant.

8. Define

$$\mathcal{M} = \{M_\lambda \mid M_\lambda(x, y) = x + \lambda y; \quad x, y, \lambda > 0\}.$$

The conventions regarding x, y and λ are as in Example 7: the quantities x and y are measured by an identical scale, denoted by λ, in a ratio scale family. Clearly, the family $\mathcal{M}$ is dimensionally invariant, but neither isotone, nor order-meaningful.

The interrelationship of these concepts is expressed in the following theorem, which also summarizes the above independence results.

THEOREM 4. The property of order-meaningfulness, isotonicity and dimensional invariance are pairwise independent. However, any two of these conditions implies the third.

Proof. Let $\mathcal{M} = \{M_{f,g} \mid fRg\}$ be a family of numerical codes.

(i) *dimensional invariance and isotonicity imply order-meaningfulness.* For any $M_{f,g} \in \mathcal{M}$, using successively dimensional invariance and isotonicity,

$$\begin{aligned} M_{f,g}[f(a), g(x)] &= Q_{f,g;f,g}[M_{f,g}(a, x)] \\ &= (Q_{f,g;f,g} \circ m_{f,g})[M(a, x)] \\ &= H_{f,g}[M(a, x)], \end{aligned}$$

with $H_{f,g} = Q_{f,g} \circ m_{f,g}$ strictly increasing and continuous. By Theorem 3, we conclude that $\mathcal{M}$ is order-meaningful.

(ii) *order-meaningfulness and isotonicity imply dimensional invariance.* Let $\mathcal{M}$ be an order-meaningful family of numerical codes. If $M_{f,g} \in \mathcal{M}$, we have by Theorem 3,

$$M_{f,g}[f(a), g(x)] = H_{f,g}[M(a, x)] \tag{11}$$

where $H_{f,g}$ is a strictly increasing, continuous function. Assuming that $\mathcal{M}$ is also isotone, we obtain

$$M_{f,g}[f(a), g(x)] = m_{f,g}\{M[f(a), g(x)]\}, \tag{12}$$

with $m_{f,g}$ again a strictly increasing, continuous function. Using successively (12) and (11), we have

$$\begin{aligned} M[f(a), g(x)] &= m_{f,g}^{-1}\{M_{f,g}[f(a), g(x)]\} \\ &= (m_{f,g}^{-1} \circ H_{f,g})[M(a, x)] \end{aligned}$$

which implies that M is dimensionally invariant, since $m_{f,g}^{-1} \circ H_{f,g}$ is strictly increasing and continuous.

It follows that any numerical code $M_{f,g} \in \mathcal{M}$ is dimensionally invariant. Indeed, whenever $f^* R g^*$,

$$\begin{aligned} M_{f,g}[f^*(a), g^*(x)] &= m_{f,g}\{M[f^*(a), g^*(x)]\} \quad \text{(isotonicity)} \\ &= Q_{f,g;\, f^*,g^*}[M_{f,g}(a, x)], \end{aligned}$$

with strictly increasing and continuous

$$Q_{f,g;\, f^*,g^*} = m_{f,g} \circ Q_{\iota,\iota;\, f^*,g^*} \circ m_{f,g}^{-1},$$

where of course by using the already established dimensional invariance of M, $Q_{\iota,\iota,\, f^*,g^*}$ is defined by

$$M(f^*(a), g^*(x)) = Q_{\iota,\iota,\, f^*,g^*}[M(a, x)].$$

(iii) *dimensional invariance and order meaningfulness imply isotonicity*. It is sufficient to assume that M is dimensionally invariant. With $J_{f,g} = Q_{\iota,\iota;\, f,g}$, we have

$$J_{f,g}[M[f(a), g(x)] = M(a, x). \tag{13}$$

Combining (11) and (13), we obtain

$$M_{f,g}[f(a), g(x)] = (H_{f,g} \circ J_{f,g})\{M[f(a), g(x)]\}$$

where $m_{f,g} = H_{f,g} \circ J_{f,g}$ is strictly increasing and continuous. Since any point (b, y) in the common domain $A \times X$ of $M_{f,g}$ and M can be written $f(a) = b$, $g(x) = y$ for some $a \in A$, $x \in X$; isotonicity follows.

Q.E.D.

REMARK. Notice that, in part (iii) of this proof, the isotonicity of $\mathcal{M}$ followed from the assumption of order-meaningfulness and the dimensional invariance of the initial code M.

4. MULTIPLICATIVE REPRESENTATIONS

Suppose that the numerical codes $M_{f,g}$ in a family $\mathcal{M}$ have multiplicative representations

$$(14) \qquad M_{f,g}(a, x) = F^*[u(a)h(x)]$$

where the functions u, h and F^* are real valued, with u, $h > 0$, and have appropriate continuity and monotonicity properties. In general, the functions u, h and F^* may depend upon the choice of the scales f, g. For isotone families of numerical codes however, it may be assumed that u, h do not depend on f, g. Indeed, with $m_{f,g}$ as in Definition 6, we have

$$\begin{aligned} M_{f,g}(a, x) &= m_{f,g}[M(a, x)] \\ &= (m_{f,g} \circ F)[u(a)h(x)], \end{aligned}$$

where

$$M(a, x) = F[u(a)h(x)]$$

yielding

$$F^* = m_{f,g} \circ F.$$

This remark justifies the following definition.

DEFINITION 7. Let $\mathcal{F}$, $\mathcal{G}$ be two scale families on A, X respectively; let $\mathcal{M} = \{M_{f,g} \mid fRg\}$, $R \subset \mathcal{F} \times \mathcal{G}$, be an isotone family of numerical codes. Then (u, h) is a *multiplicative representation* of $\mathcal{M}$ iff u, h are continuous functions taking values in the positive reals and defined on A, X respectively, such that for all $a \in A$, $x \in X$,

$$M(a, x) = F[u(a)h(x)]$$

where F is a strictly increasing, continuous function. Thus, u is strictly increasing, and h is strictly monotonic. Occasionally, when an isotone family $\mathcal{M}$ of numerical codes has a multiplicative representation, we shall simply say that $\mathcal{M}$ is *multiplicative*.

THEOREM 5. Let $\mathcal{F}$, $\mathcal{G}$ be scale families of A, X respectively and suppose that $\mathcal{M} = \{M_{f,g} \mid f \in \mathcal{F}, g \in \mathcal{G}\}$ is an isotone family of numerical codes. Then, any two of the following three conditions implies the third:

(i) $\mathcal{M}$ is order-meaningful;

(ii) $\mathcal{M}$ has a multiplicative representation u^{θ}, h^{δ} where $\theta > 0$ and $\delta \neq 0$ are constants, and both u and h are strictly increasing;

(iii) $\mathcal{F}$, $\mathcal{G}$ are respectively u-conjugate and h-conjugate to ratio scale families.

The kinds of interconnections between variables captured in Theorem 5 and its Corollary are fundamental throughout much of science. The characterizations presented here are immediate consequences of Section 4 of Narens (1981), although the methods of proof developed here appear to us to be more straightforward and likely to lead to fruitful generalizations. Other characterizations have been presented in the literature (Narens and Luce 1976; Narens 1981) for cases where naturally defined operations exist on the variables.

In the sequel, we write Ran(f) for the range of any function f.

Proof. (i), (ii) *imply* (iii). Since by Theorem 4, $\mathcal{M}$ is dimensionally invariant, we can assert the existence, for any $f \in \mathcal{F}$ and $g \in \mathcal{G}$, of a strictly increasing continuous function $K_{f,g}$ such that

$$M(a, x) = K_{f,g}\{M[f(a), g(x)]\},$$

for all $a \in A$ and $x \in X$. Since by hypothesis, (u^{θ}, h^{δ}) is a multiplicative representation of $\mathcal{M}$, we have a continuous, strictly increasing function F such that

$$F[u^{\theta}(a)h^{\delta}(x)] = (K_{f,g} \circ F)\{u^{\theta}[f(a)]h^{\delta}[g(x)]\}, \tag{15}$$

for all $a \in A$, $x \in X$, $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Let us assume (temporarily) that M is strictly increasing in the second variable. Then, $\delta > 0$. In fact since, as mentioned earlier, $\mathcal{F}$, $\mathcal{G}$ are respectively u-conjugate and h-conjugate to ratio scale families iff they are respectively u^{θ}-conjugate and h^{δ}-conjugate to ratio scale families, we may as well assume $\theta = \delta = 1$. Fixing f, g in (15) and writing

$$L_{f,g} = F^{-1} \circ K_{f,g}^{-1} \circ F$$

$$u_f = u \circ f \circ u^{-1}, \qquad h_g = h \circ g \circ h^{-1}$$

and

$$u(a) = s, \qquad h(x) = t,$$

(15) is transformed into

$$L_{f,g}(st) = u_f(s)h_g(t). \tag{16}$$

Without loss of generality, we may assume

$$1 \in \mathrm{Ran}(u) \cap \mathrm{Ran}(h),$$

where as indicated earlier, Ran denotes the range of the functions. Notice that all three functions $L_{f,g}$, u_f and h_g in (16) are measurable. Using standard functional equation results (Aczel 1966) we obtain for all $s \in \mathrm{Ran}(u) \cap \mathrm{Ran}(h)$,

(17) $\quad L_{f,g}(s) = \beta_f \gamma_g s^{\alpha_{f,g}},$

(18) $\quad u_f(s) = \beta_f s^{\alpha_{f,g}},$

(19) $\quad h_g(s) = \gamma_g s^{\alpha_{f,g}},$

for some constants β_f, γ_g, $\alpha_{f,g} > 0$ which however may depend on the choice of scales $f \in \mathscr{F}$ and $g \in \mathscr{G}$ as suggested by our notation. The results in (17), (18) and (19) are readily extended to the whole domain of the function. (We leave it to the reader to check this.) Since the left members of (18), (19) do not depend upon g, f respectively, the same property applies to the right members, which yields

$$\alpha_{f,g} = \alpha,$$

a constant, for all $f \in \mathscr{F}$ and $g \in \mathscr{G}$. We obtain for (18)

$$(u \circ f \circ u^{-1})(s) = \beta_f s^{\alpha},$$

with in particular, since $\iota_A \in \mathscr{F}$,

$$s = \beta_{\iota_A} s^{\alpha},$$

which gives $\beta_{\iota_A} = \alpha = 1$. This establishes the fact that $u\mathscr{F}u^{-1}$ is a ratio scale family. A similar argument, using Equation (19) shows that $h\mathscr{G}h^{-1}$ is also a ratio scale family. In the case where M is strictly decreasing in the second variable, we have $\delta < 0$ and a ratio representation

$$M(a, x) = F[u^{\theta}(a)/h^{-\delta}(x)].$$

We assume $\theta = -\delta = 1$, which leads to

$$L_{f,g}(s/t) = u_f(s)/h_g(t),$$

replacing (16), with a practically identical development. We leave the details to the reader.

(ii), (iii) *imply* (i). Using successively isotonicity ($M_{f,g} = m_{f,g} \circ M$, $m_{f,g}$ strictly increasing and continuous), (ii) and (iii), we have for all $f \in \mathscr{F}$, $g \in \mathscr{G}$, a, $b \in A$ and x, y, $\in X$,

$$M_{f,g}[f(a), g(x)] \leqslant M_{f,g}[f(b), g(y)]$$

iff

$$m_{f,g}\{M[f(a), g(x)]\} \leqslant m_{f,g}\{M[f(b), g(y)]\}$$

iff

$$(m_{f,g} \circ F)\{u^\theta[f(a)]h^\delta[g(x)]\} \leqslant (m_{f,g} \circ F)\{u^\theta[f(b)]h^\delta[g(y)]\} \tag{20}$$

iff

$$u^\theta(a)\beta_f^\theta h^\delta(x)\gamma_g^\delta \leqslant u^\theta(b)\beta_f^\theta h^\delta(y)\gamma_g^\delta \tag{21}$$

iff

$$u^\theta(a)h^\delta(x) \leqslant u^\theta(b)h^\delta(y),$$

which is independent of f, g. In (20), (21) we use the fact that, in view of (iii), any $f \in \mathscr{F}$ and $g \in \mathscr{G}$ are of the form

$$(u \circ f \circ u^{-1})(s) = \beta_f s, \qquad \beta_f > 0$$

$$(h \circ g \circ h^{-1})(s) = \gamma_g s, \qquad \gamma_g > 0.$$

This establishes the order-meaningfulness of $\mathscr{M}$.

(i), (iii) *imply* (ii). As a first step, we show that M satisfies Double Cancellation (F.M. I. p. 256); namely

$$M(a, x) \leqslant M(b, y) \tag{22}$$

$$M(b, z) \leqslant M(c, x) \tag{23}$$

imply

$$M(a, z) \leqslant M(c, y) \tag{24}$$

for all $a, b, c \in A$ and $x, y, z \in X$. Because $\mathscr{G}$ is conjugate to a ratio scale family, it is homogeneous. There are thus $g, g^* \in \mathscr{G}$ satisfying $g(x) = z$, $g^*(z) = g(y)$. Since $\mathscr{M}$ satisfies isotonicity and order-meaningfulness it is dimensionally invariant by Theorem 4; in particular, M is dimensionally invariant. This implies the existence of strictly increasing continuous functions Ψ_g, Ψ_{g^*} such that successively, using (22), (23),

$$\begin{aligned} M(a, z) &= M[a, g(x)] = \Psi_g[M(a, x)] \leqslant \Psi_g[M(b, y)] \\ &= M[b, g^*(z)] = \Psi_{g^*}[M(b, z)] \leqslant \Psi_{g^*}[M(c, x)] \\ &= M[c, g^*(x)] = M(c, y). \end{aligned}$$

Indeed, from the commutativity of $\mathscr{G}$ (which follows from the fact that $\mathscr{M}$ is conjugate to a ratio family) we have

$$g^*(x) = (g^* \circ g^{-1})(z) = (g^* \circ g^{-1} \circ g^{*-1} \circ g)(y) = y.$$

We conclude that (22), (23) imply (24); that is, Double Cancellation holds. In view of the isotonicity and continuity properties of M, we can assert, using standard measurement results (F.M. I.), the existence of a multiplicative representation (v, p) satisfying

$$M(a, x) = F[v(a)p(x)] \tag{25}$$

for some strictly increasing, continuous function F and all $a \in A$, $x \in X$. Using the result that (i) and (ii) imply (iii), which we established earlier, it follows that $\mathscr{F}$, $\mathscr{G}$ are respectively v-conjugate and p-conjugate to ratio scale families. We obtain in particular for each $f \in \mathscr{F}$, the existence of constants ξ_f, $\beta_f > 0$ such that

$$v[f(a)] = \xi_f v(a),$$

$$u[f(a) = \beta_f u(a),$$

which yields

$$v^{-1}[\xi_f v(a)] = u^{-1}[\beta_f u(a)]$$

with

$$\xi_{\iota_A} = \beta_{\iota_A} = 1.$$

Or letting $u(a) = s$ and noticing that $\beta_f \to \xi_f$ is a strictly increasing continuous function,

$$(v \circ u^{-1})(\beta s) = \xi(\beta)(v \circ u^{-1})(s)$$

for all positive β. Using the homogeneity of $\mathscr{F}$ and the fact that the functions $v \circ u^{-1}$ and $\beta \to \xi(\beta)$ are measurable and defined on an interval containing 1, we get

$$(v \circ u^{-1})(s) = \tau s^\theta$$

that is

$$v(a) = \tau u^\theta(a) \tag{26}$$

for some constant τ, $\delta > 0$. A similar argument, applied to the scale family $\mathcal{G}$ yields the equation

$$p(x) = \tau^* h^\delta(x), \tag{27}$$

with τ^*, $\delta \neq 0$. Substituting v, p in (25) by their expression in terms of u, h in (26), (27), we obtain

$$\begin{aligned} M(a, x) &= F[\tau u^\theta(a) \tau^* h^\delta(x)] \\ &= G[u^\theta(a) h^\delta(x)], \end{aligned}$$

with

$$G(t) = F(\tau\tau^* t).$$

This completes the proof of Theorem 5.

Q.E.D.

COROLLARY. Let $\mathcal{F}$, $\mathcal{G}$ be two ratio scale families, and suppose that $\mathcal{M} = \{M_{f,g} | f \in \mathcal{F}, g \in \mathcal{G}\}$ is an isotone, order-meaningful family of numerical codes. Then, there exist constants $\theta > 0$ and $\delta \neq 0$ such that for all $a \in A$ and $x \in X$,

$$M(a, x) = G(a^\theta x^\delta),$$

where G is a strictly increasing, continuous function.

Indeed, this is an application of the case "(i), (iii) implies (ii)" in Theorem 5, in which u, h are identity functions.

We now turn to the case in which a family $\mathcal{M} = \{M_{f,g}\}$ of numerical codes involves identical scales for the two components; namely $M_{f,g} \in \mathcal{M}$ implies $f = g$. Our results which are formulated in Theorems 6 and 7 are somewhat weaker, as we shall see.

THEOREM 6. Let $\mathcal{F}$ be a scale family, and let $\mathcal{M} = \{M_{f,f} | f \in \mathcal{F}\}$ be an isotone, order-meaningful, multiplicative family of numerical codes. Then $\mathcal{F}$ is a quasi interval scale family. In particular, if (u^θ, h^δ) is a multiplicative representation of $\mathcal{M}$, with h a strictly increasing function and $\theta > 0$ and $\delta \neq 0$, then both $u\mathcal{F}u^{-1}$ and $h\mathcal{F}h^{-1}$ are log interval scale families.

Obviously, $\mathcal{F}$ may be a quasi ratio scale family. Coulomb's Law is an example of the situation described in the Theorem (the distance between the points being kept constant) in which $\mathcal{F}$ is actually a ratio scale family.

Proof. Since the arguments are similar to those used to establish that (i), (ii) imply (iii) in Theorem 5, only a sketch of the proof will be given. Let A be the domain of $\mathscr{F}$ and let (u^θ, h^δ) be the multiplicative representation of $\mathscr{M}$. As in the proof of Theorem 5, we obtain by Theorem 4 and dimensional invariance, the equation

$$M(a, b) = K_f\{M[f(a), f(b)]\}$$

leading to

$$F[u^\theta(a)h^\delta(b)] = (K_f \circ F)\{u^\theta[f(a)]h^\delta[f(b)]\} \tag{28}$$

for all $a, b \in A$ and $f \in F$. Fixing f, and defining

$$L_f = F^{-1} \circ f \circ F$$
$$u_f = u^\theta \circ f \circ (u^\theta)^{-1}, \qquad h_f = h^\delta \circ f \circ (h^\delta)^{-1},$$
$$u^\theta(a) = s, \qquad h^\delta(b) = t,$$

we rewrite (28) as

$$L_f(st) = u_f(s)h_f(t),$$

which yields

$$u_f(s) = \beta_f s^{\alpha_f},$$
$$h_f(s) = \gamma_f s^{\alpha_f},$$

for some constants α_f, β_f, $\gamma_f > 0$. More explicitly, we have

$$(u \circ f \circ u^{-1})(s) = \beta_f^{1/\theta} s^{\alpha_f},$$
$$(h \circ f \circ h^{-1})(s) = \gamma_f^{1/\delta} s^{\alpha_f},$$

thus, both $u\mathscr{F}u^{-1}$ and $h\mathscr{F}h^{-1}$ are log interval scale families. That is, $\mathscr{F}$ is a quasi interval scale family.

Q.E.D.

We have a converse of this Theorem.

THEOREM 7. Let $\mathscr{F}$ be a scale family, and let $\mathscr{M} = \{M_{f,f} \mid f \in \mathscr{F}\}$ be an isotone family of numerical codes, with a multiplicative representation (u^θ, h^δ), $\theta > 0$, $\delta \neq 0$. Then $\mathscr{M}$ is order-meaningful if either

(i) $\mathscr{F}$ is both u-conjugate and h-conjugate to a ratio scale family; or

(ii) $h = u$ and $\mathscr{F}$ is u-conjugate to a log interval scale family.

Moreover, these are the only two possibilities.

Proof. Let A be the domain of $\mathscr{F}$ and suppose that $\mathscr{F}$ is both u-conjugate and h-conjugate to a log interval scale family; that is $f \in \mathscr{F}$ implies

$$u[f(a)] = \beta_f u(a)^{\alpha_f}$$

$$h[f(a)] = \gamma_f h(a)^{\tau_f}$$

for some constants β_f, α_f, γ_f, $\tau_f > 0$ and all $a \in A$. Notice that this assumption is compatible with both (i) and (ii). We write for simplicity

$$\iota = \iota_A, \quad M_f = M_{f,f}, \quad \text{and} \quad m_f = m_{f,f},$$

where $m_{f,f}$ is the strictly increasing, continuous function of Definition 4. Successively, for all a, b, c, $d \in A$ and $f \in \mathscr{F}$,

$$M_f[f(a), f(b)] \leqslant M_f[f(c), f(d)]$$

iff

$$m_f\{M[f(a), f(b)]\} \leqslant m_f\{M[f(c), f(d)]\}$$

iff

$$u^\theta[f(a)]h^\delta[f(b)] \leqslant u^\theta[f(c)]h^\delta[f(d)]$$

iff

$$[\beta_f u(a)^{\alpha_f}]^\theta[\gamma_f h(b)^{\tau_f}]^\delta \leqslant [\beta_f u(c)^{\alpha_f}]^\theta[\gamma_f h(d)^{\tau_f}]^\delta$$

iff

$$u(a)^{\alpha_f\theta}h(b)^{\tau_f\delta} \leqslant u(c)^{\alpha_f\theta}h(d)^{\tau_f\delta}. \tag{29}$$

In case (i), $\alpha_f = \tau_f = 1$, and order-meaningfulness follows since (29) does not depend on f.

In case (ii), (29) becomes

$$[u(a)^\theta u(b)^\delta]^{\alpha_f} \leqslant [u(c)^\theta u(d)^\delta]^{\alpha_f}$$

which is equivalent to

$$u(a)^\theta u(b)^\delta \leqslant u(c)^\theta u(d)^\delta \tag{30}$$

with the same conclusion.

To show that (i) and (ii) are the only possibilities, notice that as a consequence of Theorem 6, $\mathcal{M}$ is order-meaningful only if (29) is independent of f. In particular, we may take $f = \iota$, which implies with

$\alpha_\iota = \tau_\iota = 1$, that (29) holds iff (30) holds, or equivalently,

$$(31) \qquad \Psi_f[u(a)^\theta h(b)^\delta] = u(a)^{\alpha_f \theta} h(b)^{\tau_f \delta},$$

for some strictly increasing function Ψ_f. With $s = u(a)^\theta$, $t = h(b)^\delta$, (31) yields

$$\Psi_f(st) = s^{\alpha_f} t^{\tau_f},$$

from which we derive easily $\alpha_f = \tau_f$ for all $f \in \mathscr{F}$, and either (i) or (ii) follows.

Q.E.D.

We remark that each of Theorems 6 and 7 constitutes one of the three implications obtained in Theorem 5. In the case of a family $\mathscr{M}$ of numerical codes involving only one input scale, say $\mathscr{M} = \{M_{f,f} | f \in \mathscr{F}\}$, we cannot derive the third implication, namely: for isotone families, order-meaningfulness and ratio scalability implies multiplicativity. A counterexample is given below.

EXAMPLE 9. Define

$$\mathscr{M} = \{M_\lambda | M_\lambda(a, b) = F_\lambda[a + b + (ab)^{1/2}]; \quad a, b, \lambda > 0\}$$

where the index λ denotes the scale $a \to \lambda a$, and for all $\lambda > 0$, F_λ is some strictly increasing and continuous function. The verification is left to the reader.

We now strengthen the assumptions regarding the scale family $\mathscr{F}$ in Theorem 7, and derive the corresponding possible forms for the multiplicative representation of the family $\mathscr{M}$ of numerical codes.

THEOREM 8. Let $\mathscr{F}$ be a scale family, and let $\mathscr{M} = \{M_{f,f} | f \in \mathscr{F}\}$ be an isotone, order-meaningful, multiplicative family of numerical codes. Then:

(i) if $\mathscr{F}$ is a ratio scale family, then one of the two forms

$$(32) \qquad M(a, b) = F(a^\theta b^\delta), \quad \theta > 0, \delta \neq 0$$

$$(33) \qquad M(a, b) = F(\tau a^\theta + \xi b^\theta), \quad \tau, \theta > 0, \xi \neq 0$$

must hold, with F, a strictly increasing continuous function;

(ii) if $\mathscr{F}$ is a log interval scale family, then (32) is the only possible form;

(iii) if $\mathscr{F}$ is an interval scale family, then we must have

(34) $\qquad M(a, b) = F(\tau a + \xi b), \quad \tau > 0, \ \xi \neq 0,$

with F a strictly increasing, continuous function.

Proof. Let $\mathscr{F}$, $\mathscr{M}$ be as in the hypothesis of the Theorem, and let (u, h) be the multiplicative representation of $\mathscr{M}$. Since $\mathscr{M}$ is isotone and order-meaningful, it is dimensionally invariant by Theorem 4 – whether in case (i), (ii), or (iii) of the Theorem to be proved – and thus

(35) $\qquad M(\lambda a, \lambda b) = K_\lambda[M(a, b)],$

for some strictly increasing, continuous function K_λ. Note that λ varies in an interval containing 1, since $\mathscr{F}$ contains the identity scale. In turn (35) gives, using multiplicativity,

(36) $\qquad H[u(\lambda a)h(\lambda b)] = (K_\lambda \circ H)[u(a)h(b)],$

with H strictly increasing and continuous. Setting $s = u(a)$, $t = h(b)$, we rewrite (36) as

$$u[\lambda u^{-1}(s)]h[\lambda h^{-1}(t)] = (H^{-1} \circ K_\lambda \circ H)(st).$$

Fixing λ and using familiar functional equation arguments, this leads to

$$u[\lambda u^{-1}(s)] = \beta(\lambda)s^{\alpha(\lambda)},$$

$$h[\lambda h^{-1}(s)] = \gamma(\lambda)s^{\alpha(\lambda)},$$

with

$$\alpha(\lambda),\ \beta(\lambda),\ \gamma(\lambda) > 0.$$

These functional equations are well known, and have two solutions (cf. Aczel, 1966).

CASE 1. α is constant; thus $\alpha(\lambda) = 1$ for all λ. Then

$$u(a) = \tau a^\theta \qquad \theta,\ \tau > 0$$

$$h(a) = \xi a^\delta \qquad \xi > 0,\ \delta \neq 0$$

yielding

$$\begin{aligned} M(a, b) &= H(\tau a^\theta \xi b^\delta) \\ &= F(a^\theta b^\delta) \end{aligned}$$

with

$$F(c) = H(\tau \xi c).$$

CASE 2. α is not constant. Then we obtain the forms

$$u(a) = \tau^* \exp(\tau a^\theta),$$

$$h(a) = \xi^* \exp(\xi a^\delta),$$

yielding

$$M(a, b) = H[\tau^* \exp(\tau a^\theta)\xi^* \exp(\xi b^\delta)]$$
$$= F(\tau a^\theta + \xi b^\delta).$$

Using (35), we obtain

$$\tau(\lambda a)^\theta + \xi(\lambda b)^\delta = (F^{-1} \circ K_\lambda \circ F)(\tau a^\theta + \xi b^\delta),$$

that is, with $s = \tau a^\theta$, $t = \xi b^\delta$ and $F_\lambda = F^{-1} \circ K_\kappa \circ F$,

$$\lambda^\theta s + \lambda^\delta t = F_\lambda(s + t) = F_\lambda(t + s) = \lambda^\theta t + \lambda^\delta s.$$

Thus,

$$s(\lambda^\theta - \lambda^\delta) = t(\lambda^\theta - \lambda^\delta),$$

yielding $\theta = \delta$, and (33) follows. On the other hand, it is easy to check that (32), (33) are compatible with the hypotheses of the Theorem, in particular, $\mathscr{F}$ is a ratio scale family.

Equation (32) is also compatible with the assumption that $\mathscr{F}$ is a log interval scale, but (33) is not. Indeed, using the above argument, we would have with obvious notation, the form

$$\tau(\lambda a^\gamma)^\theta + \xi(\lambda b^\gamma)^\theta = F_{\lambda,\gamma}(\tau a^\theta + \xi b^\theta),$$

which leads easily to $\gamma = 1$.

Finally, in the case where $\mathscr{F}$ is an interval scale family, Equation (32) is easy to eliminate, while (33), again with the same argument, leads to

$$\tau(\lambda a + \gamma)^\theta + \xi(\lambda a + \gamma)^\theta = F_{\lambda,g}(\tau a^\theta + \xi b^\theta),$$

yielding $\theta = 1$ without difficulty.

Q.E.D.

A summary of some of our results is given in Table I.

TABLE I
Representations for isotone families of numerical codes

Case 1: two distinct input scales	$\mathcal{M} = \{M_{f,g} \mid f \in \mathcal{F},\ g \in \mathcal{G}\}$	
(Corollary to Theorem 5)		
$\mathcal{F}$, $\mathcal{G}$, homogeneous quasi ratio scale families	$M(a, x) = G(a^{\theta} x^{\delta})$	$\theta > 0,\ \delta \neq 0$
Case 2: two identical input scales	$\mathcal{M} = \{M_{f,f} \mid f \in \mathcal{F}\}$	
(Theorem 8)		
$\mathcal{F}$ a homogeneous scale family		
log interval $\longrightarrow$	$M(a, b) = F(a^{\theta} b^{\delta})$	$\theta > 0,\ \delta \neq 0$
ratio $\longrightarrow$	$M(a, b) = F(\tau a^{\theta} + \xi b^{\theta})$	$\tau,\ \theta > 0,\ \xi \neq 0$
interval $\longrightarrow$	$M(a, b) = F(\tau a + \xi b)$	$\tau > 0,\ \xi \neq 0.$

5. AN APPLICATION

Our results can be used to narrow down the class of numerical functions which are candidates for a description of a body of data. Such application is illustrated here by a brief discussion of an experiment of Pavel (1980; Iverson and Pavel, 1981).

In a psychoacoustic experiment, a subject was required to detect a stimulus, a faint click, embedded or preceded by a burst of white noise. Three independent variables were considered: the intensities of the click and the noise, and the delay τ between the end of the noise and the click (thus, τ can be negative). A basic notion in the analysis of the data is a probability

$$P(x, n, \tau)$$

that a click of intensity x is detected over a masking noise of intensity n, with a delay τ. The empirical results support the assumption that, over some intervals, $P(x, n, \tau)$ is strictly increasing in x, τ and strictly decreasing in n. The data are also consistent with the condition

$$P(x, n, \tau) \leq P(x', n', \tau)$$

(37) iff

$$P(\lambda x, \mu n, \tau) \leq P(\lambda x', \mu n', \tau).$$

To frame this situation in the language developed in this paper, we shall fix τ (temporarily), and assume that $P(x, n, \tau)$ – and abbreviation of $P_{\iota_A, \iota_X}(x, n, \tau)$ – is the initial code of a family $\mathcal{P}$ of numerical codes. The numbers x, n are ratio scale measurements of physical intensities. Equation (37) states that the initial code is dimensionally invariant. Making also the reasonable assumption that $\mathcal{P}$ is order-meaningful, we conclude, using the remark after Theorem 4, that $\mathcal{P}$ is isotone. In view of the ratio scale character of the physical intensities, our knowledge of the possible forms for $P(x, n, \tau)$ is now considerable. We have however two cases to examine.

CASE 1. The intensities of the click and the noise are measured by distinct ratio scales. This means that the admissible transformations of the numbers x, n in $P(x, n, \tau)$ are unrelated. Applying the Corollary to Theorem 5, (or consulting Table I), we obtain as the only possible form:

(38) $$P(x, n, \tau) = F_\tau[x/n^{\delta(\tau)}],$$

in which F_τ is a continuous, strictly increasing function. Interestingly, further analysis of Pavel's data leads him to postulate a form identical to (38) except that the function F does not depend on τ:

(39) $$P(x, n, \tau) = F[x/n^{\delta(\tau)}].$$

Notice however that (39), together with the assumption that P varies with τ, contradicts order-meaningfulness. To see this, we fix x and consider two admissible transformations $n \to \lambda n$, $\tau \to \theta\tau$. Equation (39) becomes

$$P_{\lambda,\theta}(x, n, \tau) = F_{\lambda,\theta}[x/n^{\delta_{\lambda,\theta}(\tau)}],$$

where λ, θ denote the scales used to measure n, τ. Using order-meaningfulness, we obtain via Theorem 3,

$$\begin{aligned} P_{\lambda,\theta}(x, \lambda n, \theta\tau) &= (H_{\lambda,\theta} \circ F)[x/n^{\delta(\tau)}] \\ &= F_{\lambda,\theta}[x/(\lambda n)^{\delta(\theta\tau)}] \end{aligned}$$

for some strictly increasing, continuous function $H_{\lambda,\theta}$, which yields, with $x = 1$ and

$$G_{\lambda,\theta}(s) = [F_{\lambda,\theta}^{-1} \circ H_{\lambda,\theta} \circ F)(1/s)]^{-1},$$

$$(\lambda n)^{\delta(\theta\tau)} = G_{\lambda,\theta}[n^{\delta(\tau)}]. \tag{40}$$

Setting $n = 1$ in (40) gives

$$\lambda^{\delta(\theta\tau)} = G_{\lambda,\theta}(1). \tag{41}$$

Since the right member of (41) does not depend upon τ, it follows that δ is a constant function. This conclusion is not supported by the data (cf. Fig. 1). We conclude that if an order-meaningful expression is to be chosen to describe Pavel's empirical results, it must be of the form (38), with the function F_τ effectively depending on τ. In fact, (38) is consistent with the data, some of which is displayed in Figure 1.

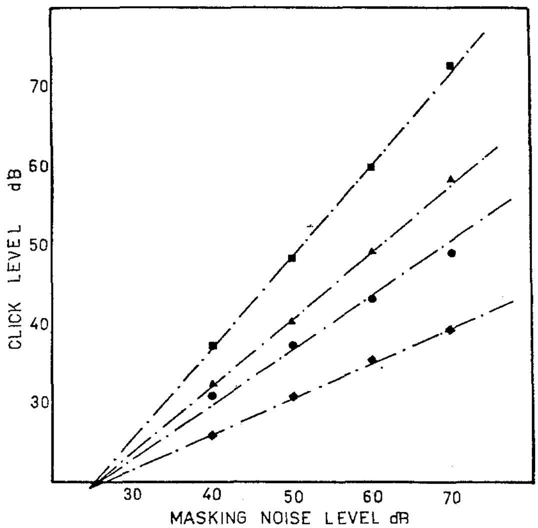

Fig. 1 Pavel's data. The straight lines are least square fit of Equation 46, plotted in decibel units.

Let us demonstrate this. Applying F_λ^{-1} on both sides of (38), we get

$$F_\tau^{-1}[P(x, n, \tau)] = x/n^{\delta(\tau)}. \tag{42}$$

setting

$$P(x, n, \tau) = \pi, \quad \gamma(\tau, \pi) = F_\tau^{-1}(\pi)$$

and solving for x in (42), we obtain the form

$$(43) \qquad x(x, \tau, \pi) = \gamma(\tau, \pi) n^{\delta(\tau)},$$

or in decibel units, taking logs on both sides of (43), with obvious notation,

$$(44) \qquad x^*(n, \tau, \pi) = \gamma^*(\tau, \pi) + \delta(\tau) n^*.$$

Thus, for any fixed values of π, τ, this predicts a linear relationship between the intensity of the click and that of the masking noise, both of these quantities being evaluated in decibel units. As indicated by the data in Figure 1, this is exactly what was observed by Pavel. Moreover, a fixed point property is apparent in Figure 1, which is expressed by the equation:

$$(45) \qquad x[n_0(\pi), \tau, \pi] = x[n_0(\pi), \tau', \pi]$$

for some $n_0(\pi)$. In words: $n_0(\pi)$ is the intensity value of the noise at which the delay τ has no effect on the intensity of the stimulus. This indicates that the parameters of the linear equation (44) are linked by a constraint of the form

$$\gamma^*(\tau, \pi) = K^*(\pi) - \delta(\tau) n_0^*(\pi),$$

for some constants $n_0^*(\pi)$ and $K^*(\pi)$ independent of τ. Going back to the initial units, this gives us

$$F_\tau^{-1}(\pi) = \frac{K(\pi)}{n_0(\pi)^{\delta(\tau)}} = \gamma(\tau, \pi),$$

yielding, as a special case of (43),

$$(46) \qquad x(n, \tau, \pi) = K(\pi)[n/n_0(\pi)]^{\delta(\pi)}.$$

We point out that, for order-meaningfulness to hold, $n_0(\pi)$ must effectively vary with π. Otherwise, as the reader can easily check, (46) becomes equivalent to (39), which is not order-meaningful.

CASE 2. The quantities x and n are physical quantities measured by the same ratio scale. That is, the admissible transformations $x \to \gamma x$, $n \to \theta n$ are linked: we must have $\gamma = \theta$. Meaningfulness arguments similar to those used for Case 1 (we omit the details), gives us two possible forms:

$$(47) \qquad P(x, n, \tau) = F_\tau[x/n^{\delta(\tau)}],$$

(48) $$P(x, n, \tau) = F_\tau[x^{\delta(\tau)} + \xi(\tau)n^{\delta(\tau)}].$$

Equation (47) was analyzed in Case 1. With, as before, $\gamma(\tau, \pi) = F_\tau^{-1}(\pi)$, (48) gives

(49) $$x(n, \tau, \pi) = [\gamma(\tau, \pi) - \xi(\tau)n^{\delta(\tau)}]^{1/\delta(\tau)}.$$

A result for which the linearity in the data of Figure 1 will create difficulties.

Much more could be said about Pavel's empirical results, which are quite extensive and analyzed in great depth in his dissertation. Our purpose here was only to illustrate, by discussing this example, the impact of considerations of meaningfulness on the search for a suitable model for a body of data.

One may ask at this point why (or whether) forms of meaningfulness should be taken as required features for a family of numerical codes. After all, a family of numerical codes purports to be a descript ı of some empirical phenomenon. Science is concerned with what is, not with what should be. We shall turn to this and other issues in the following discussion section.

6. DISCUSSION

As mentioned before, the motivation of our work is similar to that of a paper by Luce (1959). There are, however, important differences between Luce's developments and ours.

No attempt was made by Luce at that time to define the concept of "meaningfulness". Instead, central to his 1959 paper is a "Principle of Theory Construction":

A substantive theory relating two or more variables and the measurement theories for these variables should be such that:

1. (Consistency of substantive and measurement theories) Admissible transformations of one or more of the independent variables shall lead, via the substantive theory, only to admissible transformations of the dependent variables.
2. (Invariance of the substantive theory) Except for the numerical values of the parameters that reflect the effect on the dependent variables of admissible transformations of the independent variables, the mathematical structure of the substantive theory shall be independent of admissible transformations of the independent variables.

Certainly, there is no obvious conflict between the definition of "meaningfulness" proposed here and Luce's "Principle". However, as pointed out by Rozeboom (1962), and recognized by Luce (1962), the "Principle" is somewhat ambiguous. In particular, it is by no means clear that "meaningfulness" in our sense must be regarded as *the* formal interpretation of Luce's "Principle".

Another difference is that the bulk of Luce's results concerns a relation between one independent variable and one dependent variable, while, through our definition of a family of numerical codes, we are dealing here with a relation between two independent variables and one dependent variable. This may seem like a technical detail. However, a major focus of our paper (all of Sec. 4) is on the understanding of the relationship between a multiplicative form of a numerical code, and meaningfulness. For obvious reasons, such a topic has no place in Luce's 1959 paper.

The rest of this section will be devoted to a discussion of various concepts of meaningfulness currently in use.

Concepts of Meaningfulness

The most widely accepted usage of the word "meaningful" is that given by the following informal definition: "A statement involving (numerical) scales is *meaningful* if and only if its truth or falsity is unchanged under admissible transformations of all the scales in question" (Roberts 1979, p. 59).

EXAMPLE 10. The sentence

Λ: The ratio of Stendhal's weight to Jane Austen's on July 3, 1914 was 1.42.

has been called "meaningful" since its truth value is the same for whatever scale is used to measure weight. A difficulty with the definition is that the expression "involving numerical scales" is unclear. The fact is that scales can be "involved" in more than one way in a numerical statement. In Λ for instance, a particular scale has been *used* to measure the weights of Jane Austen and Stendhal. However, that scale is not *mentioned* in the statement. Can a scale be "involved" without being mentioned? To illustrate the ambiguity, it is useful to contrast two interpretations of Λ, both of which make use of a ratio scale family $\mathscr{F}$ for the measurement of weight. For concreteness, we shall suppose that the initial scaling has been made so that the identity scale ι is the pound scale.

First interpretation. The sentence Λ implicitly defines a numerical relation T such that:

> $T(a, x)$ iff a is Stendhal's weight, x is Jane Austen's weight, and $a/x = 1.42$.

The relation T could be regarded as meaningful in the sense of the definition of Roberts (1979) quoted above since for all $f \in \mathscr{F}$ and a, $x \in \mathrm{Re}_+$

$$T(a, x) \quad \text{iff} \quad T[f(a), f(x)].$$

Second interpretation. In the spirit of the language developed in this paper, consider the following family of relations, where $f \in \mathscr{F}$:

> $T_f(a, x)$ iff a, x are respectively Stendhal and Jane Austen's weights, measured on scale f; moreover, $a/x = 1.42$.

If we write $T'(f, a, x)$ for $T_f(a, x)$, then it becomes clear that this relation is different from T above: T is a first order relation between numbers whereas T' is a higher order relation between functions of numbers and pairs of numbers.

Since $\mathscr{F}$ is a ratio scale family, it follows that

$$(50) \qquad T_f[f(a), f(x)] \quad \text{iff} \quad T_{f^*}[f^*(a), f^*(x)]$$

for all a, $x \in \mathrm{Re}_+$ and all $f, f^* \in \mathscr{F}$. Note the strong resemblance between (50) and the defining property of a meaningful family of numerical codes. Natural generalizations of our Definitions 4 and 5 would lead us to consider as meaningful the family of relations

$$\mathscr{T} = \{T_f \mid f \in \mathscr{F}\}.$$

(Such generalizations will be given in Definition 8.) In this interpretation, calling Λ meaningful may be regarded as a harmless abuse of language.

In this example, it does not matter which of the two interpretations is adopted since both lead to "meaningful" Λ with the same truth value. This might suggest that there is no essential discrepancy between the two concepts of "meaningfulness". Such a conclusion, however, would overlook an essential difference between the two interpretations, which in other situations could lead to serious misunderstandings. The definition below, which generalizes the notions of dimensional invariance, (order-) meaningfulness and isotonicity, emphasizes the distinction.

DEFINITION 8. For $1 \leq i \leq n$, let $\mathscr{F}_i$ be a scale family with domain A_i; let $R \subset \mathscr{F}_1 \times \cdots \times \mathscr{F}_n$; for any $(f_1, \ldots, f_n) \in R$, let $T_{f_1,\ldots,f_n} \subset A_1 \times \cdots \times A_n$ be an n-ary relation. Let

$$\mathscr{T} = \{T_{f_1,\ldots,f_n} | (f_1, \ldots, f_n) \in R\}$$

be the family of all such relations.

Any $T_{f_1,\ldots,f_n} \in \mathscr{T}$ is called *meaningful in the first sense*, or 1-*meaningful*, (*with respect to* R) iff

$$T_{f_1,\ldots,f_n}(a_1, \ldots, a_n) \quad \text{iff} \quad T_{f_1,\ldots,f_n}[f_1^*(a_1), \ldots, f_n^*(a_n)],$$

for all $(f_1^*, \ldots, f_n^*) \in R$ and $a_i \in A_i$, $1 \leq i \leq n$. By extension, the family $\mathscr{T}$ is said to be *meaningful in the first sense*, or 1-*meaningful*, iff all its relations are 1-meaningful.

The family $\mathscr{T}$ is called *meaningful in the second sense*, or 2-*meaningful*, iff

$$T_{f_1,\ldots,f_n}[f_1(a_1), \ldots, f_n(a_n)] \quad \text{iff} \quad T_{f_1^*,\ldots,f_n^*}[f_1^*(a_1), \ldots, f_n^*(a_n)]$$

for all $(f_1, \ldots, f_n)$, $(f_1^*, \ldots, f_n^*) \in R$ and $a_i \in A_i$, $1 \leq i \leq n$.

Finally, the family $\mathscr{T}$ is called *degenerate* iff

$$T_{f_1,\ldots,f_n}(a_1, \ldots, a_n) \quad \text{iff} \quad T_{f_1^*,\ldots,f_n^*}(a_1, \ldots, a_n)$$

for all $(f_1, \ldots, f_n)$, $(f_1^*, \ldots, f_n^*) \in R$, and $a_i \in A_i$, $1 \leq i \leq n$, or equivalently, iff

$$T_{f_1,\ldots,f_n} = T_{f_n^*,\ldots,f_n^*},$$

for all $(f_1, \ldots, f_n)$, $(f_1^*, \ldots, f_n^*) \in R$.

We have the following result, generalizing Theorem 4.

THEOREM 9. The three properties of 1-meaningfulness, 2-meaningfulness and degeneracy are pairwise independent. However, any two of these three properties implies the third one.

The proofs of the three implications in the last part of the Theorem are simple, and left to the reader. To establish the independence results, we rely on Theorem 4. Let $\mathscr{F}$, $\mathscr{G}$ be two scale families, with respective domains A, X; let $R \subset \mathscr{F} \times \mathscr{G}$; let $\mathscr{M} = \{M_{f,g} | fRg\}$ be a family of numerical codes. Define a family $\mathscr{T}$ of quaternary relations $T_{f,g,f,g}$, with fRg, by:

$$T_{f,g,f,g}(a, x, b, y) \quad \text{iff} \quad M_{f,g}(a, x) \leq M_{f,g}(b, y),$$

for all a, $b \in A$, x, $y \in X$ and fRg. The following three equivalences clearly hold.

(i) $\mathcal{T}$ is 1-meaningful iff $\mathcal{M}$ is dimensionally invariant;

(ii) $\mathcal{T}$ is 2-meaningful iff $\mathcal{M}$ is order-meaningful;

(iii) $\mathcal{T}$ is degenerate iff $\mathcal{M}$ is isotone.

An application of Theorem 4 completes the proof. As a by-product of the above argument, notice in passing that, for quantitative laws, 1-meaningfulness corresponds to what we have called dimensional invariance.

We suspect that most measurement theorists would choose 1-meaningfulness as the natural interpretation of Roberts' definition. This particular version of the concept of "meaningfulness" certainly dominates the literature. As an illustration, one important example is discussed below.

When measurement theory is approached from a qualitative viewpoint, (e.g. most of F.M. I, II), meaningfulness is sometimes defined in terms of the automorphisms of the embedding real structure. To describe this idea with precision, we introduce the notion of a *qualitative structure* Ξ, i.e. a formal system of relations on a set A^0 of (empirical) objects. Measurements, or (numerical) representations of Ξ, are assignments of numbers to the elements of A^0. However, not all assignments will do. In measurement theory, the general idea is that the assignments should be "structure preserving"; that is, there should be some predetermined structure Γ of relations defined on Re_+ such that the representations of Ξ are homomorphisms into Γ. For concreteness, we shall suppose that the relations in Γ are defined on Re_+. We shall also assume, to simplify our discussion, that all the representations of Ξ are isomorphisms onto Γ. Let ρ be a representation of Ξ. For each representation θ of Ξ let f_θ be the function from Re_+ onto Re_+ such that

$$f_\theta(r) = (\theta \circ \rho^{-1})(r)$$

for each $r \in \mathrm{Re}_+$. Let

$$\mathcal{F} = \{f_\theta \mid \theta \text{ is a representation of } \Xi\}.$$

Then it is easy to show that $\mathcal{F}$ is a group under the composition of functions, with identity f_ρ. In fact, $\mathcal{F}$ is the group of automorphisms of Γ. In this context, we shall define a numerical relation as *meaningful in the third sense*, or 3-*meaningful*, (*with respect to* $\mathcal{F}$), iff it is invariant under the automorphisms of Γ. It is immediately apparent that, when $\mathcal{F}$ is a scale group (in the sense of our Definition 1), then 1 and 3-meaningfulness are very close. The only difference is that, in the latter case, some pain is taken to specify the origin of the scale family.

The importance of this concept – under the guise of 1-meaningfulness, 3-meaningfulness or dimensional invariance – is widely recognized by measurement theorists (see in particular Luce 1978) and will not be denied here. Whether it completely captures the notion of "meaningfulness" is less clear, and we shall argue that it does not.

Two features of 2-meaningfulness as a general "meaningfulness" concept are especially worthy of note in this connection.

1. In the form of one-to-one or order-meaningfulness for families of numerical codes, it provides a constraint which is strong and yet very reasonable (in fact practically unescapable, as shown in our remarks in Section 3).

2. In the same context, this concept permits the consideration of empirical laws of the form

$$M_{\lambda,\mu}(a, x) = F_{\lambda,\mu}[(a+\lambda k_0)^{\beta}(x+\mu k_1)^{\delta}], \tag{51}$$

in which $F_{\lambda,\mu}$ is a strictly increasing function, $a, x \in \mathrm{Re}_+$ are variables, β, k_0, $k_1 \in \mathrm{Re}_+$ and $\delta \neq 0$ are numerical constants, and λ, $\mu \in \mathrm{Re}_+$ denote the scales. (The conventions are as in Examples 6–9.)

In particular, notice that Equation (51) does not satisfy dimensional invariance, nor any immediate natural generalization of this concept. As far as we know, this type of law has not been analyzed from a measurement viewpoint. Its main interest lies in the special role played by the additive "dimensional" constant k_0 and k_1 in the equation. A preliminary investigation of such laws, using and extending the techniques of this paper, indicates that a satisfactory analysis is possible. A paper discussing these issues is in progress.

To our knowledge, no one has provided a satisfying definition of "numerical law" for a general scientific context. There seems to be some agreement that laws should be "meaningful" in a sense close to the one used in this paper. Almost all the laws one encounters in science are instances of meaningful families of numerical codes. Thus, a natural strategy is to identify "law" with "meaningful families of numerical codes". We would feel uneasy about an uncritical endorsement of such strategy at this stage, since we do not fully understand the exact role of "meaningfulness" in scientific formulation, deduction and communication. Presumably, the strong liking for laws that are meaningful in the sense of this paper is due to a mixture of interrelated practical and theoretical reasons. On the practical side, it is clear that the adoption of a substantial number of nonmeaningful "laws", the form of which would depend on the scale(s) used, would quickly create a scientific Tower of Babel. On the theoretical side, meaningfulness appears to favor "coherent" systems (of quantitative notation of scientific facts) over "incoherent" ones. Here we are thinking about such aspects of a "scientific law" as those emphasized by Luce (1959) in his Principle of Theory Construction.

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