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Axiomatic Measurement Theory[1]

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INTRODUCTION

Everyone is aware that measurement is a cornerstone of science, one that in some cases is highly controversial. Much complex technology underlies the refined measurement of certain physical quantities, some of which can be estimated to surprisingly large numbers of significant figures; one of the more elaborate businesses spawned by the social sciences, a business that affects all of our lives, attempts to measure intellectual ability and/or achievement; and elaborate computer programs are widely used to provide numerical representations (and simplifications), e.g., by factor analysis and multidimensional scaling, of complexes of data. Behind all of this activity is a belief, often sustained by a mixture of intuition and successful–if ill understood–procedures, that certain bodies of data can be represented in some fashion by numbers and their relations to each other. The goal of the semiphilosophical, semimathematical field of our title is to lay bare the types of empirical structures that admit such numerical representations.

The reason for the term "axiomatic" in the title is that this is how the structures involved are described. The task is to isolate axioms that, on the one hand, are empirically and/or philosophically acceptable for at least one important scientific interpretation of the primitives and that, on the other hand, permit us to prove mathematically that the structure is closely similar (usually, isomorphic or homomorphic) to some numerical structure. Ultimately, one aims for a finite collection of different classes of structures that span all the scientifically interesting cases.

At present, our knowledge appears quite adequate for the better developed parts of classical physics. It is interesting to note that many influential writers

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considered that classical physics had achieved an adequate measurement-theoretic underpinning by 1920, but that view simply was wrong and a tolerably adequate theory was only forged in the past few years and it is still being improved. It is considerably less clear that existing theory is adequate for some of the classically intractable concepts, such as those having to do with turbulence and hardness. In relativistic physics things are much worse, and it is doubtful if existing theory is suitable for such variables as relativistic velocity. More generally, any bounded variable, including probability and sensory concepts such as loudness and brightness, raises tricky problems that are not yet fully understood. Further, it is quite clear that the existing theories are not suited to the measurement questions that arise in quantum mechanics, and to date little effort has been spent on these questions.

If the field remains incompletely developed for physics, it is anyone's guess as to how adequate it will prove to be for the biological, behavioral, and social sciences. These fields, especially the latter two, have struggled for years with problems of measurement, and although much is "measured", the underlying conceptual basis is still poorly understood. In fact, a major motive for much of the work on axiomatic measurement since 1947, when the work of von Neumann and Morgenstern [**1944**, **1947**, **1953**] on utility theory became widely known, has been to clarify just what the measurement options are. There was a time–Campbell [**1920**], Cohen and Nagel [**1934**]–when measurement was said to be limited to those structures isomorphic to the additive reals and things that could be "derived" from them. This position was asserted, and asserted strongly, despite the fact that the ring of real numbers clearly played a role in the representation of physical measures–witness the additivity of length, mass, time, and the like, and the product of powers of measures to form other measures, as reflected in the units of physical measurement. Today, we know of several classes of structures with vastly richer numerical representations than the additive group of the reals, which nonetheless still plays a highly central role, and we hope that behavioral and social scientists will find useful some of the recently developed generalizations. There exists a small group of theoretical and empirical scientists who are working on the interplay of these kinds of measurement concepts with data, but it probably will take a considerable time before we have any clear sense of just how applicable this kind of "applicable mathematics" is outside of physics.

At this juncture, two quite different types of programs are needed. The one is to recast, to simplify, and to inject these ideas into the mainstream of the behavioral and social sciences, just as was done with statistics over the past 50 years. This has begun on many fronts, including texts, expository articles, empirical methods, computer programs, and the like. It will have to continue for a long time, just as it has with statistics, and it will have to be shown to make a difference. The other is to enlist the help of the mathematical community to enlarge our understanding of the relevant structures. It is not, however, just a problem of mathematical generalization and more powerful proof techniques–although both are surely needed–but of developments that are sensitive to the possible empirical interpretations that can be given to the primitives. It is, after all, applicable mathematics, and more is required of the mathematician than just mathematical skills. It is all too easy to lose sight of or to underestimate the added sensitivity that is needed if the work is to remain scientifically deep as it becomes mathematically deeper.

The body of this paper summarizes a number of the main results and approaches that have been taken. Some history and references are provided, but it is far from a scholarly survey. It is, rather, a highlighting of what we think is most important–always an idiosyncratic criterion–and is intended more as an overview and an invitation to dig more deeply and to contribute than it is a precise account of the whole field. A number of books provide more detail and depth. The most elementary and the one with the greatest number of illustrative social science examples is Roberts [**1979**]. The earliest and most compact is Pfanzagl [**1968**, **1971**]. The most comprehensive, provided one takes into account the projected second volume, is Krantz et al. [**1971**], [in preparation]. The most advanced is the nearly completed one by Narens [in preparation].

I. Structures with One Operation

All measurement rests upon having a qualitative ordering $\succsim$ of the set X of objects. It is well known (Krantz, et al. [**1971**, §2.1]) that an order preserving numerical representation exists if and only if $\langle X, \succsim \rangle$ is a total order with a finite or countable order dense subset. Moreover, any two such representations are related by a monotonically strictly increasing function. Such so-called ordinal scales are far too weak to be useful for measurement: concepts such as the derivative of a quantity are not invariant under admissible changes in the representation. In order for the representation to be firmer, it is necessary that the numerical measures preserve structure in addition to but related to the order. In practice, this has meant one of four things: either a single operation is included which is represented by some numerical operation, often addition; or the ordered elements are themselves structured in the sense that X is the Cartesian product of two or more sets, which structure is represented by some numerical operation, often multiplication; or X is a Cartesian product of the form $A \times A$ and the representation is in some geometric space with distance preserving the ordering relation; or there is even more structure such as two operations or a Cartesian product and an operation on one of the components, which is represented by two (or more) numerical operations. We deal with the first two cases, which are closely related, in the first major part of the paper, and the latter in the second major part. The third case, the geometric one, is not covered in this paper; see Beals and Krantz [**1967**], Beals, Krantz and Tversky [**1968**], Tversky and Krantz [**1970**], and Krantz et al. [in preparation].

1. Extensive structures.

DEFINITION 1. Let $\bigcirc$ be a partial binary operation on the nonempty set X (i.e., a function from a subset of $X \times X$ into X), $\succcurlyeq$ be a total ordering on X, R a subset of Re, and $\odot$ a partial operation on R. Then φ is said to be a $\odot$-*representation* for the structure $\mathfrak{X} = \langle X, \succcurlyeq, \bigcirc \rangle$ if and only if φ is an isomorphic imbedding of $\mathfrak{X}$ into the structure $\langle R, \geqslant, \odot \rangle$. If φ is a $\odot$-representation and $\odot$ is $+$, then φ is said to be an *additive representation*.

Partial operations, rather than operations (which we will often call *closed* operations to distinguish them clearly from partial operations) play a critical role in some measurement situations.

Often in measurement theory, structures of the form $\langle X, \succsim, \bigcirc \rangle$ are considered where $\succsim$ is a weak ordering (transitive and connected) rather than a total ordering (also asymmetric). The measurement theoretic results for such structures are almost identical to those of the totally ordered case. In this paper, the totally ordered case is often invoked (although not always) to simplify notation and some definitions.

The first serious results in the foundations of measurement go back to Helmholtz [**1887**] and Hölder [**1901**], who presented axiomatizations for additive physical attributes. These axiomatizations have been greatly refined by a number of researchers, and today find their most useful formulation in the following definition due to Krantz et al. [**1971**].

DEFINITION 2. Let X be a nonempty set, $\succcurlyeq$ a binary relation on X, and $\bigcirc$ a partial binary operation on X. The structure $\mathfrak{X} = \langle X, \succcurlyeq, \bigcirc \rangle$ is said to be an *extensive structure* if and only if the following eight axioms hold for all w, x, y, z in X:

1. *Total ordering*. $\succcurlyeq$ is a total ordering.

2. *Nontriviality*. There exist u, v in X such that $u \succ v$.

3. *Local definability*. If $x \bigcirc y$ is defined, $x \succcurlyeq w$, and $y \succcurlyeq z$, then $w \bigcirc z$ is defined.

4. *Monotonicity*. (1) If $x \bigcirc z$ and $y \bigcirc z$ are defined, then $x \succcurlyeq y$ iff $x \bigcirc z \succcurlyeq y \bigcirc z$, and

(2) if $z \bigcirc x$ and $z \bigcirc y$ are defined, then $x \succcurlyeq y$ iff $z \bigcirc x \succcurlyeq z \bigcirc y$.

5. *Restricted solvability*. If $x \succ y$, then there exists u such that $x \succ y \bigcirc u$.

6. *Positivity*. If $x \bigcirc y$ is defined, then $x \bigcirc y \succ x$ and $x \bigcirc y \succ y$.

7. *Archimedean*. There exists $n \in I^+$ such that either nx is not defined or $nx \succcurlyeq y$, where mx is inductively defined by $1x = x$, and if $(mx) \bigcirc x$ is defined, then $(m+1)x = (mx) \bigcirc x$.

8. *Associativity*. If $x \bigcirc (y \bigcirc z)$ and $(x \bigcirc y) \bigcirc z$ are defined, then $x \bigcirc (y \bigcirc z) = (x \bigcirc y) \bigcirc z$.

If $\bigcirc$ is a closed operation, then $\mathfrak{X}$ is said to be a *closed* extensive structure.

The theoretical measurement of length is often taken as an example of an extensive structure. Let X be a set of (straight) measuring rods. For each x, y in X, let $x \succcurlyeq y$ stand for "the rod x is at least as long as the rod y," and let $x \bigcirc y$ be the rod that is obtained by abutting x to y along a straight edge. In theoretical physics, it is assumed that $\mathfrak{X} = \langle X, \succcurlyeq, \bigcirc \rangle$ is an extensive structure and $\bigcirc$ is a closed operation. Another, slightly more subtle application, is to probability theory. Here we assume Ω to be a nonempty set, Y an algebra of subsets of Ω, and $X = Y - \{\varnothing\}$. Let $\succsim$ be a binary relation on X, to be interpreted as the concept of "at least as likely as." Thus, axiomatically, we assume $\succsim$ to be a weak ordering. A natural partial operation to define on X is $\oplus$, where for all x, y in X,

$$x \oplus y = z \quad \text{if and only if} \quad x \cap y = \varnothing \text{ and } x \cup y = z.$$

Then $\mathfrak{X} = \langle X, \succsim, \oplus \rangle$ starts to resemble an extensive structure: $\succsim$ is a weak ordering and $\oplus$ is associative for the elements for which it is defined. Furthermore, monotonicity of $\oplus$ is a natural assumption to make, i.e., for all x, y, z, w in X such that $x \cap z = \varnothing$ and $y \cap w = \varnothing$ and $z \sim w$,

$$x \succsim y \quad \text{iff} \quad x \oplus z = x \cup z \succsim y \cup w = y \oplus w.$$

What is missing is that the partial operation $\oplus$ is not defined for sufficiently many pairs of events. This can be partially rectified by letting $\boldsymbol{X} = X/\sim$, $\succcurlyeq = \succsim/\sim$, and defining $\boldsymbol{\oplus}$ by: for each A, B, C in $\boldsymbol{X}$, $A \boldsymbol{\oplus} B = C$ if and only if for some x in A, y in B, z in C, $x \oplus y = z$, and considering the totally ordered structure $\boldsymbol{\mathfrak{X}} = \langle \boldsymbol{X}, \succcurlyeq, \boldsymbol{\oplus} \rangle$. $\boldsymbol{\mathfrak{X}}$ is very close to an extensive structure. Its primary lack is that local definability and Archimedean may not hold. However, rather plausible axioms in terms of the primitives $\succsim$ and $\cup$ can be given that guarantee that $\boldsymbol{\mathfrak{X}}$ is an extensive structure. Such an extensive structure $\boldsymbol{\mathfrak{X}}$ in this paper will be called a *qualitative probability structure*. The interested reader should consult Luce **[1965]** or Krantz et al. **[1971**, Chapter 5] or Fine **[1971a]**, **[1971b]** for a detailed axiomatization. It should also be noted that it is inherent in the nature of probability, which has Ω as a maximal element, that $\oplus$ must be a partial, not a closed, operation.

The following theorem shows that extensive structures have a restricted set of additive representations, and this fact is widely used to justify and establish numerical scales of empirical variables.

THEOREM 1. *Suppose $\mathfrak{X} = \langle X, \succcurlyeq, \bigcirc \rangle$ is an extensive structure and $\langle X, \succcurlyeq \rangle$ does not have a maximal element. Then the following three statements are true:*

(i) *there exists an additive representation for $\mathfrak{X}$;*

(ii) *if φ and ψ are both additive representations of $\mathfrak{X}$, then for some r in* Re^+, $\varphi = r\psi$;

(iii) *$r\varphi$ is an additive representation for $\mathfrak{X}$ for each r in* Re^+ *and each additive representation φ of $\mathfrak{X}$.*

A proof of Theorem 1 is given in Chapters 2 and 3 of Krantz et al. **[1971]**.

Theorem 1 can be used to show for the example of length presented above that positive numbers can be assigned to measuring rods so that rod x is at least as long as y if and only if the number assigned to x is $\geqslant$ the number assigned to y, and the number assigned to the rod resulting from abutting x to y is the sum of the numbers assigned to x and y. Furthermore, any other assignment with these properties is essentially the same: it differs by at most multiplication by a positive constant. It also follows by applying Theorem 1 to the probabilistic situation discussed above with $\mathfrak{X}$ being a qualitative probability structure that a unique, finitely additive probability representation exists, i.e., there exists a unique function P from the algebra of events Y of Ω such that

(1) $P(\Omega) = 1$ and $P(\varnothing) = 0$;

(2) $P(x \cup y) = P(x) + P(y)$ for all x, y in Y such that $x \cap y = 0$; and

(3) for each x, y in X, x is at least as likely as y (i.e., $x \succsim y$) if and only if $P(x) \geqslant P(y)$.

The axioms for extensive structures are sufficient for the existence of additive representations but not necessary. For the case of a closed operation, Roberts and Luce [**1968**] have given necessary and sufficient conditions for the existence of additive representations and showed a result like Theorem 1. (These results are presented in Krantz et al. [**1971**].)

2. Generalizations of extensive structures. A number of generalizations of extensive structures have appeared in the literature. A very brief description of some of these will be now given.

Structures with weakened forms of Axiom 1, total ordering, are considered in Narens [in preparation] and Holman [**1974**]. Narens considers the case where $\bigcirc$ is a closed operation and $\succcurlyeq$ is a transitive and reflexive relation, and gives necessary and sufficient conditions for such structures to have an additive representation. Holman considers a case that has an equivalence relation instead of an ordering relation. By considerably strengthening the Archimedean axiom, he shows a theorem analogous to Theorem 1.

Falmagne [**1971**], [**1975**] considers structures which have additive representations, but in which local definability (Axiom 3) is weakened so that arbitrarily small elements need not exist. "Arbitrarily small" here means arbitrarily small in terms of some additive representation rather than in terms of the ordering relation $\succcurlyeq$, i.e., in terms of some additive representation of $\mathfrak{X}$ assuming values arbitrarily close to 0. Falmagne's axiomatization yields a theorem analogous to Theorem 1.

Structures without Archimedean axioms are considered in Narens [**1974a**], [**1974b**], [in preparation]. In general, such structures do not have additive representations in the reals. However, Narens shows that they have additive representations in certain structures richer than the reals, namely the nonstandard reals and structures that resemble lexicographically ordered vector spaces. Skala [**1975**] has collected together various results about nonarchimedean measurement.

Perhaps the most important generalization of extensive structures comes from deleting Axiom 8, associativity. This structure, which was first considered in

Narens and Luce [**1976**], has surprisingly strong measurement theoretic properties.

DEFINITION 3. $\mathfrak{X} = \langle X, \succcurlyeq, \bigcirc \rangle$ is said to be a *positive concatenation structure* if and only if $\mathfrak{X}$ satisfies all the axioms for an extensive structure except possibly Axiom 8, associativity.

Narens and Luce [**1976**] showed that positive concatenation structures have $\odot$-representations for some $\odot$, and that such representations have strong uniqueness properties. Cohen and Narens [**1979**] gave a slightly different version of uniqueness for these structures, and their version is given in statement (ii) of the following theorem.

THEOREM 2. *Suppose* $\mathfrak{X} = \langle X, \succcurlyeq, \bigcirc \rangle$ *is a positive concatenation structure. Then the following two statements are true*:

(i) $\mathfrak{X}$ *has a* $\odot$*-representation for some* $\odot$;

(ii) *if* φ *and* ψ *are* $\odot$*-representations for* $\mathfrak{X}$ *such that* $\varphi(X) = \psi(X)$ *and if for some* x *in* X, $\varphi(x) = \psi(x)$, *then* $\varphi = \psi$.

There are some scientifically important concatenation operations, such as temperature and averaging, that do not satisfy positivity of Definition 2. In these cases Axioms 6, 7, and 8 do not hold, but they can be replaced by another Archimedean axiom and the following property (called intern):

$$\text{if } x \succ y, \text{ then } x \succ x \bigcirc y \succ y.$$

Such concatenation structures are called *intensive*. Some work on a special case, satisfying a property called bisymmetry, was reported by Pfanzagl [**1959a**], [**1959b**] (see Krantz et al. [**1971**, §6.9]). Narens and Luce [**1976**] showed that a broad class of intensive structures is closely related to positive concatenation structures.

3. Conjoint structures. Structures of the form $\langle X, \succsim, \bigcirc \rangle$ where $\succsim$ is an ordering and $\bigcirc$ is a partial operation naturally arise in physical science with $\succsim$ and $\bigcirc$ being directly observable relations on physical variables. The corresponding situation of directly observable concatenation operations happens rarely in the behavioral sciences. Still they play an important, indirect role as follows: A prevalent type of structure both in the physical and behavioral sciences is a directly observable ordering on a Cartesian product–e.g., the ordering by energy over mass-velocity pairs or by loudness over energy pairs to the two ears. Krantz [**1964**] first showed how extensive structures arise in the simplest such cases, and Narens and Luce [**1976**] showed that more general ordered structures are often transformable into positive concatenation structures. In these cases, the concatenation operation $\bigcirc$ results from the interaction of objects in different components of the Cartesian product. The following definition describes two types of such structures.

DEFINITION 4. $\mathcal{C} = \langle X \times P, \succsim, ab \rangle$ is said to be a *conjoint structure solvable with respect to the element ab* if and only if ab is in $X \times P$ and the following seven axioms hold:

1. *Weak ordering.* $\succsim$ is a weak ordering on $X \times P$.

2. *Nontriviality.* There exists xp in $X \times P$ such that $xp \succ ab$.

3. *Density.* For each xp, yp in $X \times P$, if $xp \succsim yp$, then for some z in X, $xp \succ zp \succ yp$.

4. *Solvability (with respect to ab).* For each xp in $X \times P$, there exist z and q such that $xp \sim zb$ and $xb \sim aq$.

5. *Archimedean.* For each x, y in X such that $xb \succ ab$, there exists n in I^+ such that $(nx)b \succ yb$, where nx is defined inductively as follows: $1x = x$, and if nx is defined and s is such that $xb \sim as$, then $(n+1)x$ is some u such that $ub \sim (nx)s$.

6. *Independence.* For each x, y in X and p, q in P, (i) if $xs \succsim ys$ for some s, then $xp \succsim yp$; and (ii) if $wp \succsim wq$ for some w, then $xp \succsim xq$.

From Axiom 6, independence, it easily follows that the relations $\succsim_X$ and $\succsim_P$ defined on X and P respectively by: for each x, y in X and each p, q in P,

$$x \succsim_X y \text{ iff for some } s, \quad xs \succsim ys,$$

and

$$p \succsim_P q \text{ iff for some } w, wp \succsim wq,$$

are weak orderings on X and P respectively. Once again, to simplify notation and some definitions, we will assume the following axiom.

7. *Component total ordering.* $\succsim_X$ and $\succsim_P$ are total orderings, which will be written as $\geqslant_X$ and $\geqslant_P$.

If in addition to Axioms 1–7 above, ab is the minimal element of $X \times P$ (i.e., $xp \succsim ab$ for all xp in $X \times P$), then $\mathcal{C}$ is said to be a conjoint structure *solvable with respect to a minimal element ab*.

Let $\mathcal{C} = \langle X \times P, \succsim \rangle$ be a conjoint structure solvable with respect to a *minimal element ab* (i.e., $xp \succsim ab$ for all xp in $X \times P$). We will now sketch the construction of Narens and Luce [**1976**] which shows how to code $\mathcal{C}$ as a positive concatenation structure; it generalizes the proof for the additive case given in Holman [**1971**]. By solvability and component total ordering, let ξ: $X \times P \to X$ and σ: $X \to P$ be defined as the unique solutions to the following equations for each xp in $X \times P$,

$$xp \sim \xi(xp)b \quad \text{and} \quad xb \sim a\sigma(x).$$

Let $X^+ = \{x \mid x \in X \text{ and } x \succ_X a\}$ and for each x, y in X^+, let $x \bigcirc y = \xi[x\sigma(y)]$. Then it follows from results in Narens and Luce [**1976**] that $\mathcal{X}^+ = \langle X^+, \geqslant_X, \bigcirc \rangle$ is a positive concatenation structure. Note that for each x, y, u, v in X^+, $x \bigcirc y \geqslant_X u \bigcirc v$ iff $\xi[x\sigma(y)] \geqslant_X \xi[u\sigma(v)]$ iff $x\sigma(y) \succsim u\sigma(v)$.

For $\bigcirc$ to be associative, the following condition on $\mathcal{C}$ is necessary and sufficient.

The Thompsen condition. For each x, y, z in X and each p, q, r in P, if $xp \sim yq$ and $yr \sim zp$, then $xr \sim zq$.

THEOREM 3. *Suppose* $\mathcal{C}$ *is a solvable conjoint structure with a minimal element* ab *and satisfies the Thompsen condition. Then* $\bigcirc$ *is associative and there exist functions* $\varphi: X \to \mathrm{Re}^+ \cup \{0\}$, $\psi: P \to \mathrm{Re}^+ \cup \{0\}$ *such that*

(i) $\varphi(a) = \psi(b) = 0$,

(ii) *for each* xp, yq *in* $X \times P$,

$$xp \succsim yq \text{ iff } \varphi(x) + \psi(p) \geq \varphi(y) + \psi(q), \tag{1}$$

and

(iii) *if* φ', ψ' *is another pair of functions on* X *and* P *respectively that satisfy* (i) *and* (ii) *above, then for some* r *in* Re^+, $\varphi' = r\varphi$ *and* $\psi' = r\psi$.

The interested reader should consult Narens and Luce [**1976**] for the proof of Theorem 3 and details of the above construction. (Structures $\mathcal{C}$ that satisfy Equation 1 for some φ, ψ are called *additive conjoint structures*.)

The solvability condition in Definition 3 requires $\bigcirc$ to be a closed operation. A weaker form of solvability that yields partial operations is considered in Narens and Luce [**1976**], and a weaker form for conjoint structures satisfying the Thompsen condition is considered in Luce [**1966**] and Chapter 6 of Krantz et al. [**1971**].

4. Uniqueness of positive concatenation structures. The uniqueness of representations of positive concatenation structures, as given in statement (ii) of Theorem 2, takes a form different from that of extensive structures, as given in statements (ii) and (iii) of Theorem 1. The two kinds of uniqueness are equally "unique" in the sense that a value at one point determines the representation. However, Theorem 1 also tells how any two representations are related to each other, whereas Theorem 2 does not.

To clarify that question for a positive concatenation structure $\mathcal{X} = \langle X, \succcurlyeq, \bigcirc\rangle$, we consider the automorphism group of $\mathcal{X}$. Suppose φ is a $\odot$-representation and α is an automorphism of $\mathcal{X}$. It is easy to verify that $\varphi\alpha$ is a $\odot$-representation of $\mathcal{X}$. Furthermore, each $\odot$-representation ψ of $\mathcal{X}$ such that $\psi(X) = \varphi(X)$ is of this form since it easily follows that $\varphi^{-1}\psi$ is an automorphism of $\mathcal{X}$. Thus to understand how $\odot$-representations of $\mathcal{X}$ with the same range are related, it is sufficient to understand how automorphisms are related.

Cohen and Narens [**1979**] showed that the group of automorphisms $\langle A, *\rangle$ of a positive concatenation structure $\mathcal{X} = \langle X, \succcurlyeq, \bigcirc\rangle$ has a natural ordering $\succcurlyeq$ defined on it by: for each α, β in A,

$$\alpha \succcurlyeq \beta \text{ iff for some } x \text{ in } X, \alpha(x) \succcurlyeq \beta(x).$$

They showed that the structure $\mathcal{G} = \langle A, \succcurlyeq, *\rangle$ is an Archimedean, totally ordered group. It is well known that such groups are of the following three types. Let ι be the identity automorphism, and $A^+ = \langle \alpha \in A | \alpha \succ \iota\rangle$. Then $\mathcal{G}$ is *trivial* if $A^+ = \varnothing$, *discrete* if A^+ has a least element, and *dense* if A^+ has no least element. There are positive concatenation structures with automorphism groups of each type: Consider $\langle \mathrm{Re}^+, >, \oplus\rangle$ and its group of automorphisms $\mathcal{G}$. In each of the following choices for $\oplus$, the structure is a positive concatenation structure.

1. $x \oplus y = x + y$. $\oplus$ is commutative and associative. $\mathcal{G}$ is dense and consists of multiplication by every positive real.

2. $x \oplus y = x + y + x^{1/2}y^{1/2}$. $\oplus$ is commutative and nonassociative. $\mathcal{G}$ is dense and consists of multiplication by every positive real.

3. $x \oplus y = x + y + x^{1/3}y^{2/3}$. $\oplus$ is noncommutative and nonassociative. $\mathcal{G}$ is dense and consists of multiplication by every positive real.

4. $x \oplus y = x + y = (xy)^{1/2}[2 + \sin(\frac{1}{2}\log xy)]$. $\oplus$ is commutative and nonassociative. Cohen and Narens **[1979]** showed $\mathcal{G}$ to be discrete and consist of multiplication by $e^{2\pi n}$, $n = 0, 1, 2, \ldots$.

5. $x \oplus y = x + y + x^2y^2$. $\oplus$ is commutative and nonassociative. Cohen and Narens **[1979]** showed $\mathcal{G}$ to be trivial.

All of the above structures are Dedekind complete (every nonempty bounded subset has a least upper bound). Narens and Luce **[1976]**, Cohen and Narens **[1979]** and Narens **[1981]** investigate conditions under which positive concatenation structures are extendable to Dedekind complete ones. The arguments are much more complicated and subtle than those familiar from the associative case, and several results are established concerning what sort of measurement-theoretic properties are inherited by such Dedekind completions. The interested reader should consult the above papers. Throughout the rest of this part we consider only the Dedekind complete case with a dense automorphism group.

5. Homogeneous structures.

Definition 5. Let $\mathcal{X} = \langle X, R_1, R_2, \ldots \rangle$ be a relational structure (i.e., X is a nonempty set and $R_1, R_2, \ldots$ are relations and/or functions on X). Then $\mathcal{X}$ is said to be *homogeneous* if and only if for each x, y in X there exists an automorphism α of $\mathcal{X}$ such that $\alpha(x) = y$.

Cohen and Narens **[1979]** showed the following theorem.

Theorem 4. *Suppose $\mathcal{X} = \langle X, \succsim, \bigcirc \rangle$ is a Dedekind complete positive concatenation structure. Then the following three conditions are equivalent:*

(i) *$\mathcal{X}$ is homogeneous;*

(ii) *$\mathcal{X}$ has a dense automorphism group;*

(iii) *for each n in I^+, $n(x \bigcirc y) = (nx) \bigcirc (ny)$.*

Definition 6. Dedekind complete positive concatenation structures that satisfy one of the conditions of Theorem 4 are called *fundamental unit structures*.

Condition (iii) in Theorem 4 is of particular interest since it formulates in the language of the first order predicate calculus what is meant by $\mathcal{X}$ being homogeneous despite the fact that the concept of "automorphism" is a higher order concept, not formulable in the first order predicate calculus. In an extensive structure, condition (iii) of Theorem 4 follows from associativity, and it amounts to an interesting generalization of associativity, as the following theorem shows.

THEOREM 5. *Suppose* $\mathfrak{X} = \langle X, \succcurlyeq, \bigcirc \rangle$ *is a fundamental unit structure. Then there exists a* $\odot$*-representation* φ *of* $\mathfrak{X}$ *such that the following three statements are true*:

(1) *There exists* $f\colon X \to \mathrm{Re}^+$ *such that, for all* x, y *in* X, $\varphi(x) \odot \varphi(y) = \varphi(y) f[\varphi(x)/\varphi(y)]$;

(2) *for each* $\odot$*-representation* ψ *of* $\mathfrak{X}$, *there exists* r *in* Re^+ *such that* $\psi = r\varphi$; *and*

(3) *for each* r *in* Re^+, $r\varphi$ *is a* $\odot$*-representation of* $\mathfrak{X}$.

DEFINITION 7. $\odot$-representations φ of fundamental unit structures $\mathfrak{X}$ that satisfy statements (1), (2) and (3) of Theorem 5 are called *unit representations* of $\mathfrak{X}$.

Properties of fundamental unit structures are thoroughly explored in Cohen and Narens [**1979**], and we know a great deal about this class of structures. In particular the form of the function f in statement (1) of Theorem 5 is highly constrained.

The method of establishing Theorem 5 extends to other kinds of structures, and Narens [**1981**] has exploited this fact to show that ratio scalability–uniqueness of the representation up to multiplication by positive reals–holds in a variety of structures. The following theorem, which generalizes Theorem 5, is an instance of this approach.

THEOREM 6. *Suppose* X *is a nonempty set*, $\succcurlyeq$ *is a binary relation on* X, *and* $\mathfrak{X} = \langle X, \succcurlyeq, R_1, R_2, \dots \rangle$ *is a relational structure that has the following four properties*:

(1) $\langle X, \succcurlyeq \rangle$ *is a totally ordered and Dedekind complete.*

(2) $\mathfrak{X}$ *is homogeneous* (*Definition* 4).

(3) *The group of automorphisms of* $\mathfrak{X}$ *is commutative.*

(4) $\langle X, \succcurlyeq \rangle$ *is dense in the sense that if* x, y *are in* X *and* $x \succ y$, *then there is a* z *in* $\mathfrak{X}$ *with* $x \succ z \succ y$.

Then there exists a structure $\mathfrak{N} = \langle \mathrm{Re}^+, >, S_1, \dots, S_n, \dots \rangle$ *that is isomorphic to* $\mathfrak{X}$ *and is such that for all isomorphisms* φ, ψ *of* $\mathfrak{X}$ *onto* $\mathfrak{N}$, (i) *there exists* r *in* Re^+ *such that* $\psi = r\varphi$, *and* (ii) *for each* s *in* Re^+, $s\varphi$ *is an isomorphism of* $\mathfrak{X}$ *onto* $\mathfrak{N}$.

II. MORE THAN ONE OPERATION

6. Introduction. It is easy enough to speak of studying general relational structures of the form $\langle A, \succsim, R_2, \dots, R_n \rangle$, where $\succsim$ is a weak (or quasi) order. However, until recently (Narens [**1981**]), little measurement research has been done on the general case, and a good deal of attention has been concentrated on structures with two operations (or the equivalent thereof) that in one way or another can be mapped into addition and multiplication.

Perhaps the most natural example for mathematicians is the concept of a ring that can be mapped into subrings of $\langle \mathrm{Re}, >, +, \cdot \rangle$, and a generalization of this will be considered in §7 with applications to two different measurement problems: qualitative conditional probability and polynomial conjoint measurement.

To a physicist, it probably seems more natural to consider an ordering on the Cartesian product of two (or more) distinct sets, which under certain assumptions induces an operation on each component, together with an explicit operation on either the product itself or on one of the components. Here the explicit operation is represented by addition and the Cartesian product by multiplication. A typical example is the measure of kinetic energy, $\frac{1}{2}mv^2$, where mass and velocity are the component attributes, and mass, at least, possesses a concatenation operation. Much the same sort of representation occurs when we think of the probability measure over independent repetitions of an identical experiment (as in a random sample). We take up the latter in §8 and the former in §9.

§10 is devoted to the philosophical topic of meaningful statements in measurement contexts and, in particular, the relationship between this topic and the concept of dimensionally invariant laws in physics.

To a social scientist or a statistician, still another role for addition and multiplication comes to mind, namely, in the computation of expected values or, more generally, weighted averages. Much work along this line has centered on the specific economic problem of subjective expected utility theory, but the formalism can be interpreted more generally. This we take up in §11.

7. Semirings. In formulating the concept of a semiring in a form suitable for use in measurement theory, it is again necessary to work with partial operations.

DEFINITION 8. Suppose A is a set, $\succsim$ a binary relation on A, $\bigcirc$ and $*$ partial binary operations with domains $B^{\bigcirc}$ and $B^{*} \subseteq A \times A$. Then $\langle A, \succsim, B^{\bigcirc}, B^{*}, \bigcirc, *\rangle$ is said to be a *positive, regular Archimedean ordered local semiring* if and only if

1. $\langle A, \succsim, B^{\bigcirc}, \bigcirc\rangle$ is an extensive structure (Definition 2).
2. $\langle A, \succsim, B^{*}, *\rangle$ satisfies Axioms 1–4 and 8 of Definition 2.
3. (i) If $(b, c) \in B^{\bigcirc}$, $(a, b\bigcirc c) \in B^{*}$ then (a, b), $(a, c) \in B^{*}$ and $(a * b)\bigcirc(a * c) \in B^{\bigcirc}$ and $a*(b\bigcirc c) = (a * b)\bigcirc(a * c)$.
   (ii) The right distributive analogue of (i).
4. For $a \in A$, there exist $b, c \in A$ such that $(b, c) \in B^{\bigcirc}$ and $(a, b\bigcirc c) \in B^{*}$.

(The notion of an Archimedean ordered semiring presented here is a little more restricted than the one presented in Chapter 2 of Krantz et al. which does not assume that $\langle A, \succsim, B^{\bigcirc}, \bigcirc\rangle$ satisfies positivity as given in Definition 2 and assumes a weaker form of restrictive solvability than given in Definition 2.)

This generalizes the concept of an Archimedean ordered ring in which both operations are closed, $\langle A, \succsim, \bigcirc\rangle$ is an Archimedean ordered group, $\langle A, \bigcirc, *\rangle$ is a ring with zero element e, and if $a \succ e$, $b \succ c$, then $a * b \succ a * c$ and $b * a \succ c * a$. The following theorem generalizes the classic result that any

Archimedean ordered ring is uniquely isomorphic to a subring of $\langle \mathrm{Re}, >, +, \cdot \rangle$ (see Krantz et al. [1971, 2.27]).

THEOREM 7. *Suppose* $\langle A, \succsim, B^{\circ}, B^{*}, \circ, * \rangle$ *is a positive, regular Archimedean ordered local semiring. Then there is a unique homomorphism of* $\langle A, \succsim, B^{\circ}, B^{*}, \circ, * \rangle$ *into* $\langle \mathrm{Re}^{+}, >, \mathrm{Re}^{+2}, \mathrm{Re}^{+2}, +, \cdot \rangle$.

To date, this has been used in the proof of two measurement theorems. The first arises in the study of qualitative conditional probability.

Let $\mathcal{E}$ be an algebra of subsets of X, $\mathfrak{N} \subset \mathcal{E}$, and $\succsim$ a relation on $\mathcal{E} \times (\mathcal{E} - \mathfrak{N})$. The interpretation of $(A, B) \succsim (C, D)$ is that event A given event B is at least as probable as event C given event D. The following theorem can be shown.

THEOREM 8. *Suppose* $\langle X, \mathcal{E}, \mathfrak{N}, \succsim \rangle$ *satisfies the following eight axioms: for all* $A, A' \in \mathcal{E}$, $B, B', C, C' \in \mathcal{E} - \mathfrak{N}$,

1. $\succsim$ *is a weak order.*
2. $X \in \mathcal{E} - \mathfrak{N}$ *and* $A \in \mathfrak{N}$ *iff* $(A, X) \sim (\varnothing, X)$.
3. $(X, X) \sim (C, C)$ *and* $(X, X) \succsim (A, B)$.
4. $(A, B) \sim (A \cap B, B)$.
5. *Suppose* $A \cap B = A' \cap B' = \varnothing$. *If* $(A, C) \succsim (A', C')$ *and* $(B, C) \succsim (B', C')$, *then* $(A \cup B, C) \succsim (A' \cup B', C)$; *and if* $\succ$ *holds in either antecedent, it holds in the conclusion.*
6. *Suppose* $A \subseteq B \subseteq C$ *and* $A' \subseteq B' \subseteq C$. *If* $(B, C) \succsim (A', B')$ *and* $(A, B) \succsim (B', C')$, *then* $(A, C) \succsim (A', C')$.
7. (*Archimedean*) *Every standard sequence is finite, where* $\{A_i\}$ *is a standard sequence iff* $A_i \in \mathcal{E} - \mathfrak{N}$, $A_{i+i} \supseteq A_i$, *and* $(X, X) \succ (A_i, A_{i+i}) \sim (A_1, A_2)$.
8. (*Solvability*) *If* $(A, B) \succsim (A', C)$, *there exists* $A'' \in \mathcal{E}$ *such that* $A' \cap C \subseteq A''$ *and* $(A, B) \sim (A'', C)$.

There then exists a unique real-valued function P *on* $\mathcal{E}$ *such that for all* $A, A' \in \mathcal{E}$, $B, B' \in \mathcal{E} - \mathfrak{N}$,

(i) $\langle X, \mathcal{E}, P \rangle$ *is a finitely additive probability space.*
(ii) $N \in \mathfrak{N}$ *iff* $P(N) = 0$.
(iii) $(A, B) \succsim (A', B')$ *iff* $P(A \cap B)/P(B) > P(A' \cap B')/P(B')$.

The proof of Theorem 8 is given in Krantz et al. [1971, §5.6], and involves defining $\succsim'$ on $\mathcal{E}$ by $(A, X) \succsim (A', X)$ which with the union of disjoint sets as $\circ$ leads to an Archimedean ordered local ordered semigroup on $\mathcal{E}/\sim'$. Define $*$ on $\mathcal{E}/\sim'$ by $[A] * [B] = [C]$ iff $(A, X) \sim (C, B)$. Then one can show the conditions of Theorem 7 hold, from which the representation follows readily.

The other application of Theorem 7, which will be described in less detail, concerns the generalization of additive conjoint measurement to representation by simple polynomials (see Krantz et al. [1971, Chapter 7]). For $n = 3$, these are

$x + y + z$, xyz, $x(y + z)$ and $xy + z$ and all permutations of the symbols. More generally, a polynomial on n factors is *simple* if and only if, for some m, $0 < m < n$, it is the sum or the product of a simple polynomial on a set of m factors and another simple polynomial on the remaining $n - m$ factors. In some cases, such as $x(y + z)$, Theorem 7 is used to arrive at the representation from axioms somewhat like, but more complex than, those for additive conjoint measurement.

8. Independence in qualitative probability. In §1, the reduction of qualitative probability to extensive structures required the assumption of solvability conditions. This is also true for the reduction just given of conditional probability to local semirings.

The major idea to overcome the assumption of strong solvability conditions has been to incorporate, in one way or another, independence of events as a primitive of the structure. This is, of course, contrary to the spirit of Kolmogorov's [**1933**] axiomatization of numerical probability in which independence is a defined concept [A and B are independent iff $P(A \cap B) = P(A)P(B)$]. However, since there are qualitative probability structures with nonunique representations, it is clear that, in general, independence cannot be defined in terms of the qualitative ordering of the events. Moreover, in scientific practice, the independence of many events is assumed on considerations such as physical isolation, and not just through the above numerical definition, and this kind of postulated independence is widely used in arriving at estimates of probabilities from relative frequencies.

Two directions for incorporating independence have been tried. The first, and the one that initially seems more straightforward, is simply to add a binary relation $\perp$ on $\mathcal{E}$ to the structure $\langle X, \mathcal{E}, \succsim \rangle$, where X is a nonempty set, $\mathcal{E}$ an algebra of subsets on X, and $\succsim$ the "at least as likely as" ordering on $\mathcal{E}$, and then to search for axioms sufficient to yield a unique probability measure P on $\mathcal{E}$ such that P preserves the ordering $\succsim$ and if $A \perp B$, then $P(A \cap B) = P(A)P(B)$. At first one might hope for a local ring to be involved, but that hope is dashed when one realizes that the relation of independence is really quite irregular; in particular, it utterly fails the property that if $A \perp B$ and $A \succsim A'$ and $B \succsim B'$, then $A' \perp B'$, which is part of the ring definition. No one has yet seen how to make effective use of the extra independence primitive without imposing strong structural conditions that are unacceptable to many researchers (see Domotor [**1970**], and Krantz et al. [**1971**, §5.8]).

The second and far more successful approach is to model qualitatively the idea of an indefinite number of independent repetitions of an experiment. This is, after all, what lies qualitatively beneath the idea of independent repetitions of identical random variables, the resulting central limit theorem, and our standard procedure for estimating probabilities from data. True, there are experiments that cannot be repeated, and those await a better theory. For those that can be repeated, it is really quite surprising how strong the results are–in essence, they amount to the joint qualitative space being rich enough that the solvability conditions are met and so Theorem 1 is applicable, and this leads to a unique probability measure that is multiplicative over the repetitions. This, and more, were worked out in Kaplan's **[1971]** important master's thesis which was exposited in Kaplan and Fine **[1977]**. We summarize one of the key results here.

Suppose $\langle X, \mathcal{E}, \succsim \rangle$ is a nontrivial (there exists $A \in \mathcal{E}$ with $X \succ A \succ \varnothing$) qualitative probability structure, and $\langle X_i, \mathcal{E}_i, \succsim_i \rangle$, $i = 1, 2, \ldots,$ are isomorphic copies of it. Let $X^* = \times_{i=1}^{\infty} X_i$. For any $A \in \mathcal{E}$, let A^i denote that subset of elements in X^* such that the ith component falls in the isomorphic image of A. For any $\alpha \subseteq \{1, 2, \ldots\}$, denote by $\mathcal{E}(\alpha)$ the σ-algebra generated by A^i for all $A \in \mathcal{E}$ and all $i \in \alpha$. Let $\mathcal{E}^* = \mathcal{E}(\{1, 2, \ldots\})$. Finally, let $\succsim^*$ be a binary relation on $\mathcal{E}^*$.

DEFINITION 9. $\langle X^*, \mathcal{E}^*, \succsim^* \rangle$ is an *infinite, independent, identically distributed product space* for $\langle X, \mathcal{E}, \succsim \rangle$ if and only if the following three conditions hold for all $A, B \in \mathcal{E}$:

1. $A^i \succsim^* B^i$ iff $A \succsim B$.

2. Suppose $\alpha, \beta, \gamma, \delta \subseteq \{1, 2, \ldots\}$, $\alpha \cap \beta = \gamma \cap \delta = \varnothing$, $A \in \mathcal{E}(\alpha)$, $B \in \mathcal{E}(\beta)$, $C \in \mathcal{E}(\gamma)$, and $D \in \mathcal{E}(\delta)$. Then $A \succsim^* C$ and $B \succsim D$ implies $A \cap B \succsim^* C \cap D$, and if $\succ^*$ holds in either antecedent, then it holds in the conclusion.

3. $A^i \sim^* A^j$ *for all* $i, j \in \{1, 2, \ldots\}$.

THEOREM 9. *Suppose* $\langle X^*, \mathcal{E}^*, \succsim^* \rangle$ *is an infinite, independent, identically distributed product space for* $\langle X, \mathcal{E}, \succsim \rangle$ *with the property that* $\succsim^*$ *is monotonely continuous. Then there exists a unique countably additive probability measure* P^* *on* $\mathcal{E}^*$ *that preserves the ordering* $\succsim^*$ *and, for* $i \neq j$,

$$P(A^i \cap B^j) = P(A^i)P(B^j).$$

The proof rests on showing that $\langle X, \mathcal{E}^*, \succsim \rangle$ has no atoms and then invoking Villegas' **[1964]** theorem (for that case see Krantz et al. **[1971**, p. 216]). The multiplicative property then falls out moderately easily.

Luce and Narens **[1978]** present axioms for $\succsim_2$ on $\mathcal{E} \times \mathcal{E}$ that are sufficient to insure a finitely additive probability representation with the multiplicative representation of $\succsim_2$. They also prove that if $\succsim_n$ are orderings on $\mathcal{E}^n$, $n = 1, 2, \ldots,$ which all can be represented multiplicatively by a probability measure, then that measure is necessarily unique.

9. Distributive triples and dimensional analysis. Next we turn to the physicist's simultaneous use of addition and multiplication, where the former represents the combining operation within a dimension and the latter the combining operation between dimensions, as in expressions such as $\frac{1}{2}mv^2$. This is of considerable importance since, as is well known, the interaction of the major dimensions of classical physics is as products of powers of additive scales. This structure is reflected in the nature of the units of physical scales and it underlies the surprisingly powerful method of dimensional analysis. Although a variety of axiomatizations exist for the mathematical structure representing the dimension of physics, the most satisfactory is Whitney's [**1968**], which is reproduced in §10.2 of Krantz et al. [**1971**]. Within that framework, one formulates the idea of a similarity transformation, imposes the requirement that any numerical law of physics be invariant under those transformations, which is referred to as the property of *dimensional invariance*, and then proves Buckingham's [**1914**] π-theorem establishing that any such law is some function of the maximal number of independent dimensionless quantities that can be formed from the variables involved in the law (see Theorem 10.4 of Krantz et al. [**1971**, p. 466]).

From the point of view of a measurement theorist, two major questions must be considered. First, what is the qualitative nature of the interlock among dimensions that permits the representation of each dimension as the product of powers of a limited number of dimensions each of which has an associative operation? That is the topic of this section. Second, why should physical laws be dimensionally invariant? That is the topic of the next section.

DEFINITION 10. Let $\succsim$ be a reflexive relation on $A \times P$, and suppose $\bigcirc_A$ is a closed binary operation on A. Then $\bigcirc_A$ is said to be *distributive* if and only if for all $a, b, c, d \in A$, $p, q \in P$, whenever $(a, p) \sim (c, q)$ and $(b, p) \sim (d, q)$, then $(a \bigcirc_A b, p) \sim (c \bigcirc_A d, q)$.

As the following theorem of Narens [**1981**] shows, distributivity seems to be the key interlock exploited in the dimensional structures of physics.

THEOREM 10. *Suppose $\langle A \times P, \succsim \rangle$ is a conjoint structure that satisfies weak ordering, independence, and solvability with respect to ap for each ap in $A \times P$ (Definition 3), and suppose $\bigcirc_A$ is a closed operation on A, and that $\mathcal{Q} = \langle A, \succsim_A, \bigcirc_A \rangle$ is a fundamental unit structure with unit representation φ_A (Definition 7).*

1. *If $\bigcirc_A$ is distributive, there exists φ_P on P such that, for all $a, b \in A$, $p, q \in P$,*

$$(a, p) \succsim (b, q) \quad \text{iff} \quad \varphi_A(a)\varphi_P(p) \geq \varphi_A(b)\varphi_P(q).$$

2. *$\bigcirc_A$ is distributive if and only if, for all automorphisms θ of $\mathcal{Q}$ and all $a, b \in A, p, q \in P$,*

$$(a, p) \sim (b, q) \quad \text{iff} \quad (\theta(a), p) \sim (\theta(b), q).$$

3. *If, in addition $\bigcirc_P$ is a closed operation on P such that $\langle P, \succsim_P, \bigcirc_P \rangle$ is a fundamental unit structure with unit representation φ_P, then there is some constant α such that, for every $\beta > 0$, $\varphi_A^{\beta}\varphi_P^{\alpha\beta}$ preserves the ordering relation $\succsim$ on $A \times P$.*

Parts 1 and 3 of Theorem 10 were established for associative operations in Narens and Luce [**1976**]. Proofs of Theorem 10 can be found in Narens [**1981**], Krantz et al. [in preparation], and Narens [in preparation].

In order to develop the usual numerical representation of a large number of interlocked physical dimensions, it is necessary to postulate an adequate density of distributive triples of the sort described in parts 1 and 4 of Theorem 10. When done properly–the details are given in Luce [1978] and Krantz et al. [in preparation]–one is able to show the existence of a finite basis in fundamental unit structures such that all other scales are represented as products of powers of these scales.

From the point of view of a nonphysical scientist, the major interest in these results is that they show exactly what is involved in adding a new dimension to those already discovered by physics. Whether or not such a fundamental dimension will be discovered in the biological realm is conjectural. What is not conjectural is the qualitative conditions that such a new dimension would have to exhibit in order to become a part of the existing structure of dimensions.

10. Qualitative and quantitative meaningfulness. Once qualitative information is recast numerically–when measurement is possible–one needs to consider carefully just which numerical statements do and do not correspond to something meaningful in the underlying structure. As a trivial example, if only order is preserved, there is nothing in the qualitative structure corresponding to $x + y = z$. This problem has been explored with some care, but a complete consensus as to its solution does not yet exist. The source of the uncertainty centers on exactly what is meant by saying that a relation is "meaningful" in a relational structure that is characterized axiomatically. Intuitively, one would like to say that a relation is meaningful if and only if it can be defined in terms of the primitives of the structure, but there is at this time no appropriate formal definition of "defined" that is useful in general measurement contexts.

This inability to define directly these kinds of relevant concepts by some procedure of formal logic forces one to consider some sort of indirect procedure. The one exploited by measurement theorists rests on the idea that if a relation is definable in terms of the defining relations of a structure, then adding it to the structure does not further restrict the structure from the point of view of its measurement theoretic properties. Looking just at a structure, this implies that (1) it should remain invariant under the same transformations of the structure into itself that leave invariant the defining relations of the structure, i.e., the *endomorphisms* of the structure. But it also implies that (2) it does not alter the homomorphisms into numerical structures, which in turn suggests that (3) the homomorphic images of the relation should be invariant under the endomorphisms of the representing structure. However, it is by no means obvious which of these three conditions is the strongest or when they agree. We examine this more carefully.

Suppose $\mathcal{A} = \langle A, S_1, S_2, \ldots, S_n \rangle$ is a relational structure and $\mathcal{R} = \langle R, T_1, T_2, \ldots, T_n \rangle$ is a numerical relational structure with T_i being of the same order as S_i, $i = 1, 2, \ldots, n$. Suppose there is at least one homomorphism

φ from $\mathcal{A}$ into $\mathcal{R}$, i.e., for each $i = 1, 2, \ldots, n$, $(a_1, a_2, \ldots, a_{k_i}) \in S_i$ iff $(\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_{k_i})) \in T_i$. Let $\Phi(\mathcal{A}, \mathcal{R})$ denote the set of all homomorphisms. For any relation T of order k on R and any $\varphi \in \varphi(\mathcal{A}, \mathcal{R})$ define

$$S(\varphi, T) = \{(a_1, \ldots, a_k) | (\varphi(a_1), \ldots, \varphi(a_k)) \in T\}.$$

DEFINITION 11. A relation T on R is *$\mathcal{A}$-reference invariant relative* to $\mathcal{R}$ if and only if, for all $\varphi, \psi \in \Phi(\mathcal{A}, \mathcal{R})$, $S(\varphi, T) = S(\psi, T)$, in which case the common value is denoted $S(T)$. A relation S on A that is $\mathcal{A}$-reference invariant relative to $\mathcal{A}$ is said to be *structure invariant*.

Put another way, S is structure invariant if and only if it is invariant under the endomorphisms of $\mathcal{A}$. The term "reference invariant" is due to Adams, Fagot and Robinson [**1965**], and the general definition, to Pfanzagl [**1968, 1971**].

In order to avoid pointless distinctions, let us suppose that $\mathcal{A}$ is *irreducible* in the sense that all homomorphisms are one-to-one.

The following theorem is shown in Krantz et al [in preparation].

THEOREM 11. *Suppose $\mathcal{A}$ is irreducible and $\Phi(\mathcal{A}, \mathcal{R}) \neq \varnothing$. Let T be a relation of order k on R. If T is $\mathcal{A}$-reference invariant relative to $\mathcal{R}$, then $S(T)$ is structure invariant in $\mathcal{A}$. If, further, $T \subseteq \bigcup_{\varphi \in \Phi} \Phi(A^k)$, where A^k is the Cartesian product of A with itself k times, then T is structure invariant in $\mathcal{R}$.*

One can show that if a relation S on A is structure invariant in $\mathcal{A}$, it is not necessary for there to exist T on R such that T is $\mathcal{A}$-reference invariant relative to $\mathcal{R}$ and $S = S(T)$. Since, however, if such a T were to exist, the three notions of invariance would agree, it is thus interesting to know when this occurs. We formulate a simple, intuitively natural sufficient condition for such an occurrence.

Let $\varphi \in \Phi(\mathcal{A}, \mathcal{R})$ and let γ be a mapping of $\varphi(A)$ into R that leaves invariant the restriction of $\mathcal{R}$ to $\varphi(A)$; we speak of this as a *partial endomorphism*. It is not difficult to see that $\varphi' \in \Phi(\mathcal{A}, \mathcal{R})$ if and only if there is a partial endomorphism γ such that $\varphi' = \gamma\varphi$. The converse--given φ and γ, there exists the endomorphism α of $\mathcal{A}$ such that $\varphi\alpha = \gamma\varphi$--is not generally true. In case it is, we say $\mathcal{R}$ is *compatible* with $\mathcal{A}$. Krantz et al. [in preparation] show the following result:

THEOREM 12. *Suppose $\mathcal{A}$ is irreducible and $\mathcal{R}$ is compatible with $\mathcal{A}$.*

1. *A relation T on R is $\mathcal{A}$-reference invariant relative to $\mathcal{R}$ iff T is structure invariant in $\mathcal{R}$.*

2. *A relation S on A is structure invariant if and only if there exists a relation T on R such that T is $\mathcal{A}$-reference invariant relative to $\mathcal{R}$ and $S(T) = S$.*

Under the conditions of Theorem 12, it is widely believed that the common concept of invariance correctly captures the idea of a meaningful relation. When they do not agree, the strongest--reference invariance--is probably the appropriate one, if any are, for most measurement situations.

An easy to remember and often applicable sufficient condition for compatibility, and so for Theorem 12, is that for every $\varphi \in \Phi(\mathcal{A}, \mathcal{R})$, $\varphi(A) = R$; in this case all endomorphisms are automorphisms.

There have been two major applications of these ideas. The first and perhaps the most important for the behavioral and social sciences is to characterize the types of statistical hypotheses that are meaningful in different measurement structures. Indeed, the whole problem of meaningfulness was initiated when Stevens [**1946**], [**1951**] pointed out that various familiar statistics are invariant under some groups of automorphisms that arise in well-known measurement systems but are not invariant under other groups. This observation and the structures he drew from it have led to a somewhat confused literature on the subject, but we need not go into that here.

The other application (Luce [**1978**]) is to the problem of dimensionally invariant laws in the structure of dimensional quantities. In essence, one proceeds as follows. The qualitative development based on distributive triples, outlined in the last section, is assumed, and it has a homomorphic representation in Whitney's structure which in turn is isomorphic to a multiplicative vector space over the reals. Both of these are then mapped in a quite natural way into relational structures, which it turns out are compatible. In that reformulation, dimensionally invariant laws correspond to relations that are structure invariant, and so by Theorem 12 they correspond exactly to qualitative relations that are structure invariant and hence to qualitatively meaningful ones. Thus, according to Luce, the answer to the question, "Why are physical laws dimensionally invariant?" is that this class of laws corresponds exactly to the class of all meaningful qualitative relations, provided we accept reference invariance or its equivalent in this context, structure invariance, as defining meaningfulness.

The concepts of meaningfulness discussed above are explored more fully in Narens [**1981**], Krantz et al. [in preparation] and Narens [in preparation]. The relationship between dimensional invariance and meaningfulness is discussed in Luce [**1978**] and Krantz et al. [in preparation].

11. Averages and expected utility theory. Our final examples of a numerical representation involving both addition and multiplication are averages of the form $\Sigma w_i \varphi_i / \Sigma w_i$, $w_i > 0$. In economics, psychology, and statistics this representation occurs as the expectation of random variables, especially in the theories of subjective expected utility. In psychology Anderson [**1974a**], [**1974b**] has also successfully applied averaging representations to category scale data from a wide variety of substantive areas.

The literature includes two somewhat different approaches which, fortunately, can be described in closely parallel terms. Let $\mathcal{C}$ be a set (of outcomes), X a set (sample space), and $\mathcal{E}$ an algebra of subsets of X. One type of theory concerns a weak ordering of the set $\mathcal{A}$ of all functions f: X into $\mathcal{C}$ subject to the restriction that, for all $c \in f(X)$, $f^{-1}(c) \in \mathcal{E}$. Such functions are called "acts" because of the decision theory interpretation. The most important examples of such a theory is Savage's [**1954**] subjective (personalistic) expected utility.

The other type of theory concerns a weak order of a set $\mathcal{D}$ of functions of the form: for $A \in \mathcal{E}$, $A \neq \varnothing$, $f_A: A \to \mathcal{C}$ subject to the following two restrictions: for all $A, B \in \mathcal{E} - \{\varnothing\}, f_A, g_B \in \mathcal{D}$,

(i) if $A \cap B = \varnothing, f_A \cup g_B \in \mathcal{D}$;

(ii) if $A \supseteq B$, the restriction of f_A to B is $\in \mathcal{D}$.

These functions are called "conditional acts". Pfanzagl **[1959]**, **[1967b]** examined a special case of such a theory, and Luce and Krantz **[1971]** studied the general case (also see Krantz et al. **[1971**, Chapter 8]).

The axioms, especially the structural and Archimedean ones, are sufficiently complex in both cases that we do not present them here. Rather we discuss the two most important necessary conditions, describe the line of proof, and state the representations obtained.

Both theories capture two essential facts about averages. The one is that on a *fixed* domain it looks just like measurement on additive conjoint structures. The other is that on disjoint domains, the average of two equivalent conditional acts is equivalent to them. To state these in the unconditional theory, one in essence has to define what is meant by a conditional act. By contrast, they are quite direct in the conditional theory, so we state them explicitly.

For all $A, B \in \mathcal{E} - \{\varnothing\}, A \cap B = \varnothing, f_A, f'_A, g_B \in \mathcal{D}$,

(i) $f_A \succsim f'_A$ iff $f_A \cup g_B \succsim f'_A \cup g_B$;

(ii) if $f_A \sim g_B$, then $f_A \cup g_B \sim f_A$.

The line of argument in the unconditional theory is to let $\succsim$ on $\mathcal{Q}$ induce an ordering on $\mathcal{E}$ and to assume an axiom adequate to prove the existence of a unique probability measure P that preserves the induced ordering. At that point the proof follows that of von Neumann and Morgenstern **[1947**, p. 617] leading to a real-valued function u on $\mathcal{C}$ such that, for all $f, g \in \mathcal{Q}, f \succsim g$ iff $E_P[u(f)] \geq E_P[u(g)]$, where E_P is the expectation operator with respect to P.

The line of argument in the conditional theory is to note that any partition of X into three or more nonnull subevents generates an additive conjoint structure. Using the results about uniqueness of representations for this structure, one is able to show simultaneously the existence of a real-valued function v on $\mathcal{D}$, unique up to positive linear transformations, and a unique probability measure P on $\mathcal{E}$ such that, for all $f_A, g_B \in \mathcal{D}$,

(i) $f_A \succsim g_B$ iff $v(f_A) \geq v(g_B)$, and

(ii) if $A \cap B = \varnothing$, then $v(f_A \cup g_B) = v(f_A)P(A|A \cup B) + v(g_B)P(B|A \cup B)$.

If, in addition, for each $c \in \mathcal{C}$ there exists $A \in \mathcal{E} - \{\varnothing\}$ and $c_A \in \mathcal{D}$ with $c_A(a) = c$ for $a \in A$, and if whenever $c_A, c_B \in \mathcal{D}$, $c_A \sim c_B$, then one can show there exists a real-valued function u on $\mathcal{C}$ such that $v(f_A) = E_P[u(f_A)]$.

For finite X, it has been shown how to map either expectation representation into the other (Krantz et al. **[1971**, §8.6.4]).

As theories of decision making, both suffer from problems of interpretation. The inclusion of all possible acts, including all constant ones, in the unconditional theory is highly unrealistic. The meaning of $f_A \cup g_B$ in the conditional theory is obscure since the choice of a conditional act appears to place $\cup$ under the control of the decision maker, whereas the axioms force a fixed probability on $\mathcal{E}$. For an extended discussion of these matters, see Balch [1974], Balch and Fishburn [1974], Krantz and Luce [1974], and Spohn [1977].

A general averaging model of the type assumed by Anderson [1974a], [1974b] arises from the conditional theory as follows. Let $X = \{1, 2, \ldots, n\}$, $\mathcal{E} = 2^X$, $\mathcal{C}_i$, $i \in X$, be sets and define

$$\mathfrak{D} = \{f_A | A \in \mathcal{E} - \{\varnothing\}, f_A(i) \in \mathcal{C}_i \text{ for } i \in A\}.$$

Observe that if $A \cap B = \varnothing$ and $f_A, g_B \in \mathfrak{D}$, then, automatically, $f_A \cup g_B \in \mathfrak{D}$. Assuming the axioms of the conditional theory, it follows readily that there exist nonnegative weights $w_i = P(\{i\})$ and a real-valued function φ_i on $\mathcal{C}_i$ such that

$$u(f_A) = \sum_{i \in A} w_i \varphi_i[f_A(i)] / \sum_{i \in A} w_i$$

preserves the ordering relation $\succsim$. For details, see Luce [1981].

Conclusions

Our understanding of positive concatenation and of conjoint structures is reasonably adequate when the automorphism group is dense and especially so for fundamental unit structures. In contrast, we know very little about the discrete and trivial cases. Undoubtedly, many of these are so irregular as to be of no conceivable scientific interest, but some are clearly of importance, witness the case of probability.

Our understanding of the interplay of solvable conjoint structures having at least one component that is a fundamental unit structure is adequate when they satisfy distributivity, as appears to be the case for classical physics. It is totally unsatisfactory when distributivity does not hold, as arises with velocity in relativistic physics. The problem evidences itself in our inability to relate closely the automorphisms of the conjoint structure and that of the positive concatenation one. Closely related to these problems of dimensional interlocks is the general conceptual issue of meaningfulness and how it should be defined in terms of automorphisms and/or endomorphisms and/or some other invariance concept. Judging by the importance of dimensional analysis, this issue is of rather more than just philosophical interest.

The study of structures with more than one operation, including the case just mentioned, is probably susceptible to considerable generalization, just as fundamental unit structures generalized extensive ones. At the moment, all of the theories lead only to polynomial or metric representations. Almost certainly, more powerful algebraic proof techniques will be required since the existing methods seem to be leading to ever more complex, not easily generalized proofs.

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