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Note

A Note on Weber's Law for Conjoint Structures

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Throughout this paper the following definitions and conventions will be observed: Re^+ will denote the positive reals and I^+ the positive integers. $\gtrsim_X$ will denote a binary relation on the set X, and for each x, y in X, (i) $x \sim_X y$ iff $x \gtrsim_X y$ and $y \gtrsim_X x$, and (ii) $x >_X y$ iff $x \gtrsim_X y$ and not $y \gtrsim_X x$. If $\succ$ is a binary relation, then for all u, v, $u \not\succ v$ iff it is not the case that $u \succ v$. If $\bigcirc$ is an associative operation, then $1x = x$, and for each $n \in \mathrm{I}^+$, $(n + 1)x = (nx) \bigcirc x$.

In the measurement of a set X of physical quantities, one usually has a qualitative ordering relation $\gtrsim_X$ and a qualitative concatenation operation $\bigcirc$ defined on the physical quantities. A fundamental measurement on X then consists of finding a real value function that maps $\gtrsim_X$ into $\geqslant$ and $\bigcirc$ into $+$. Qualitative axioms in terms of the primitives $\gtrsim_X$ and $\bigcirc$ have been given which guarantee that fundamental measurements on X exist. Such axiomatic structures are often called in the literature—especially the psychological literature—"extensive structures".

DEFINITION 1. Let X be a nonempty set, $\gtrsim_X$ be a binary relation on X, and $\bigcirc$ be a binary operation on X. Then $(X, \gtrsim_X, \bigcirc)$ is said to be a (dense, positive, and closed) *extensive* structure if an only if the following six axioms hold for all x, y, z, w in X:

1. $\gtrsim_X$ is a weak ordering, i.e., $\gtrsim_X$ is transitive and connected.
2. $\bigcirc$ is an associative operation.
3. $x \gtrsim_X y$ iff $x \bigcirc z \gtrsim_X y \bigcirc z$ iff $z \bigcirc x \gtrsim_X z \bigcirc y$.
4. $x \bigcirc y >_X x$.
5. If $x >_X y$ then there exists u such that $x >_X y \bigcirc u >_X y$.
6. There exist $n \in \mathrm{I}^+$ such that $nx \gtrsim_X y$. ▮

DEFINITION 2. Let $\mathscr{X} = \langle X, \gtrsim_X, \bigcirc\rangle$ be an extensive structure. Then φ is said to be a *representation* of $\mathscr{X}$ if and only if φ is a function from X into Re^+ such that the following two conditions hold for each x, y in X:

(i) $x \gtrsim_X y$ iff $\varphi(x) \geqslant \varphi(y)$;
(ii) $\varphi(x \bigcirc y) = \varphi(x) + \varphi(y)$.

THEOREM 1. *Let $\mathscr{X} = \langle X, \gtrsim_X, \bigcirc\rangle$ be an extensive structure. Then there exists a*

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representation of $\mathscr{X}$. Furthermore, if φ and ψ are two representations of $\mathscr{X}$, then for some r in Re$^+$, $\varphi = r\psi$.

Proof. Chapter 3 of Krantz *et al.* (1971). ∎

In many psychophysical experiments, various pairs from a set X of physical stimuli are presented to a subject who judges whether one of the pair is "noticeably greater than" (i.e., "noticeably brighter than", or "noticeably louder than", etc.) than the other of the pair. The set of physical stimuli, X, has a naturally define binary relation $\gtrsim_X$ and a naturally defined concatenation operation on it so that (according to physics) $\mathscr{X} = \langle X, \gtrsim_X, \bigcirc\rangle$ is an extensive structure. The qualitative "noticeably greater than" relation, $\succ$, belongs to the subject and is psychological. In the Nineteenth Century, Gustav Fechner suggested that much of psychophysical scaling could be accounted for by what he called "Weber's Law" which states that associated with each fundamental measurement φ of $\mathscr{X}$ is a real valued function $\Delta\varphi$ on X such that for each pair of stimuli x, y in X, the following two conditions hold:

(i) $x \succ y$ iff $\varphi(x) > \varphi(y) + \Delta_\varphi(y)$;

(ii) $\dfrac{\Delta_\varphi(x)}{\varphi(x)} = \dfrac{\Delta_\varphi(y)}{\varphi(y)}$.

We will now give a qualitative characterization of Weber's Law.

Definition 3. By Definition, and throughout the rest of this paper, let $\mathscr{F}$ be the following structure: $\mathscr{F} = \langle X, \gtrsim_X, \bigcirc, \succ\rangle$ where $\succ$ is a binary relation on X and the following four axioms hold for each x, y, z, w in X:

Ax. 1. $\langle X, \gtrsim_X, \bigcirc\rangle$ is an extensive structure.

Ax. 2. $\succ$ is transitive and irreflexive.

Ax. 3. (i) There exists u in X such that $x \succ u$; (ii) if $x \not\succ y$ and $x \gtrsim_X w$, then $w \not\succ y$; (iii) if $x \succ y$ and $z \gtrsim_X x$, then $z \succ y$; (iv) if $x \gtrsim_X y$ and $y \bigcirc z \not\succ y$, then $x \bigcirc z \not\succ x$; (v) if $x \succ y \succ z$ and $z \bigcirc w \not\prec z$, then $x \gtrsim_X w$.

Ax. 4. (i) If $x \succ y$ and $z \succ w$, then $x \bigcirc z \succ y \bigcirc w$; (ii) if $x \not\succ y$ and $z \not\succ w$, then $x \bigcirc z \not\succ y \bigcirc w$.

In Definition 3, Axiom 1 states a physical fact, Axiom 2 is a mathematical formulation of some of the semantical properties that "noticeably greater than" should have, Axiom 3 are some properties that relations which give rise to threshold structures should have, and Axiom 4 captures the essence of Weber's Law.

Holman [1974] has given an axiomatization of Weber's law that does not use order information.

Convention. Throughout the rest of this paper, let φ be a representation of $\langle X, \gtrsim_X, \bigcirc\rangle$.

DEFINITION 4. Define the function Δ_φ on X as follows: for each x in X, if for some w in X, $x \bigcirc w \not\succ x$, then

$$\Delta_\varphi(x) = \sup\{\varphi(u) \mid x \bigcirc u \not\succ x\},$$

and if for all w in X, $x \bigcirc w \succ x$, then

$$\Delta_\varphi(x) = 0.$$

THEOREM 2. *The following three conditions hold for each x, y in X:*

(i) *if $\varphi(x) > \varphi(y) + \Delta_\varphi(y)$, then $x \succ y$;*

(ii) *if $\varphi(y) + \Delta_\varphi(y) > \varphi(x)$, then $x \not\succ y$;*

(iii) $$\frac{\Delta_\varphi(x)}{\varphi(x)} = \frac{\Delta_\varphi(y)}{\varphi(y)}.$$

The proof of Theorem 2 is not difficult and will be omitted. (Also, the proof of Holman's (1974) related result can be modified to produce a proof of this theorem.)

The qualitative conditions for Weber's Law (Definition 3) interact nicely with "distributivity," a qualitative condition for conjoint structures which was first investigated in Narens (1976) and then more fully in Narens & Luce (1976). There are various kinds of distributive structures, and Theorem 3 gives a representation theorem for one of these. Theorem 4 illustrates the type of interaction that takes place between distributivity and Weber's Law.

DEFINITION 5. $\langle Y \times P, \succsim', \bigcirc' \rangle$ is said to be a $Y \times P$-*distributive* structure if and only if the following four axioms hold for all x, y, z in Y and all p, q, r in P:

1. $\langle Y \times P, \succsim', \bigcirc' \rangle$ is an extensive structure (Definition 1).

2. *Independence:* (i) if for some u in Y, $up \succsim' uq$, then $xp \succsim' xq$; and (ii) if for some s in P, $ys \succsim' zs$, then $yr \succsim' zr$.

3. *Solvability:* there exists u in Y and s in P such that $xp \sim' uq$ and $xp \sim' ys$.

4. $Y \times P$-*distributivity:* $xp \bigcirc' xq \sim' xr$ if and only if $yp \bigcirc' yq \sim' yr$.

THEOREM 3. *Let $\mathscr{D} = \langle Y \times P, \succsim', \bigcirc' \rangle$ be a $Y \times P$-distributive structure and F be a representation of $\mathscr{D}$. Then there exist functions ζ on Y and ψ on P such that for each yp in $X \times P$, $F(yp) = \zeta(y)\,\psi(p)$.*

Proof. Section 5 of Narens & Luce (1976). ∎

THEOREM 4. *Let $\mathscr{D} = \langle Y \times P, \succsim', \bigcirc' \rangle$ be a $Y \times P$-distributive structure, F be a representation of $\mathscr{D}$, and $\langle Y \times P, \succsim', \bigcirc', \succ' \rangle$ satisfy Axioms 1-4 of Definition 3. Then there exist a positive real number c and functions Δ_F on F, ζ and Δ_ζ on Y, and ψ and Δ_ψ on P such that the following six conditions hold for each xp, yq in $Y \times P$:*

(i) *if* $F(xp) > F(yq) + \Delta_F(yq)$, *then* $xp \succ' yq$;

(ii) *if* $F(yq) + \Delta_F(yq) > F(xp)$, *then* $xp \not\succ' yq$;

(iii) $\Delta_F(xp)/F(xp) = c$;

(iv) $F(xp) = \zeta(x)\,\psi(p)$;

(v) $\Delta_\zeta(x)\,\Delta_\psi(p) = c\Delta_F(xp)$;

(vi) $\Delta_\zeta(x)/\zeta(x) = \Delta_\psi(p)/\psi(p) = c$.

Proof. Let Δ_F be defined in an analogous manner to Δ_φ in Definition 4. Let $c = \Delta_F(zr)/F(zr)$ for some zr in $Y \times P$. Then (i), (ii), and (iii) follow from Theorem 2. Then by Theorem 3, let ζ and ψ be functions on Y and P respectively such that for each uv in $Y \times P$, $F(uv) = \zeta(u)\,\psi(p)$. Thus (iv) has been shown. Let xp, yq be arbitrary elements of $Y \times P$. Define Δ_ζ on Y and Δ_ψ on P as follows:

$$\Delta_\zeta(x) = c\zeta(x),$$

$$\Delta_\psi(p) = c\psi(p).$$

Then

$$\Delta_\zeta(x)\,\Delta_\psi(p) = c^2\zeta(x)\,\psi(p) = c^2F(xp) = \left[\frac{\Delta_F(xp)}{F(xp)}\right]^2 F(xp) = c\,\Delta_F(xp).$$

Thus we have shown (v). Also,

$$\frac{\Delta_\zeta(x)}{\zeta(x)} = c = \frac{\Delta_\psi(p)}{\psi(p)},$$

and thus (vi) holds. ▌

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