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Fundamental Unit Structures: A Theory of Ratio Scalability[1]

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A theory of nonassociative measurement structures is developed which produces a natural generalization of associative measurement (i.e., extensive structures), and representation and uniqueness theorems are established for these generalized structures, and it is shown that in many cases these representations are ratio scales. The methods of proof strongly relate the structure of the automorphism group of the nonassociative structure to its underlying concatenation operation.

1. INTRODUCTION

Measurement theory is concerned with the nature of numerical representations of empirical structures. It strives to give clear descriptions of the forms of numerical representations in terms of axiomatizations of the empirical structures and to give criteria for the drawing of proper inferences from the numerical representations. It is also concerned with the theory of error. Although the empirical structures ordinarily encountered in psychology, physics, and other sciences may be complex, the construction of representations for these structures can usually be resolved in terms of certain basic "fundamental" structures. This is the approach of Krantz, Luce, Suppes, & Tversky [1971], and the mechanisms for resolving complex measurement structures in terms of "fundamental" ones is explicitly laid out in Narens and Luce [1976].

In the Krantz *et al.* approach, complex structures are resolved into what they call *extensive structures*, which are structures of the form $\mathscr{X} = \langle X, \succsim, \bigcirc \rangle$ which, for the purposes of this introduction, we may take X to be a nonempty set of empirical objects, $\succsim$ to be a total ordering on X, and $\bigcirc$ to be an operation on X (which is sometimes called a *concatenation* operation). A (numerical) representation for $\mathscr{X}$ is then an isomorphic imbedding of $\mathscr{X}$ into the positive reals, where $\succsim$ is mapped into $\geqslant$ and $\bigcirc$ is mapped into

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some binary associative operation $\odot$ on the positive reals. If φ is the isomorphic imbedding described above, then we say that φ is a *$\odot$-representation* for $\mathscr{X}$. Note that the isomorphism φ and the associativity of $\odot$ forces $\bigcirc$ to be associative. The definition of $\odot$-representation for structures with nonassociative $\odot$ is identical to the associative case except $\odot$ is not assumed to be associative.

Krantz *et al.* give axioms for $\mathscr{X}$ that are very weak but nevertheless yield the existence of additive representations (i.e., $+$-representations for $\mathscr{X}$) that are "unique" in the sense that if φ and ψ are two additive representations for $\mathscr{X}$ then for some positive real r, $r\varphi = \psi$. The associativity of $\bigcirc$ is the crucial empirical axiom that makes the additivity of representations possible. In the traditional approach employed by Krantz *et al.*, heavy use of associativity is made for showing both the existence of a $\odot$-representation and the uniqueness of such representations. Narens and Luce (1976) introduced a generalization of extensive structures called ***positive concatenation structures*** that satisfies all the Krantz *et al.* (1971) axioms for extensive structures except possibly associativity. It turns out that these structures have $\odot$-representations for some $\odot$ (where of course in this case $\odot$ may not be associative), and these representations are "unique" in the sense that if φ and ψ are two $\odot$-representations for $\mathscr{X}$ such that for some x, $\varphi(x) = \psi(x)$, then $\varphi = \psi$. (See Theorem 2.3 for an exact statement of this result.)

Extensive structures naturally appear in theoretical physics, where all the basic units of measurement form extensive structures. In psychology, direct applications of extensive structures to empirical phenomena is far more rare: Psychological concatenations of stimuli are not common, and when they do occur, are usually nonassociative. However, psychologists have successfully utilized "indirect concatenations" through the techniques of conjoint measurement. It was probably Luce and Tukey's 1964 paper on additive conjoint measurement which stimulated research on the idea that interactions between variables can be viewed as a positive concatenation structure. We will now briefly outline this procedure.

Let $\gtrsim$ be a weak ordering (i.e., a transitive and connected) relation on the nonempty set $Y \times P$. Assume ab is the smallest element in $Y \times P$, i.e., $yp \gtrsim ab$ for all yp in $Y \times P$. By assuming a condition called ***independence*** (if $xb \gtrsim yb$ then for all p in P, $xp \gtrsim yp$; and if $aq \gtrsim ar$, then for all z in Y, $zq \gtrsim zr$), $\gtrsim$ naturally induces weak orderings $\gtrsim_Y$ and $\gtrsim_P$ on Y and P, respectively. To conform to our previous notation, we will assume the two latter orderings are total orderings and write them as $\succcurlyeq_Y$ and $\succcurlyeq_P$. By assuming a condition called *local solvability* (for each xp in $Y \times P$, there exist y and q such that $yb \sim xp$ and $xb \sim aq$), a function f from Y onto P and an operation $\bigcirc_Y$ on Y can be defined such that for all x, y in Y,

$$(x \bigcirc_Y y)b \sim xf(y), \tag{1.1}$$

so that in the sense of Eq. (1.1), the concatenation operation $\bigcirc_Y$ captures the interaction between the Y and P dimensions of the structure $\langle Y \times P, \gtrsim\rangle$. With a few very plausible assumptions about the ordering $\gtrsim$, $\mathscr{Y} = \langle Y, \succcurlyeq_Y, \bigcirc_Y\rangle$ becomes a positive concatenation structure. (The interested reader should consult Narens & Luce [1976] for a correct and precise statement of this result.) Now for $\mathscr{Y}$ to be an extensive structure, $\bigcirc_Y$ must be associative, and this can only happen if additional conditions are imposed on the ordering $\succsim$. One frequently used condition that is equivalent to the associativity of $\bigcirc_r$ is called *double cancellation* and is extensively discussed in Krantz *et al.* (1971). However, in most interesting psychological situations such conditions seem to fail.

Extensive structures have two important properties in measurement: They have additive representations and have $\odot$-representations (e.g., $\odot = +$) that are ratio scalable. Positive concatenation structures with nonassociative operations cannot have additive representations; however, as will be shown in this paper, they can have $\odot$-representations that are ratio scalable. Since ratio scalability rather than additivity is the essential ingredient in many measurement situations (e.g., dimensional analysis in physics), these types of positive concatenation structures provide the basis for a natural generalization of much of current measurement theory.

To summarize to this point, extensive structures have wide use in physical measurement but little in psychology. In fact, empirical structures of the form $\langle X, \succcurlyeq, \bigcirc \rangle$ are not very important in psychology since natural psychological concatenations of stimuli are difficult to find. In psychology, conjoint structures are much more natural and prevalent, and in a natural way, important psychological conjoint structures can be *interpreted* as structures of the form $\mathscr{X} = \langle X, \succcurlyeq, \bigcirc \rangle$, but where $\mathscr{X}$ is a positive concatenation structure rather than an extensive structure. Although positive concatenation structures do not have additive representations unless they are extensive structures, they do have representations with strong uniqueness results and may have $\odot$-representations that form ratio scales. All of this suggests that a closer look be given to positive concatenation structures, and this is what we do in this paper.

In Section 2, positive concatenation structures are studied in terms of their automorphism groups, and it is shown that the total ordering relation $\succcurlyeq$ of a positive concatenation structure $\mathscr{X}$ naturally induces a total ordering $\succcurlyeq'$ on the automorphism group of $\mathscr{X}$ by the definition

$$\alpha \succcurlyeq' \beta \text{ iff for some } x,\ \alpha(x) \succcurlyeq \beta(x).$$

What is probably the most surprising result of this section is that the resulting ordered group of automorphisms is Archimedean and thus any two automorphisms must commute.

In Section 3, fundamental unit structures are investigated. These are positive concatenation structures $\mathscr{X} = \langle X, \succcurlyeq, \bigcirc \rangle$ such that $\langle X, \succcurlyeq \rangle$ is Dedekind complete and $\mathscr{X}$ is homogeneous in the sense that for each x, y in X there is an automorphism α of $\mathscr{X}$ such that $\alpha(x) = y$. The principal results of this section are that (1) such structures have $\odot$-representations that are ratio scalable for some $\odot$, and (2) if φ and ψ are $\odot$ and $\odot'$-representations for $\mathscr{X}$, respectively, that are ratio scalable, then for some positive reals r and s, $\varphi = r\psi^s$. Alternative characterizations of fundamental unit structures are also considered in this section.

Section 4 is concerned with numerical fundamental unit structures. The principal result of this section is that explicit methods can be given for transforming such structures into representations that are ratio scalable provided that certain weak differentiability conditions hold.

Section 5 gives necessary and sufficient conditions for the imbeddability of positive concatenation structures into fundamental unit structures.

2. Positive Concatenation Structures

Throughout this paper, Re will denote the real numbers, Re^+ the positive reals, I the integers, and I^+ the positive integers. A function $\bigcirc : Y \times Z \to X$ is said to be a *partial* (*binary*) *operation* on X if Y and Z are subsets of X, and a *closed* (*binary*) *operation* (or just *operation*) if $Y = Z = X$. If $\bigcirc$ is a partial operation, then $x \bigcirc y$ is said to be *defined* if (x, y) is in the domain of $\bigcirc$, and otherwise $x \bigcirc y$ is said to be *undefined*. As usual, $1x = x$, and if $(nx) \bigcirc x$ is defined for some n in I^+, then $(n + 1)x = (nx) \bigcirc x$.

Definition 2.1. Let X be a nonempty set, $\succsim$ a binary relation on X, and $\bigcirc$ a partial binary operation on X. The structure $\mathscr{X} = \langle X, \succsim, \bigcirc\rangle$ is a *totally ordered, positive concatenation structure* if and only if the following seven axioms hold for all w, x, y, z in X:

Axiom 1. *Total ordering*: $\succsim$ is a total ordering.

Axiom 2. *Nontriviality*: There exist u, v in X such that $u \succ v$.

Axiom 3. *Local definability*: If $x \bigcirc y$ is defined, $x \succsim w$, and $y \succsim z$, then $w \bigcirc z$ is defined.

Axiom 4. *Monotonicity*: (i) if $x \bigcirc z$ and $y \bigcirc z$ are defined, then,

$$x \succsim y \quad \text{iff} \quad x \bigcirc z \succsim y \bigcirc z,$$

and (ii) if $z \bigcirc x$ and $z \bigcirc y$ are defined, then

$$x \succsim y \quad \text{iff} \quad z \bigcirc x \succsim z \bigcirc y.$$

Axiom 5. *Restricted solvability*: If $x \succ y$, then there exists u such that $x \succ y \bigcirc u$.

Axiom 6. *Positivity*: If $x \bigcirc y$ is defined, then $x \bigcirc y \succ x$ and $x \bigcirc y \succ y$.

Axiom 7. *Archimedean*: There exists $n \in I^+$ such that either nx is not defined or $nx \succsim y$.

Convention. Throughout the rest of this paper let $\mathscr{X} = \langle X, \succsim, \bigcirc\rangle$ be a totally ordered, positive concatenation structure.

Definition 2.2. x_i is said to be a *net* in $\mathscr{X}$ if and only if x_i is a sequence of elements of X such that for each x in X there exists n in I^+ such that for all $i \geqslant n$, $x \succ x_i$.

LEMMA 2.1. *The following three propositions are true*:

(i) *For each x in X there exists y in X such that $x \succ y \bigcirc y$.*

(ii) *There exists a net in $\mathscr{X}$.*

(iii) *For each net x_i in $\mathscr{X}$, $\{nx_i \mid n, i \in I^+\}$ is a dense subset of $\langle X, \succcurlyeq \rangle$.*

Proof. Lemmas 2.1 and the proof of Lemma 2.2 of Narens and Luce (1976). ∎

DEFINITION 2.3. α is said to be an automorphism of $\mathscr{X}$ iff $\alpha: X \to X$ is onto X and for each x, y in X, $x \succcurlyeq y$ iff $\alpha(x) \succcurlyeq \alpha(y)$, and $\alpha(x \bigcirc y) = \alpha(x) \bigcirc \alpha(y)$.

CONVENTION. Throughout the rest of this paper let ι denote the identity automorphism of $\mathscr{X}$.

THEOREM 2.1. *Let α be an automorphism of $\mathscr{X}$. Then the following three propositions are true*:

(i) *If $\alpha(x) = x$ for some x, then $\alpha = \iota$.*

(ii) *If $\alpha(x) \succ x$ for some x, then $\alpha(y) \succ y$ for all y in X.*

(iii) *If $\alpha(x) \prec x$ for some x, then $\alpha(y) \prec y$ for all y in X.*

Proof. *Case* 1. $Y = \{x \mid \alpha(x) = x\}$ contains a net. Then $\alpha(x) = x$ for some x. Suppose u is such that $\alpha(u) \neq u$. We need to only show a contradiction. If $u \succ \alpha(u)$, then by Lemma 2.1, part (iii), let y in Y and n in I^+ be such that

$$u \succ ny \succ \alpha(u). \tag{2.1}$$

Then by Definition 2.3,

$$\alpha(u) \succ \alpha(ny) = n\alpha(y) = ny,$$

which contradicts Eq. (2.1). Similarly $\alpha(u) \succ u$ leads to a contradiction. Thus $\alpha(u) = u$ for all u in X, i.e., $\alpha = \iota$.

Case 2. $B = \{x \mid \alpha(x) \succ x\}$ contains a net. Then $\alpha(x) \succ x$ for some x. We will show $\alpha(y) \succ y$ for all y in X by contradiction. First suppose u is such that $u \succ \alpha(u)$. Then by Lemma 2.1, part (iii), let n in I^+ and y in B be such that

$$u \succ ny \succ \alpha(u). \tag{2.2}$$

Then by Eq. 2.2,

$$\alpha(u) \succ \alpha(ny) = n\alpha(y) \succ ny,$$

which contradicts Eq. (2.2). Next suppose that v is such that $\alpha(v) = v$. Since B contains a net, let w in B be such that $w \prec v$. Then since w is in B, $\alpha(w) \succ w$. By restricted solvability, let z be such that $\alpha(w) \succ w \bigcirc z$. Let t in B be such that $z \succ t$. Then $\alpha(t) \succ t$ and by monotonicity,

$$\alpha(w) \succ w \bigcirc t.$$

Thus

$$\alpha^2(w) \succ \alpha(w) \bigcirc \alpha(t) \succ (w \bigcirc t) \bigcirc t \succ 2t,$$

and

$$\alpha^3(w) \succ \alpha^2(w) \bigcirc \alpha^2(t) \succ (2t) \bigcirc \alpha^2(t) \succ (2t) \bigcirc t = 3t,$$

and by induction,

$$\alpha^n(w) \succ nt,$$

for each n in I^+. Thus by Archimedean, let m in I^+ be such that

$$\alpha^m(w) \succ v. \tag{2.3}$$

Since $v \succ w$ and $\alpha(v) = v$,

$$\alpha(v) = v \succ \alpha(w),$$
$$\alpha^2(v) = \alpha(v) = v \succ \alpha^2(w),$$

and thus by induction,

$$v \succ \alpha^m(w),$$

which contradicts Eq. (2.3). Thus, in summary, we have shown $\alpha(y) \succ y$ for each y in X.

Case 3. $C = \{x \mid \alpha(x) \prec x\}$ contains a net. Then $C = \{x \mid \alpha^{-1}(x) \succ x\}$ contains a net, and thus by *Case* 2, $\alpha^{-1}(y) \succ y$ for all y in X, i.e., $y \succ \alpha(y)$ for all y in X.

Since by Lemma 2.1, X contains a net, it follows that Case 1, 2, or 3 must hold, and the theorem immediately follows. ∎

DEFINITION 2.4. φ is said to be a $\odot$-*representation* for $\mathscr{X}$ if and only if $\varphi: X \to \mathrm{Re}^+$ such that the following three conditions hold for all x, y in X:

(i) $\langle \varphi(X), \geqslant, \odot \rangle$ is a positive concatenation structure;

(ii) $x \succsim y$ iff $\varphi(x) \geqslant \varphi(y)$;

(iii) $\varphi(x \bigcirc y) = \varphi(x) \odot \varphi(y)$.

THEOREM 2.2. *There exist* $\odot$ *and* φ *such that* φ *is a* $\odot$-*representation for* $\mathscr{X}$.

Proof. Theorem 2.1 of Narens and Luce (1976). ∎

LEMMA 2.2. *The following two propositions are equivalent:*

(i) *For all $\odot$-representations φ and ψ of $\mathscr{X}$ such that $\varphi(X) = \psi(X)$, if for some x, $\varphi(x) = \psi(x)$, then $\varphi = \psi$.*

(ii) *For all automorphisms α of $\mathscr{X}$, if for some x, $\alpha(x) = x$, then $\alpha = \iota$.*

Proof. Assume (i). Let α be an automorphism of $\mathscr{X}$ and x be such that $\alpha(x) = x$. By Theorem 2.2, let φ and $\odot$ be such that φ is a $\odot$-representation for $\mathscr{X}$. Then $\varphi\alpha$ is a $\odot$-representation for $\mathscr{X}$ and $\varphi(X) = \varphi\alpha(X)$. Thus, by assumption, $\varphi\alpha = \varphi$. Since α is a one-to-one function, it follows that α is the identity, ι.

Assume (ii). Let φ, ψ be $\odot$-representation for $\mathscr{X}$ and let x be such that $\varphi(x) = \psi(x)$. Then $\varphi^{-1}\psi$ is an automorphism and $x = \varphi^{-1}\psi(x)$. Thus, by assumption, $\varphi^{-1}\psi = \iota$, and therefore $\varphi = \psi$. ∎

THEOREM 2.3. *Suppose φ, ψ are $\odot$-representations for $\mathscr{X}$, $\varphi(X) = \psi(X)$, and $x \in X$ is such that $\varphi(x) = \psi(x)$. Then $\varphi = \psi$.*

Proof. Let $\alpha = \varphi^{-1}\psi$. Then α is an automorphism of $\mathscr{X}$ and $\alpha(x) = x$. Thus by Theorem 2.1, $\alpha = \iota$, and thus, since φ and ψ are one-to-one functions, $\varphi = \psi$. ∎

Narens and Luce (1976, p. 201) prove Theorem 2.3 with different assumptions, namely, that the assumption $\varphi(X) = \psi(X)$ is replaced by: For each x in X there exists y such that $x = y \bigcirc y$.

DEFINITION 2.5. Define the binary relation $\succcurlyeq'$ on the set of automorphisms of $\mathscr{X}$, A, as follows: For each α, β in A, $\alpha \succcurlyeq' \beta$ iff for some x in X, $\alpha(x) \succcurlyeq \beta(x)$.

CONVENTION. Throughout the rest of this paper we shall confuse notation a little and often write $\succcurlyeq$ for $\succcurlyeq'$ (as defined in Definition 2.5). We shall also let A denote the set of automorphisms of X, and often we shall denote composition of members of A by an asterisk. (Thus $\alpha[\beta(x)] = (\alpha * \beta)(x)$.) Furthermore, $\mathscr{G}$ will denote the structure $\langle A, \succcurlyeq, *\rangle$ and will be called *the (totally) ordered group of automorphisms of* $\mathscr{X}$. We will often follow the practice in mathematics of confusing A and $\mathscr{G}$, i.e., $\alpha \in \mathscr{G}$ will mean that α is an automorphism of $\mathscr{X}$. We will also often write integral multiples of elements of A in exponential notation, $\alpha = \alpha^1$, and $\alpha^{n+1} = \alpha^n * \alpha$.

THEOREM 2.4. *$\mathscr{G}$ is an Archimedean, totally ordered group.*

Proof. It is well known that $\langle A, *\rangle$ is a group.

Let α, β, γ be arbitrary elements of A. Note that by Theorem 2.1,

$$\begin{aligned} \alpha \succcurlyeq \beta \text{ iff } \alpha(x) &\succcurlyeq \beta(x) \qquad \text{for some } x \\ \text{iff } \alpha(y) &\succcurlyeq \beta(y) \qquad \text{for all } y \text{ in } X. \end{aligned} \tag{2.4}$$

We will first show that $\langle A, \succcurlyeq\rangle$ is totally ordered.

(*Transitivity*) Suppose $\alpha \succsim \beta$ and $\beta \succsim \gamma$. Then by Eq. (2.4), $\alpha(y) \succsim \beta(y)$ and $\beta(y) \succsim \gamma(y)$ and thus $\alpha(y) \succsim \gamma(y)$ for all y in X. Therefore $\alpha \succsim \gamma$.

(*Connectivity*) By Eq. (2.4), either for each y in X, $\alpha(y) \succsim \beta(y)$, or for each y in X, $\beta(y) \succsim \alpha(y)$, and thus either $\alpha \succsim \beta$ or $\beta \succsim \alpha$. Now suppose $\alpha \succsim \beta$ and $\beta \succsim \alpha$. Then for each y in X, $\alpha(y) \succsim \beta(y)$ and $\beta(y) \succsim \alpha(y)$, i.e., $\alpha(y) = \beta(y)$, and thus $\alpha = \beta$. Thus $\succsim$ is a total ordering on A.

We next show that $\mathscr{G}$ is an ordered group: Let y be an arbitrary element of X. Then by Eq. (2.4),

$$\begin{aligned} \alpha \succsim \beta &\text{ iff } \alpha(y) \succsim \beta(y) \\ &\text{ iff } \gamma[\alpha(y)] \succsim \gamma[\beta(y)] \\ &\text{ iff } \gamma * \alpha \succsim \gamma * \beta, \end{aligned}$$

and

$$\begin{aligned} \alpha \succsim \beta &\text{ iff } \alpha(y) \succsim \beta(y) \\ &\text{ iff } \alpha[\gamma(y)] \succsim \beta[\gamma(y)] \\ &\text{ iff } \alpha * \gamma \succsim \beta * \gamma. \end{aligned}$$

Finally, we will show that $\mathscr{G}$ is Archimedean. Suppose $\alpha \succ \iota$. We need only show that $\alpha^n \succsim \beta$ for some n in I^+. Suppose not; i.e., suppose that $\beta \succ \alpha^n$ for all n in I^+. A contradiction will be shown. Let $x \in X$. Since $\alpha \succ \iota$, it follows from Eq. (2.4) that

$$\alpha(x) \succ \iota(x) = x.$$

By *restricted solvability* (Axiom 5, Definition 2.1), let u be such that

$$\alpha(x) \succ x \bigcirc u.$$

$$\alpha^2(x) \succ \alpha(x \bigcirc u) = \alpha(x) \bigcirc \alpha(u) \succ (x \bigcirc u) \bigcirc u \succ 2u,$$

and

$$\alpha^3(x) \succ \alpha^2(x) \bigcirc \alpha^2(u) \succ (2u) \bigcirc u = 3u,$$

and by induction, for each n in I^+,

$$\alpha^n(x) \succ nu. \tag{2.5}$$

However, since $\beta \succ \alpha^n$ for each n in I^+,

$$\beta(x) \succ \alpha^n(x) \succ nu$$

for each n in I^+, and this contradicts that $\mathscr{X}$ is Archimedean. ▌

DEFINITION 2.6. α in A is said to be *positive* iff $\alpha \succ \iota$. $\mathscr{G}$ is said to be *trivial* iff $A = \{\iota\}$.

$\mathscr{G}$ is said to be *discrete* if and only if $\mathscr{G}$ has a smallest positive automorphism. $\mathscr{G}$ is said to be *dense* if and only if $\mathscr{G}$ is nontrivial and nondiscrete.

THEOREM 2.5. *If $\mathscr{G}$ is nontrivial then $\bigcirc$ is closed.*

Proof. Suppose $\mathscr{G}$ is nontrivial. Let α be a positive automorphism of $\mathscr{X}$ and x, y be elements of X. By Lemma 2.1 and restricted solvability, let u, v in X and m in I^+ be such that

$$\alpha(u) \succ u \bigcirc v \succ u, \tag{2.6}$$

$$\alpha(y) \succ mv \succ y, \tag{2.7}$$

and

$$\alpha(x) \succ x \bigcirc u \succ x. \tag{2.8}$$

It then follows from the proof that $\mathscr{G}$ is Archimedean (Eq. (2.5) in Theorem 2.4),

$$\alpha^m(u) \succ mv. \tag{2.9}$$

Thus by Eqs. (2.6)–(2.9) and local definability,

$$\alpha^{m+1}(x) \succ \alpha^m(x \bigcirc u) = \alpha^m(x) \bigcirc \alpha^m(u) \succ x \bigcirc \alpha^m(u) \succ x \bigcirc mv \succ x \bigcirc y,$$

i.e., $x \bigcirc y$ is defined. ∎

THEOREM 2.6. *Suppose $\bigcirc$ is not closed and φ, ψ are $\bigcirc$-representation for $\mathscr{X}$ such that $\varphi(X) = \psi(X)$. Then $\varphi = \psi$.*

Proof. By Theorem 2.5, the identity, ι, is the only automorphism of $\mathscr{X}$. Since $\varphi^{-1}\psi$ is an automorphism of $\mathscr{X}$, $\varphi^{-1}\psi = \iota$, and thus $\varphi = \psi$ since φ and ψ are one-to-one functions. ∎

EXAMPLE 2.1. (Examples of positive concatenation structures with dense groups of automorphisms.) For each r, s in Re^+, let $\alpha_r(s) = r \cdot s$. Let $\bigcirc_1$, $\bigcirc_2$, $\bigcirc_3$ be defined on Re^+ as follows: For each x, y in Re^+,

$$x \bigcirc_1 y = x + y,$$
$$x \bigcirc_2 y = x + y + x^{1/2} \cdot y^{1/2},$$
$$x \bigcirc_3 y = x + y + x^{1/4}y^{3/4}.$$

Then it is easy to verify that $\bigcirc_1$ is associative, O_2 is commutative and nonassociative, $\bigcirc_3$ is nonassociative and noncommutative, and $\mathscr{X}_1 = \langle \mathrm{Re}^+, \succsim, \bigcirc_1 \rangle$, $\mathscr{X}_2 = \langle \mathrm{Re}^+, \succsim, \bigcirc_2 \rangle$, $\mathscr{X}_3 = \langle \mathrm{Re}^+, \succsim, \bigcirc_3 \rangle$ are positive concatenation structures, and for each r in Re^+, α_r is an automorphism for $\mathscr{X}_1$, $\mathscr{X}_2$, $\mathscr{X}_3$.

Examples of positive concatenation structures with discrete and trivial groups of automorphisms will be given later.

The following lemmas are used in the subsequent development.

LEMMA 2.3. *Suppose α is a positive automorphism of $\mathscr{X}$, J is an infinite subset of I^+, and $y \in X$. Then $\{\alpha^{-n}(y) \mid n \in J\}$ is a net.*

Proof. Suppose not. Let v in X be such that $\alpha^{-n}(y) \succcurlyeq v$ for each n in J. Then it follows that there exists x such that $\alpha^{-n}(y) \succ x$ for each n in J since X contains a net. Thus $y \succ \alpha^n(x)$ for each n in J. By *restricted solvability*, let u *be* such that $\alpha(x) \succ x \bigcirc u \succ x$. Then by the proof that $\mathscr{G}$ is Archimedean (Eq. (2.5) in Theorem 2.4), it follows that for each n in J,

$$y \succ \alpha^n(x) \succ nu,$$

which contradicts that $\mathscr{X}$ is Archimedean. ∎

LEMMA 2.4. *Suppose $\mathscr{G}$ is dense and x, y are elements of $\mathscr{X}$ such that $x \succ y$. Then there exist α, β in A such that*

$$x \succ \alpha(x),\ \beta(y) \succ y.$$

Proof. We will first show that there exists a positive β in A such that $x \succ \beta(y) \succ y$. Suppose not, i.e., suppose for all positive β in A, $\beta(y) \succcurlyeq x$. A contradiction will be shown. Let z be such that $x \succ y \bigcirc z$. Then for each positive γ in A,

$$\begin{aligned}
&\gamma(y) \succcurlyeq x \succ y \bigcirc z \succ z,\\
&\gamma^2(y) \succ \gamma(y \bigcirc z) = \gamma(y) \bigcirc \gamma(z) \succ \gamma(y) \bigcirc z \succ z \bigcirc z = 2z,\\
&\gamma^3(y) \succ \gamma^2(y \bigcirc z) = \gamma^2(y) \bigcirc \gamma^2(z) \succ \gamma^2(y) \bigcirc z \succ (2z) \bigcirc z = 3z,
\end{aligned}$$

and by induction, for each n in I^+,

$$\gamma^n(y) \succ nz. \tag{2.10}$$

Let ξ be a positive element of A. Since $\mathscr{X}$ is Archimedean, let m in I^+ be such that

$$mz \succ \xi(y). \tag{2.11}$$

Since $\mathscr{G}$ is an Archimedean ordered group and is dense, let η be a positive element of A such that

$$\xi \succ \eta^m. \tag{2.12}$$

Then it follows from Eqs. (2.12), (2.10), and (2.11) that

$$\xi(y) \succ \eta^m(y) \succ mz \succ \xi(y),$$

which is a contradiction. Thus for some positive β in A, $x \succ \beta(y) \succ y$.

By the above, let β be such that $x \succ \beta(y) \succ y$. Then

$$x \succ \beta^{-1}(x) \succ \beta^{-1}[\beta(y)] = y. \quad \blacksquare$$

LEMMA 2.5. *$\mathscr{G}$ is commutative.*

Proof. By Theorem 2.4, $\mathscr{G}$ is an Archimedean, totally ordered group, and it is well known that all Archimedean, totally ordered groups are commutative. $\blacksquare$

LEMMA 2.6. *Suppose $\mathscr{G}$ is dense and z is an arbitrary element of X. Then $\{\theta(z) \mid \theta \in A\}$ is a dense subset of $\langle X, \succsim \rangle$.*

Proof. Let x, y be arbitrary elements of X such that $x \succ y$. There are three cases to consider.

Case 1. $x \succsim z \succsim y$. Then by Lemma 2.4, for some θ in A, $x \succ \theta(z) \succ y$.

Case 2. $z \succ x$. By Lemma 2.4, let α in A be such that $x \succ \alpha(x) \succ y$. Then $\iota \succ \alpha$. Thus by Lemma 2.3, $\{\alpha^n(z) \mid n \in I^+\}$ is a net. Thus let n be the largest positive integer such that $\alpha^n(z) \succsim x$. Then

$$x \succ \alpha^{n+1}(z) \succ \alpha(x) \succ y.$$

Case 3. $y \succ z$. By Lemma 2.4, let β in A be such that $x \succ \beta(y) \succ y$. Then $\beta \succ \iota$. Thus by the proof of Theorem 2.4, particularly Eq. (2.5), let w in X be such that for all n in I^+, $\beta^n(z) \succ nw$. Since $\mathscr{X}$ is Archimedean and $\beta^{n+1}(z) \succ \beta^n(z)$, it then follows that there exists a maximal m in I^+ such that $y \succsim \beta^m(z)$. Then

$$x \succ \beta(y) \succsim \beta^{m+1}(z) \succ y. \quad \blacksquare$$

3. FUNDAMENTAL UNIT STRUCTURES

DEFINITION 3.1. $\mathscr{X}$ is said to be ***homogeneous*** if and only if for each x, y in X, there exist α in $\mathscr{G}$ such that $\alpha(x) = y$.

DEFINITION 3.2. For each α, β in $\mathscr{G}$, let $\alpha \bigcirc' \beta$ be the function from X into X defined as follows: For each x in X,

$$(\alpha \bigcirc' \beta)(x) = \alpha(x) \bigcirc \beta(x).$$

LEMMA 3.1. *Suppose $\mathscr{X}$ is homogeneous and α, β are elements of $\mathscr{G}$. Then $\alpha \bigcirc' \beta$ is in $\mathscr{G}$.*

Proof. Let a be a fixed element of X. Since $\mathscr{X}$ is homogeneous, let γ in $\mathscr{G}$ be such that

$$\gamma(a) = (\alpha \bigcirc' \beta)(a).$$

We will show that $\gamma = \alpha \bigcirc' \beta$ and thus show $\alpha \bigcirc' \beta$ is in $\mathscr{G}$. Let x be an arbitrary element of X. Since $\mathscr{X}$ is homogeneous, let θ be an element of $\mathscr{G}$ such that $\theta(a) = x$. Then, by the commutativity of $\mathscr{G}$,

$$\begin{aligned} \gamma(x) &= \gamma[\theta(a)] \\ &= \theta[\gamma(a)] \\ &= \theta[(\alpha \bigcirc' \beta)(a)] \\ &= \theta[\alpha(a) \bigcirc \beta(a)] \\ &= \theta[\alpha(a)] \bigcirc \theta[\beta(a)] \\ &= \alpha[\theta(a)] \bigcirc \beta[\theta(a)] \\ &= \alpha(x) \bigcirc \beta(x). \quad \blacksquare \end{aligned}$$

LEMMA 3.2. *Suppose $\mathscr{X}$ is homogeneous. Then nx is an automorphism of $\mathscr{X}$ for all n in I^+.*

Proof. $1x$ is the identity automorphism, ι, in $\mathscr{G}$. Suppose $m \in I^+$ and mx is in $\mathscr{G}$. Then by Lemma 3.1, $(m + 1)x = (mx) \bigcirc x$ is in $\mathscr{G}$. Thus by induction, nx is in $\mathscr{G}$ for all n in I^+. $\blacksquare$

LEMMA 3.3. *Suppose for each n in I^+, nx is in $\mathscr{G}$. Then $\mathscr{G}$ is dense.*

Proof. $\mathscr{G}$ is nontrivial. Suppose $\mathscr{G}$ is discrete. Let α be the smallest positive automorphism of $\mathscr{G}$. For each n in I^+, let $\lambda(n)$ be the positive integer such that $\alpha^{\lambda(n)} = nx$. Since $(n + 1)x \succ nx$ and α is the smallest positive automorphism in $\mathscr{G}$, it follows that for each n in I^+,

$$\alpha^{\lambda(n+1)} = (n + 1)x \succcurlyeq \alpha * \alpha^{\lambda(n)} = \alpha^{\lambda(n)+1}.$$

Let z be an arbitrary element of X. Then since $\alpha^{\lambda(n)}(z) = nz$, $\alpha^{\lambda(n)}(z)$ becomes arbitrarily large for large n, and thus $\alpha^{-\lambda(n)}(z)$ becomes arbitrarily small for large n. Thus by restricted solvability, let m in I^+ be such that

$$\alpha(z) \succ z \bigcirc \alpha^{-\lambda(m)}(z).$$

Then

$$\begin{aligned} \alpha^{\lambda(m)+1}(z) &= \alpha^{\lambda(m)}[\alpha(z)] \succ \alpha^{\lambda(m)}(z) \bigcirc z \\ &= mz \bigcirc z = (m + 1)z = \alpha^{\lambda(m+1)}(z), \end{aligned}$$

i.e., $\alpha^{\lambda(m)+1} \succ \alpha^{\lambda(m+1)}$. But this contradicts that $\alpha^{\lambda(m+1)} \succcurlyeq \alpha^{\lambda(m)+1}$. $\blacksquare$

LEMMA 3.4. *Suppose $\mathscr{G}$ is dense and $\mathscr{X}$ is Dedekind complete. Then $\mathscr{G}$ is Dedekind complete.*

Proof. Let B be a nonempty subset of automorphisms of $\mathscr{X}$ that is bounded above by the automorphism α.

Then for each x in X, $\{\beta(x) \mid \beta \in B\}$ is a nonempty subset of X bounded by $\alpha(x)$. Thus by Dedekind completeness in $\langle X, \succcurlyeq \rangle$, for each x in X, let

$$\theta(x) = \text{l.u.b.}\{\beta(x) \mid \beta \in B\}.$$

We will show that θ is an automorphism of $\mathscr{X}$. Let x, y be arbitrary elements of X.

1. We will show $\theta(x \bigcirc y) = \theta(x) \bigcirc \theta(y)$. Suppose $\theta(x \bigcirc y) \succ \theta(x) \bigcirc \theta(y)$. By Lemma 2.6, let γ in $\mathscr{G}$ be such that

$$\theta(x \bigcirc y) \succ \gamma(x \bigcirc y) \succ \theta(x) \bigcirc \theta(y). \tag{3.1}$$

Since for each β in B $\theta(x) \succcurlyeq \beta(x)$ and $\theta(y) \succcurlyeq \beta(y)$,

$$\theta(x) \bigcirc \theta(y) \succcurlyeq \beta(x) \bigcirc \beta(y) = \beta(x \bigcirc y)$$

for each β in B. Thus for each β in B, $\gamma(x \bigcirc y) \succ \beta(x \bigcirc y)$ and thus $\gamma \succ \beta$. Therefore,

$$\gamma(x \bigcirc y) \succcurlyeq \text{l.u.b.}\{\beta(x \bigcirc y) \mid \beta \in B\} = \theta(x \bigcirc y),$$

and thus contradicts Eq. (3.1).

Suppose $\theta(x) \bigcirc \theta(y) \succ \theta(x \bigcirc y)$. Let γ in $\mathscr{G}$ be such that

$$\theta(x) \bigcirc \theta(y) \succ \gamma(x \bigcirc y) \succ \theta(x \bigcirc y). \tag{3.2}$$

For all β in B,

$$\gamma(x \bigcirc y) \succ \theta(x \bigcirc y) \succcurlyeq \beta(x \bigcirc y),$$

and thus $\gamma \succ \beta$. Therefore, for each β in B,

$$\gamma(x) \succcurlyeq \beta(x) \qquad \text{and} \qquad \gamma(y) \succcurlyeq \beta(y),$$

from which it follows

$$\gamma(x) \bigcirc \gamma(y) \succcurlyeq \theta(x) \bigcirc \theta(y),$$

and since $\gamma(x) \bigcirc \gamma(y) = \gamma(x \bigcirc y)$, it follows that

$$\gamma(x \bigcirc y) \succcurlyeq \theta(x) \bigcirc \theta(y),$$

and this contradicts Eq. (3.2).

Since not $\theta(x \bigcirc y) \succ \theta(x) \bigcirc \theta(y)$ and not $\theta(x) \bigcirc \theta(y) \succ \theta(x \bigcirc y)$, it follows from the fact that $\succcurlyeq$ is a total ordering that $\theta(x \bigcirc y) = \theta(x) \bigcirc \theta(y)$.

2. We will now show that $x \succcurlyeq y$ iff $\theta(x) \succcurlyeq \theta(y)$. It is sufficient to show that if $x \succ y$ then $\theta(x) \succ \theta(y)$. Thus suppose $x \succ y$. By Lemma 2.1, let z in X and m in I^+ be such that

$$x \succ (m+1)z \succ mz \succ y. \tag{3.3}$$

From Eq. (3.3) and the definition of θ,

$$\theta(x) \succcurlyeq \theta[(m+1)z] \succcurlyeq \theta(mz) \succcurlyeq \theta(y). \tag{3.4}$$

However, by part 1 above,

$$\theta[(m+1)z] = (m+1)\,\theta(z) \qquad \text{and} \qquad \theta(mz) = m\theta(z).$$

Thus since $(m+1)\,\theta(z) \succ m\theta(z)$, by Eq. (3.4), $\theta(x) \succ \theta(y)$.

3. We will now show that θ is onto X. Let u be an arbitrary element of X. Let

$$Y = \{\beta^{-1}(u) \mid \beta \in B\}.$$

Then Y is bounded below by $\alpha^{-1}(u)$. Since $\langle X, \succcurlyeq \rangle$ is Dedekind complete, let v be the greatest lower bound of Y. For each β in B, since $\beta^{-1}(u) \succcurlyeq v$, it follows that $u \succcurlyeq \beta(v)$, and thus $u \succcurlyeq \theta(v)$. Suppose that $u \succ \theta(v)$. A contradiction will be shown. For each β in B,

$$u \succ \theta(v) \succcurlyeq \beta(v).$$

Let γ in $\mathscr{G}$ be such that

$$u \succ \gamma[\theta(v)] \succ \theta(v).$$

Then, for each β in B,

$$u \succ \gamma * \theta(v) \succcurlyeq \gamma * \beta(v),$$

and thus since $*$ is commutative,

$$u \succ \beta * \gamma(v),$$

which yields

$$\beta^{-1}(u) \succ \gamma(v) \succ v.$$

This contradicts v being the g.l.b. of Y. Thus $u = \theta(v)$, and since u is an arbitrary element of X, it follows that θ is onto X.

Parts 1, 2, and 3 above establish that θ is an automorphism of $\mathscr{X}$. θ is an upper bound of B. Suppose the automorphism γ is another upper bound of B. Then for each β in B and each w in X, $\gamma(w) \succcurlyeq \beta(w)$ and thus $\gamma(w) \succcurlyeq \theta(w)$, i.e., $\gamma \succcurlyeq \theta$. Thus θ is the l.u.b. of Y. ▮

DEFINITION 3.3. $\mathscr{X}$ is said to be a *fundamental unit structure* if and only if $\mathscr{X}$ is Dedekind complete and homogeneous.

LEMMA 3.5. *Suppose $\mathscr{X}$ is Dedekind complete and $\mathscr{G}$ is dense. Then $\mathscr{X}$ is a fundamental unit structure.*

Proof. Let x, y be arbitrary elements of X. Let

$$B = \{\beta \mid \beta \text{ is in } A \text{ and } \beta(x) \preccurlyeq y\}$$

and

$$C = \{\beta \mid \beta \text{ is in } A \text{ and } \beta(x) \succ y\}.$$

Then B and C are nonempty subsets of A and (B, C) is a Dedekind cut of $\langle A, \succcurlyeq\rangle$. Let θ be the cut element of (B, C). Suppose $\theta(x) \neq y$. A contradiction will be shown.

Case 1. $\theta(x) \prec y$. Let γ in A be such that

$$\theta(x) \prec \gamma[\theta(x)] \preccurlyeq y.$$

Then $\gamma * \theta$ is in B and $\theta \prec \gamma * \theta$ which contradicts that θ is the cut element of (B, C)

Case 2. $y \prec \theta(x)$. Let γ in A be such that

$$y \prec \gamma[\theta(x)] \prec \theta(x).$$

Then $\gamma * \theta$ is in C and $\gamma * \theta \prec \theta$ which contradicts that θ is the cut element of (B, C). ▮

THEOREM 3.1. *Let $\mathscr{X}$ be Dedekind complete. Then the following three propositions are equivalent*:

(i) *$\mathscr{G}$ is dense*;

(ii) *$\mathscr{X}$ is a fundamental unit structure*;

(iii) *nx is in $\mathscr{G}$ for each n in I^+.*

Proof. (i) implies (ii) by Lemma 3.5; (ii) implies (iii) by Lemma 3.2; and (iii) implies (i) by Lemma 3.3. ▮

EXAMPLE 3.1. (An example of a Dedekind complete, commutative positive concatenation structure with a discrete group of automorphisms.) Let $\mathscr{X} = \langle \mathrm{Re}^+, \succsim, \bigcirc \rangle$, where $\bigcirc$ is defined as follows: For each $x, y > 0$,

$$x \bigcirc y = x + y + (xy)^{1/2}(2 + \sin[\tfrac{1}{2}\log(xy)]).$$

Then $\mathscr{X}$ is a positive concatenation structure, and α_n defined by

$$\alpha_n(z) = ze^{2\pi n}$$

is an automorphism of $\mathscr{X}$ for each nonnegative integer n. Thus $\mathscr{X}$ has a nontrivial group of automorphisms. Now, if $\mathscr{X}$ had a dense group of automorphisms, then by Theorem 3.1, $\beta_n(x) = nx$ is an automorphism of $\mathscr{X}$ for each n in I^+. However, in general

$$\beta_2(x \bigcirc y) \not\sim \beta_2(x) \bigcirc \beta_2(y),$$

as one can easily verify by taking $x = 1$ and $y = 2$. Thus $\mathscr{X}$ must have a discrete group of automorphisms.

LEMMA 3.6. *Suppose $\mathscr{X}$ is a fundamental unit structure, $a \in X$, and F is a function from A into X which is defined by: For each α in A,*

$$F(\alpha) = \alpha(a).$$

Then F is an isomorphism from $\langle A, \succsim', \bigcirc' \rangle$ onto $\langle X, \succsim, \bigcirc \rangle$.

Proof. It immediately follows that for each α, β in A,

$$\alpha \succsim \beta \quad \text{iff} \quad F(\alpha) \succsim F(\beta). \tag{3.5}$$

Let α, β, γ be arbitrary elements of A and suppose $\alpha \bigcirc' \beta = \gamma$. Then by Lemma 3.1,

$$\alpha(a) \bigcirc \beta(a) = \gamma(a).$$

In other words,

$$F(\gamma) = F(\alpha \bigcirc' \beta) = F(\alpha) \bigcirc F(\beta). \tag{3.6}$$

F is also onto X, since if γ is an arbitrary element of X, and η in A is such that $\eta(a) = \gamma$, then $F(\eta) = y$. Thus by Eqs. (3.5) and (3.6), F is an isomorphism. ∎

LEMMA 3.7. *Suppose $\mathscr{X}$ is a fundamental unit structure. Then for each α, β, γ in A,*

$$\alpha * (\beta \bigcirc' \gamma) = (\alpha * \beta) \bigcirc' (\alpha * \gamma).$$

Proof. Let α, β, γ be arbitrary elements of A. Let x in X be such that $(\beta \bigcirc' \gamma)(x) = \beta(x) \bigcirc \gamma(x)$. Then

$$\begin{aligned}[\alpha * (\beta \bigcirc' \gamma)](x) &= \alpha[(\beta \bigcirc' \gamma)(x)] \\ &= \alpha[\beta(x) \bigcirc \gamma(x)] \\ &= \alpha[\beta(x)] \bigcirc \alpha[\gamma(x)] \\ &= [\alpha * \beta(x)] \bigcirc [\alpha * \gamma(x)] \\ &= [(\alpha * \beta) \bigcirc' (\alpha * \gamma)](x).\end{aligned}$$

Thus

$$\alpha * (\beta \bigcirc' \gamma) = (\alpha * \beta) \bigcirc' (\alpha * \gamma). \quad ▮$$

DEFINITION 3.4. $\langle \varphi, f \rangle$ is said to be a *unit representation* for $\mathscr{X}$ if and only if φ is a function from X onto Re^+ and f is a function from Re^+ into Re^+ and the following two conditions hold for all x, y in X:

(i) $x \succcurlyeq y$ iff $\varphi(x) \geqslant \varphi(y)$;
(ii) $\varphi(x \bigcirc y) = f[\varphi(x)/\varphi(y)] \cdot \varphi(y)$.

THEOREM 3.2. *Suppose $\mathscr{X}$ is a fundamental unit structure. Then there exists a unit representation for $\mathscr{X}$.*

Proof. Since $\langle A, \succcurlyeq, * \rangle$ is a Dedekind complete, Archimedean ordered group, and since it is well known that all Dedekind complete, Archimedean ordered groups are isomorphic, let φ be an isomorphism of $\langle A, \succcurlyeq, * \rangle$ onto the multiplicative group of the positive reals, $\langle \mathrm{Re}^+, \geqslant, \cdot \rangle$. Then for each α, β in A,

$$\alpha \succcurlyeq \beta \quad \text{iff} \quad \varphi(\alpha) \geqslant \varphi(\beta). \tag{3.7}$$

Let H: $\mathrm{Re}^+ \times \mathrm{Re}^+ \to \mathrm{Re}^+$ be defined as follows: for each α, β in A,

$$H[\varphi(\alpha), \varphi(\beta)] = \varphi(\alpha \bigcirc' \beta). \tag{3.8}$$

Let r, s, t be arbitrary elements of Re^+. Since φ is onto Re^+, let α, β, γ in A be such that $\varphi(\alpha) = r$, $\varphi(\beta) = s$, and $\varphi(\gamma) = t$. Then

$$\begin{aligned}H(rs, rt) &= H[\varphi(\alpha) \cdot \varphi(\beta), \varphi(\alpha) \cdot \varphi(\gamma)] \\ &= H[\varphi(\alpha * \beta), \varphi(\alpha * \gamma)] \\ &= \varphi[(\alpha * \beta) \bigcirc' (\alpha * \gamma)] \\ &= \varphi[\alpha * (\beta \bigcirc' \gamma)] \\ &= \varphi(\alpha) \cdot \varphi(\beta \bigcirc' \gamma) \\ &= \varphi(\alpha) \cdot H[\varphi(\beta), \varphi(\gamma)] \\ &= r \cdot H(s, t).\end{aligned}$$

Thus for each r, s, t in Re$^+$,

$$H(rs, rt) = rH(s, t). \tag{3.9}$$

Let $f(s/t) = H(s/t, 1)$. Then by Eq. (3.9), for each s, t in Re$^+$,

$$H(s, t) = H\left(\frac{ts}{t}, t \cdot 1\right) = tH\left(\frac{s}{t}, 1\right) = f\left(\frac{s}{t}\right) \cdot t. \tag{3.10}$$

Thus by Eqs. (3.8) and (3.10) let f be such that for each α, β in A,

$$\varphi(\alpha \bigcirc' \beta) = f\left[\frac{\varphi(\alpha)}{\varphi(\beta)}\right] \cdot \varphi(\beta). \tag{3.11}$$

Since $\langle X, \succcurlyeq, \bigcirc \rangle$ and $\langle A, \succcurlyeq, \bigcirc' \rangle$ are isomorphic, it follows from Eqs. (3.7) and (3.11) that $\mathscr{X}$ has a unit representation. ▮

THEOREM 3.3. *Suppose $\mathscr{X}$ is a fundamental unit structure and $\langle \varphi, f \rangle$ is a unit representation for $\mathscr{X}$. Then the following six statements are true for each r, s in* Re$^+$:

(i) $f(r) > 1$;
(ii) $r \geqslant 1$ iff $f(rs) \leqslant rf(s)$;
(iii) $f(r) > r$;
(iv) $r \geqslant s$ iff $f(r) \geqslant f(s)$
(v) $\lim_{u \to \infty} f(us)/u = s$;
(vi) $\lim_{n \to \infty} f^{[n]}(1) = \infty$,

where $f^{[n]}(r)$ is defined for n in I^+ as follows:

$$f^{[1]}(r) = f(r) \qquad \text{and} \qquad f^{[n+1]}(r) = f[f^{[n]}(r)].$$

Proof. Let r, s be arbitrary elements of Re$^+$.

(i) Since φ is onto Re$^+$, let u, v in X be such that $r = \varphi(u)/\varphi(v)$. Then, since $u \bigcirc v \succ v$,

$$f\left[\frac{\varphi(u)}{\varphi(v)}\right] \cdot \varphi(v) > \varphi(v),$$

and thus $f(r) > 1$.

(ii) Let a, b, c in X and t in Re$^+$ be such that

$$\varphi(a) = rt, \qquad \varphi(b) = t, \qquad \varphi(c) = rst.$$

Then

$$
\begin{aligned}
r \geqslant 1 \quad &\text{iff} \quad rt \geqslant \\
&\text{iff} \quad \varphi(a) \geqslant \varphi(b) \\
&\text{iff} \quad a \succsim b \\
&\text{iff} \quad c \bigcirc a \succsim c \bigcirc b \\
&\text{iff} \quad \varphi(c \bigcirc a) \geqslant \varphi(c \bigcirc b) \\
&\text{iff} \quad f\left[\frac{\varphi(c)}{\varphi(a)}\right] \cdot \varphi(a) \geqslant f\left[\frac{\varphi(c)}{\varphi(b)}\right] \cdot \varphi(b) \\
&\text{iff} \quad f\left(\frac{rst}{rt}\right) rt \geqslant f\left(\frac{rst}{t}\right) t \\
&\text{iff} \quad f(s)r \geqslant f(rs).
\end{aligned}
$$

(iii) Let a, b in X be such that $\varphi(a) = rs$, $\varphi(b) = s$. Then since $a \bigcirc b \succ a$,

$$\varphi(a \bigcirc b) > \varphi(a),$$

and thus

$$f\left[\frac{\varphi(a)}{\varphi(b)}\right] \cdot \varphi(b) > \varphi(a),$$

and therefore

$$f\left(\frac{rs}{s}\right) \cdot s > rs,$$

i.e., $f(r) > r$.

(iv) Let u, v, t in Re^+ be such that $r = u/v$ and $s = t/v$. Let x, y, z in X be such that $\varphi(x) = u$, $\varphi(y) = t$, and $\varphi(z) = v$. Then

$$
\begin{aligned}
r \geqslant s \quad &\text{iff} \quad \frac{u}{v} \geqslant \frac{t}{v} \\
&\text{iff} \quad u \geqslant t \\
&\text{iff} \quad x \succsim y \\
&\text{iff} \quad x \bigcirc z \succsim y \bigcirc z \\
&\text{iff} \quad \varphi(x \bigcirc z) \geqslant \varphi(y \bigcirc z) \\
&\text{iff} \quad f\left[\frac{\varphi(x)}{\varphi(z)}\right] \varphi(z) \geqslant f\left[\frac{\varphi(y)}{\varphi(z)}\right] \varphi(z) \\
&\text{iff} \quad f\left(\frac{u}{v}\right) v \geqslant f\left(\frac{t}{v}\right) v \\
&\text{iff} \quad f(r) \geqslant f(s).
\end{aligned}
$$

(v) Let $\epsilon > 0$. Let y and z in X be such that $\varphi(y) = s$ and $\varphi(z) = s(1 + \epsilon)$. Then $z \succ y$. Let u be an arbitrarily large element of Re$^+$ such that (by restricted solvability)

$$z \succ y \bigcirc \varphi^{-1}\left(\frac{1}{u}\right) \succ y.$$

Then

$$\varphi(z) > \varphi\left[y \bigcirc \varphi^{-1}\left(\frac{1}{u}\right)\right] > \varphi(y),$$

i.e.,

$$s(1 + \epsilon) > f\left[\frac{\varphi(y)}{\varphi\varphi^{-1}(1/u)}\right] \cdot \varphi\varphi^{-1}\left(\frac{1}{u}\right) > s,$$

and thus

$$s(1 + \epsilon) > f(su) \cdot \frac{1}{u} > s. \tag{3.12}$$

Since Eq. (3.12) is true for arbitrarily large u for each $\epsilon > 0$, it follows that

$$\lim_{u \to \infty} \frac{f(us)}{u} = s.$$

(vi) Let x in X be such that $\varphi(x) = r$. Since $\mathscr{X}$ is Archimedean and $\bigcirc$ is a closed operation (Theorem 2.5), nx becomes arbitrarily large for arbitrarily large n in I^+, and thus, since φ is onto Re$^+$, $\varphi(nx)$ becomes arbitrarily large for arbitrarily large n in I^+. Now,

$$\varphi(2x) = \varphi(x \bigcirc x) = f(1) \cdot \varphi(x) = f^{[1]} \cdot r,$$

$$\varphi(3x) = \varphi(2x \bigcirc x) = f\left[\frac{\varphi(2x)}{\varphi(x)}\right] \cdot \varphi(x)$$

$$= f\left(\frac{f^{[1]} \cdot r}{r}\right) \cdot r = f^{[2]}(1) \cdot r,$$

and by induction,

$$\varphi(nx) = f^{[n-1]}(1) \cdot r.$$

Since $\varphi(nx) \to \infty$ as $n \to \infty$, $\lim_{n\to\infty} f^{[n]}(1) = \infty$. ∎

THEOREM 3.4. *Suppose* f: Re$^+ \to$ Re$^+$ *is such that statements* (i)–(vi) *of Theorem* 3.3 *hold for all* r, x *in* Re$^+$. *Let* $\odot$ *be the binary operation on* Re$^+$ *such that for all* r, s *in* Re$^+$,

$$r \odot s = f\left(\frac{r}{s}\right) \cdot s.$$

Then $\langle \text{Re}^+, \geqslant, \odot \rangle$ *is a fundamental unit structure.*

Proof. We will first show that $\langle \mathrm{Re}^+, \geqslant, \odot \rangle$ is a positive concatenation structure. Let r, s, t be arbitrary elements of Re^+.

Total ordering, nontriviality, and local definability immediately follow.

Monotonicity.

$$s \odot r \geqslant t \odot r \quad \text{iff} \quad f\left(\frac{s}{r}\right) \cdot r \geqslant f\left(\frac{t}{r}\right) \cdot r \qquad \text{(i)}$$

$$\text{iff} \quad f\left(\frac{s}{r}\right) \geqslant f\left(\frac{t}{r}\right)$$

$$\text{iff} \quad \frac{s}{r} \geqslant \frac{t}{r}$$

$$\text{iff} \quad s \geqslant t.$$

$$r \odot s \geqslant r \odot t \quad \text{iff} \quad f\left(\frac{r}{s}\right) \cdot s \geqslant f\left(\frac{r}{t}\right) t$$

$$\text{iff} \quad f\left(\frac{r}{s}\right) \cdot \frac{s}{t} \geqslant f\left(\frac{r}{t}\right), \qquad \text{(ii)}$$

but by statement (ii) of Theorem 3.3,

$$\text{iff} \quad \frac{s}{t} \geqslant 1$$

$$\text{iff} \quad s \geqslant t.$$

Restricted solvability. Suppose $r > s$. Since $f(u) > u$ for all u in Re^+ and $\lim_{u\to\infty} f(us)/u = s$, let v in Re^+ be such that $r > f(vs)/v > s$. Then since $s \odot 1/v = f(vs) \cdot 1/v$, it follows that $r > s \odot 1/v > s$.

Positivity. Since $f(r/s) > 1$,

$$r \odot s = f\left(\frac{r}{s}\right) \cdot s > s.$$

Since $f(r/s) > r/s$,

$$r \odot s = f\left(\frac{r}{s}\right) \cdot s > r.$$

Archimedean. Let $[1]r = r$, and for each n in I^+, $[n+1]r = ([n]r) \odot r$. Then $[2r] = r \odot r = f^{[1]}(1) \cdot r$. Suppose for $n \geqslant 2$, $[n]r = f^{[n-1]}(1) \cdot r$. Then

$$[n+1]r = [nr] \odot r = f\left[\frac{[nr]}{r}\right] \cdot r = f\left[\frac{f^{[n-1]}(1) \cdot r}{r}\right] \cdot r$$

$$= f^{[n]}(1) \cdot r.$$

Thus since $\lim_{n\to\infty} f^{[n]}(1) = \infty$, $[nr]$ becomes arbitrarily large for arbitrarily large n in I^+, and therefore Archimedean is satisfied.

We will now show that $\langle \text{Re}^+, \geqslant, \odot \rangle$ is homogeneous. Let p, r, s be arbitrary elements of Re^+. Define α_p: $\text{Re}^+ \to \text{Re}^+$ as follows: For each t in Re^+, $\alpha_p(t) = p \cdot t$. We will show that α_p is an automorphism of $\langle \text{Re}^+, \geqslant, \odot \rangle$. It is immediate that α_p is crder preserving. Furthermore,

$$\alpha_p(r \odot s) = p(r \odot s) = pf\left(\frac{r}{s}\right) s = f\left(\frac{pr}{ps}\right) \cdot ps$$
$$= (pr) \odot (ps) = \alpha_p(r) \odot \alpha_p(s).$$

Thus α_p is an automorphism. Since $\alpha_{s/r}(r) = s$, $\langle \text{Re}^+, \geqslant, \odot \rangle$ is homogeneous. ∎

THEOREM 3.5. *Suppose* $\langle \varphi, f \rangle$ *and* $\langle \psi, g \rangle$ *are unit representations for* $\mathscr{X}$. *Then there exist* s, t *in* Re^+ *such that for each* r *in* Re^+,

$$\varphi = s\psi^{1/t} \qquad \text{and} \qquad f(r) = g(r^t)^{1/t}.$$

Proof. We will first consider the case where z is an element of X such that $\varphi(z) = \psi(z) = 1$. Define $\odot$ and $\odot'$ on Re^+ by $u \odot v = f(u/v)v$ and $u \odot' v = g(u/v)v$. Let $L = \psi\varphi^{-1}$. Then it is easy to show that L is an isomorphism from $\mathscr{R}_1 = \langle \text{Re}^+, \geqslant, \odot \rangle$ onto $\mathscr{R}_2 = \langle \text{Re}^+, \geqslant, \odot' \rangle$. It is easy to show that multiplications by reals are automorphisms of $\mathscr{R}_1$ and $\mathscr{R}_2$ and that $\langle \text{Re}^+, \geqslant, \cdot \rangle$ is the group of automorphisms for both $\mathscr{R}_1$ and $\mathscr{R}_2$. For each r in Re^+, let $h(r) = L^{-1}rL$, where $h(r)$ is to be interpreted as the automorphism p of $\mathscr{R}_1$ (interpreted as multiplication by p) such that for each u in Re^+, $pu = L^{-1}[rL(u)]$. Then it is easy to verify that h is an automorphism of $\langle \text{Re}^+, \geqslant, \cdot \rangle$. Since it follows from Hölder's theorem that all automorphisms of $\langle \text{Re}^+, \cdot, \geqslant \rangle$ are positive real powers, let t' in Re^+ be such that for all r in Re^+, $h(r) = r^{t'}$. Then for each x in X,

$$L^{-1}rL\varphi(x) = r^{t'}\varphi(x),$$

i.e.,

$$rL\varphi(x) = L[r^{t'}\varphi(x)],$$

which by substituting $\psi\varphi^{-1}$ for L yields

$$r\psi\varphi^{-1}\varphi(x) = \psi\varphi^{-1}[r^{t'}\varphi(x)],$$
$$r\psi(x) = \psi\varphi^{-1}[r^{t'}\varphi(x)]. \tag{3.13}$$

Now taking $x = z$ in Eq. (3.13) and remembering that we assume $\varphi(z) = \psi(z) = 1$, it then follows that

$$r = \psi\varphi^{-1}(r^{t'}),$$

i.e., that

$$\psi^{-1}(r) = \varphi^{-1}(r^{t'}). \tag{3.14}$$

Since Eq. (3.14) is true for all r in Re^+, it follows that for each x in X, $\psi(x)^{t'} = \varphi(x)$. Letting $t = 1/t'$, we then get

$$\varphi = \psi^{1/t}. \tag{3.15}$$

Now since we assume $\varphi(z) = \psi(z) = 1$, it follows from Eq. (3.15) that for each v in X,

$$\begin{aligned} \psi(v \bigcirc z) &= \varphi(v \bigcirc z)^t \\ &= \left(f\left[\frac{\varphi(v)}{\varphi(z)}\right]\varphi(z)\right)^t \\ &= f[\varphi(v)]^t. \end{aligned} \tag{3.16}$$

However, it is also the case that for each v in X,

$$\begin{aligned} \psi(v \bigcirc z) &= g\left[\frac{\psi(v)}{\psi(z)}\right]\psi(z) \\ &= g[\psi(v)]. \end{aligned} \tag{3.17}$$

Thus by Eqs. (3.15)–(3.17), for each v in X,

$$f[\varphi(v)]^t = g[\psi(v)] = g[\varphi(v)^t]. \tag{3.18}$$

Since φ is onto Re^+, Eq. (3.18) yields for each r in Re^+,

$$f(r) = g(r^t)^{1/t}. \tag{3.19}$$

We will now consider the case where it is not necessarily true that $\varphi(z) = \psi(z) = 1$ for some z in X. Let z be an arbitrary element of X. Let $\varphi' = \varphi/\varphi(z)$ and $\psi' = \psi/\psi(z)$. Then it is easy to verify that $\langle\varphi', f\rangle$ and $\langle\psi', g\rangle$ are unit representation for $\mathscr{X}$ and $\varphi'(z) = \psi'(z) = 1$. Thus by Eq. (3.19),

$$f(r) = g(r^t)^{1/t}.$$

Furthermore, by Eq. (3.15), $\varphi' = \psi'^{1/t}$, i.e.,

$$\frac{\varphi}{\varphi(z)} = \left[\frac{\psi}{\varphi(z)}\right]^{1/t} = \left[\frac{1}{\psi(z)}\right]^{1/t}\psi^{1/t}.$$

Thus letting $s = \varphi(z)/\psi(z)^{1/t}$, we get

$$\varphi = s\psi^{1/t}. \quad \blacksquare$$

Unit representations for fundamental unit structures have multiplications by certain reals which are automorphisms of a positive concatenation structure, as Example 3.2 clearly demonstrates. However, having automorphisms as multiplications by reals severely restricts the forms of the positive concatenation structure, especially if smooth differentiability conditions also hold. This is clearly shown in the next example and is worked out in detail in the next section.

EXAMPLE 3.2. Suppose $\langle \varphi, \odot \rangle$ is a $\odot$-representation for $\mathscr{X}$, $\varphi(X) = \mathrm{Re}^+$, and $\odot$ has an extension to a function H that has a power series expansion around the origin for nonnegative reals, i.e., for each $x, y \geqslant 0$,

$$H(x, y) = \sum_{i,j} a_{ij} x^i y^j,$$

where i, j range over nonnegative integers and for each x, $y > 0$, $H(x, y) = x \odot y$. Also suppose $H(0, 0) = 0$ and $\langle \mathrm{Re}^+, \geqslant, \odot \rangle$ has a nontrivial automorphism that acts as multiplication, i.e., suppose r in Re^+ is such that $r \neq 1$ and for all x, y in Re^+,

$$r(x \odot y) = rx \odot ry.$$

Then from $H(0, 0) = 0$, it follows that $a_{00} = 0$, and from $r(x \odot y) = rx \odot ry$ that

$$rH(x, y) - H(rx, ry) = 0$$

for all $x, y > 0$. In terms of the power series representation, this yields

$$r \sum_{i,j} a_{ij} x^i x^j - \sum_{i,j} (rx)^i (ry)^j = \sum_{i,j} a_{ij} x^i y^j (r - r^i r^j) = 0,$$

which can only happen if for all i, j

$$a_{ij} x^i y^j (r - r^i r^j) = 0.$$

We have already shown $a_{00} = 0$. If either $i > 1$ or $j > 1$, then $r - r^i r^j \neq 0$, and thus $a_{ij} = 0$. Therefore the only possible nonzero terms are when $i = 0$, $j = 1$, and $i = 1$, $j = 0$. Therefore, for all $x, y > 0$,

$$H(x, y) = x \odot y = a_{01} x + a_{10} y. \tag{3.20}$$

Since, by calculation, $\odot$ satisfies the bisymmetric law, $(x \odot y) \odot (u \odot v) = (x \odot u) \odot (y \odot v)$, by isomorphism $\bigcirc$ must also satisfy this law, and thus $\mathscr{X}$ is a bisymmetric structure, i.e., for all p, q, w, z in X, $(p \bigcirc q) \bigcirc (w \bigcirc z) = (p \bigcirc w) \bigcirc (q \bigcirc z)$. However, there is an asymmetry in the axioms for a positive concatenation structure, namely, that restricted solvability and Archimedean (Definition 2.1) are defined using the "right side" of the operation. In terms of Eq. (3.20), a_{01} must be 1 since if $a_{01} < 1$, then for sufficiently large x and sufficiently small y,

$$x \odot y = a_{01}x + a_{10}y < x,$$

and this contradicts positivity, and if $a_{01} > 1$, then for sufficiently large x and y such that $x - y$ is a sufficiently small positive number, $x > y$, and for all z in Re^+, $y \bigcirc z > x$, and this contradicts restricted solvability. Now it follows from an argument similar to the one for the impossibility of $a_{01} < 1$ that it is impossible for $a_{10} < 1$. However, by the asymmetry in the definition of restricted solvability, it is possible for $a_{10} < 1$. In fact $\langle \mathrm{Re}^+, \geqslant, \oplus \rangle$, where $\oplus$ is defined by $x \oplus y = x + 2y$ is a fundamental unit structure. Now "left restricted solvability," i.e., for each x, y in X, if $x \succ y$, then for some z, $x \succ z \bigcirc y$, is a reasonable assumption for measurement theory, and if added as an assumption for this case yields the conclusion $a_{10} = 1$, i.e., $x \bigcirc y = x + y$, i.e., $\bigcirc$ is an associative operation. Thus to summarize, if $\mathscr{X}$ has a representation $\langle \varphi, \odot \rangle$ with a power series expansion for the operation $\odot$, and if $\langle \mathrm{Re}^+, \geqslant, \odot \rangle$ has a nontrivial automorphism that acts as multiplication, then $\bigcirc$ is a bisymmetric operation, and if in addition "left restricted solvability" holds for $\mathscr{X}$, then $\bigcirc$ is associative.

4. Characterizations of Automorphisms of $\mathscr{X}$

Throughout this section we will assume X is the set of positive real numbers, the relation $\succsim$ is $\geqslant$, and $\bigcirc$ is a closed operation.

The basic goal of this section is to find methods for transforming $\mathscr{X}$ into a representation whose automorphisms can be characterized as multiplications by positive reals. Unit representations have this property; however, some positive concatenation structures with a discrete set of automorphisms also have this property, e.g., see Example 3.1. To accomplish this goal, we assume that $\bigcirc$ has certain differentiability properties and then give the explicit form of the appropriate transformation as an integral equation involving the first partials of $\bigcirc$. This method can also be used to show that certain positive concatenation structures have only the trivial automorphisms.

Convention. Throughout this section we will often write $x \bigcirc y$ as $H(x, y)$. We will also often write partial derivatives as subscripts, e.g.,

$$H_{xy}(a, b) = \left. \frac{\partial^2 H}{\partial x \, \partial y} \right|_{a,b} .$$

Convention. Throughout this section, it is convenient to sometimes appropriately extend the automorphisms of $\mathscr{X}$ and the operation H so that they take values at 0. This is done as follows: For each α in A, let $\alpha(0) = 0$, and for each x in X, let $H(x, 0) = \lim_{y \to 0} H(x, y) = x$ and let $H(0, 0) = 0$. However, by the asymmetry between x and y in the restricted solvability condition, $\lim_{x \to 0} H(x, y)$ need not be y; thus we define

$H(0, y) = \lim_{x \to 0} H(x, y)$. Then for each α in A, α and H are monotone on $\{0\} \cup X$, α is onto $\{0\} \cup X$, $\alpha(0) = 0$, and for all x, y in $\{0\} \cup X$, $\alpha[H(x, y)] = H[\alpha(x), \alpha(y)]$.

Throughout this section, we use the following theorem of real analysis:

THEOREM 4.1 (The Inverse Function Theorem). *Let $f\colon \mathrm{Re}^+ \to \mathrm{Re}$ be continuously differentiable in an open interval containing a. Then there exists an open interval S containing $f(a)$ such that the inverse of f, f^{-1}, exists on S and is continuously differentiable there, and is such that for all y in S,*

$$(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}.$$

DEFINITION 4.1. H is said to be *smooth* if and only if H, H_x, H_y are continuously differentiable on X. H is said to be *smooth at* 0 if and only if for each $x, x' \in X$ $\lim_{t \to 0} H_y(x, t)/H_y(x', t)$ exists and is finite.

THEOREM 4.2. *Suppose H is smooth and α in A is continuously differentiable on X. Then the following three statements are true:*

(i)
$$\alpha'(x) = \lim_{t \to 0} \frac{\alpha'(t)\, H_y(\alpha(x), \alpha(t))}{H_y(x, t)} \tag{4.1}$$

(ii) *If in addition H is smooth at 0, then $\lim_{t \to 0} H_y(x, t)/H_y(x', t)$ may be written as $g(x)/g(x')$ for some function g. Furthermore, in this case for each α in A and x in X,*

$$\alpha'(x)\, g(x)/g(\alpha(x)) = K_\alpha \tag{4.2}$$

for some constant K_α dependent only on α.

(iii) *Let H be smooth on X and at* 0, *let g be as in part* (ii), *and*

$$J(z) = \int_0^z dt/g(t). \tag{4.3}$$

Then for each continuously differentiable α in A, $\alpha(z) = J^{-1}(K_\alpha J(z))$. The map $\omega\colon A \to \mathrm{Re}^+$, defined by $\omega(\alpha) = K_\alpha$, is an isomorphism from the continuously differentiable automorphisms on A onto a multiplicative subgroup of the positive reals. Furthermore, if we define $\odot$ on $J(X)$ by

$$x \odot y = J(J^{-1}(x) \bigcirc J^{-1}(y))$$

and for each α continuously differentiable in A,

$$\bar{\alpha}(z) = K_\alpha z,$$

then J is an isomorphism between x and $\bar{\mathscr{X}} = \langle J(X), \geqslant, \odot \rangle$ and $\bar{\alpha}$ is an automorphism of $J(X)$.

Proof.[2] (i) Since α is an automorphism of X,

$$\lim_{u\to t}\frac{H(\alpha(x),\alpha(t))-H(\alpha(x),\alpha(u))}{t-u}=\lim_{u\to t}\frac{\alpha(H(x,t))-\alpha(H(x,u))}{t-u}. \tag{4.4}$$

However, by the chain rule we obtain

$$\lim_{u\to t}\frac{H(\alpha(x),\alpha(t))-H(\alpha(x),\alpha(u))}{t-u}=\alpha'(t)\,H_y(\alpha(x),\alpha(t)). \tag{4.5}$$

Similarly by the chain rule we get

$$\lim_{u\to t}\frac{\alpha(H(x,t))-\alpha(H(x,u))}{t-u}=\alpha'(H(x,t))\,H_y(x,t). \tag{4.6}$$

Combining (4.4), (4.5), and (4.6) we get

$$\alpha'(H(x,t))=\frac{\alpha'(t)\,H_y(\alpha(x),\alpha(t))}{H_y(x,t)}. \tag{4.7}$$

Letting $t\to 0$ and noting that α' is continuously differentiable, we obtain

$$\alpha'(x)=\lim_{t\to u}\frac{\alpha'(t)\,H_y(\alpha(x),\alpha(t))}{H_y(x,t)},$$

which is (4.1).

(ii) By hypothesis for each x, x' in X, $\lim_{t\to 0} H_y(x,t)/H_y(x',t)$ exists and is finite. It then follows by elementary properties of limits that

$$\lim_{t\to 0} H_y(x,t)/H_y(x',t)=g(x)/g(x'),$$

where

$$g(x)=\lim_{t\to 0} H_y(x,t)/H_y(x_0,t)$$

for some fixed x_0. Since $H_y(x,t)$ is strictly positive for each x, t (i.e., since $\bigcirc$ is strictly monotonic in both coordinates), $g(x)\geqslant 0$. From the assumption that H is smooth at zero, it immediately follows that $g(x)>0$.

Choose x, x' arbitrarily. Then by elementary properties of limits and the fact that $\alpha'(x)$ and $\alpha'(x')>0$,

[2] Ideas for part of the proof of this theorem were suggested by Geoffrey Iverson.

$$\frac{\alpha'(x)}{\alpha'(x')} = \frac{\lim\limits_{t\to 0} \dfrac{\alpha'(t)\, H_y(\alpha(x), \alpha(t))}{H_y(x, t)}}{\lim\limits_{t\to 0} \dfrac{\alpha'(t)\, H_y(\alpha'(x'), \alpha(t))}{H_y(x', t)}}$$

$$= \lim_{t\to 0} \frac{H_y(x', t)}{H_y(x, t)} \frac{H_y(\alpha(x), \alpha(t))}{H_y(\alpha(x'), \alpha(t))}$$

$$= \lim_{t\to 0} \frac{H_y(\alpha(x), \alpha(t))}{H_y(\alpha(x'), \alpha(t))} \cdot \lim_{t\to 0} \frac{H_y(x', t)}{H_y(x, t)}$$

$$= \frac{g(\alpha(x))\, g(x')}{g(\alpha(x'))\, g(x)}. \tag{4.8}$$

Regrouping in (4.8) yields

$$\frac{\alpha'(x)\, g(x)}{g(\alpha(x))} = \frac{\alpha'(x')\, g(x')}{g(\alpha(x'))}. \tag{4.9}$$

However, since x and x' are arbitrary, it follows that

$$\frac{\alpha'(x)\, g(x)}{g(\alpha(x))} = K_\alpha,$$

where K_α depends only on α. Furthermore, $K_\alpha > 0$ since $g(x)$, $g(\alpha(x))$, and $\alpha'(x)$ are positive.

(iii) Regrouping in (4.2), integrating, and utilizing the fact that $\alpha(0) = 0$, we obtain after a change of variables,

$$\int_0^t \frac{\alpha'(x)\, dx}{g(\alpha(x))} = K_\alpha \int_0^t \frac{dx}{g(x)},$$

$$\int_0^{\alpha(t)} \frac{du}{g(u)} = K_\alpha \int_0^t \frac{dx}{g(x)},$$

$$J(\alpha(t)) = K_\alpha J(t).$$

Since g is strictly positive it follows that J is differentiable and strictly monotonic and therefore one to one. Therefore,

$$\alpha(t) = J^{-1}(K_\alpha J(t)).$$

To show that ω is an isomorphism from the set of continuously differentiable automorphisms of A onto a multiplicative subgroup of positive reals, first note that

$$\begin{aligned} \alpha * \beta(t) &= \alpha(\beta(t)) \\ &= \alpha J^{-1}(K_\beta J(t)) \\ &= J^{-1} K_\alpha J(J^{-1} K_\beta J(t)) \\ &= J^{-1}(K_\alpha \cdot K_\beta J(t)). \end{aligned}$$

Therefore $\omega[\alpha * \beta] = \omega(\alpha) \cdot \omega(\beta)$. Furthermore since α is strictly monotonic on Re^+, by the inverse function theorem, α^{-1} is continuously differentiable if α is, and hence

$$\omega[\alpha^{-1}(\alpha)] = \omega(\alpha^{-1}) \cdot \omega(\alpha),$$
$$\omega(\iota) = 1,$$

where ι is the identity. Thus

$$(\omega(\alpha))^{-1} = \omega(\alpha^{-1}).$$

Furthermore ω is order preserving since

$$\begin{aligned} \alpha \geqslant \beta &\text{ iff for some } x,\ \alpha(x) \geqslant \beta(x) \\ &\text{ iff for some } x\ J^{-1}(\omega(\alpha)\ J(x)) \geqslant J^{-1}(\omega(\beta)\ J(x)) \\ &\text{ iff } \omega(\alpha) \geqslant \omega(\beta). \end{aligned}$$

The above observations show that ω is an order-preserving isomorphism from the continuously differentiable automorphisms of A onto a multiplicative subgroup of positive reals.

That J is an isomorphism from $\langle X, \geqslant, \bigcirc\rangle$ onto $\langle J(X), \geqslant, \odot\rangle$ follows immediately from the definition of $\odot$ and the monotone continuity of J. Furthermore,

$$\begin{aligned} \bar{\alpha} \cdot J(x) \odot J(y) &= K_\alpha J(x \bigcirc y) \\ &= J(\alpha(x \bigcirc y)) \\ &= J(\alpha(x) \bigcirc \alpha(y)) \\ &= J(\alpha(x)) \odot J(\alpha(y)) \\ &= K_\alpha J(x) \odot K_\alpha J(y). \end{aligned}$$

Thus

$$K_\alpha J(x) \odot J(y) = K_\alpha J(x) \odot K_\alpha J(y).$$

Since J is monotone continuous the result has been proved.

Theorem 4.2, especially part (ii), gives powerful methods for determining the form and structure of differentiable automorphisms. However, to fully utilize this theorem, we need to know general sets of conditions for which all automorphisms are continuously differentiable on X. This is done in Theorem 4.5, where it is shown that if H is smooth, then all automorphisms of $\mathscr{X}$ are continuously differentiable. The method of proof of Theorem 4.5 uses ideas developed by Aczél (1966) for the solutions of functional equations.

THEOREM 4.3 (Differentiation of Integrals Theorem).

Suppose $f(x, y)$ and $\partial f/\partial x$ are continuous in the rectangle $(x - \epsilon, x + \epsilon) \times [a, b]$. Then

$$\frac{d}{dx} \int_a^b f(x, y)\, dy \bigg|_z = \int_a^b \frac{\partial}{\partial x} f(x, y) \bigg|_{z,y} dy,$$

and the derivative is continuous.

LEMMA 4.1 *(Implicit Function Theorem). Define the function G as follows: For each x, y, v in X, $G(x, y) = v$ iff $y = x \bigcirc v$. Suppose $\mathscr{X}$ is smooth and $G(a, b) = c$. Then G_x and G_y exist and are continuous at (a, b).*

Proof. Left to reader. ∎

LEMMA 4.2. *Suppose $\mathscr{X}$ is smooth, α is an automorphism of $\mathscr{X}$, $u, v \in X$, and $u < v$. Then*

$$W(x) = \int_u^v \alpha[H(x, y)]\, dy \tag{4.10}$$

is a continuously differentiable function of x.

Proof. Since α is monotonic and onto Re^+, α is continuous. First we will make a change of variables in Eq. (4.10). Let $Q = H(x, y)$. Let $G(x, y)$ be defined as in Lemma 4.1. Then $G(x, Q) = y$. By Lemma 4.1, G is continuously differentiable at points for which it is defined. Since $dQ = H_y(x, y)\, dy$ and H is monotonic in both variables, and thus $H_y(x, y) > 0$, it follows that

$$dy = \frac{dQ}{H_y[x, G(x, Q)]}.$$

Thus by changing variables we get

$$W(x) = \int_{H(x,u)}^{H(x,v)} \frac{\alpha(Q)\, dQ}{H_y[x, G(x, Q)]}.$$

We will now show that W is differentiable in x. We first write $W(x)$ in the form

$$W(x) = L[V(x), B(x), C(x)],$$

where

$$L(a, b, c) = \int_a^b \frac{\alpha(Q)\, dQ}{H_y[c, G(c, Q)]}, \qquad a < b,\ c < Q,$$

$$U(x) = H(x, v),$$

$$B(x) = H(x, u),$$

$$C(x) = x.$$

In order to apply the chain rule to L to show that $W'(x)$ exists, we first must show that

U, B, C are differentiable in x and L is continuously differentiable in each partial. U and B are differentiable in x by assumption, and it is immediate that C is differentiable in x. Let us consider $L(a, b, c)$. $\partial L/\partial b$ exists and is continuous since

$$E(Q, c) = \frac{\alpha(Q)}{H_y[c, G(c, Q)]} \tag{4.11}$$

is continuous on $(-\epsilon + H(x, u), \epsilon + H(x, v)) \times (x - \epsilon, x + \epsilon)$ for some $\epsilon > 0$ and since, by the fundamental theorem of calculus,

$$\frac{d}{dz} \int_a^z E(Q, c)\, dQ = E(x, c).$$

Similarly, $\partial L/\partial a$ exists and is continuous. To show $\partial L/\partial c$ exists and is continuous, the differentiation of integrals theorem is used. Letting $E(Q, c)$ be as in Eq. (4.11), we note that $E(Q, c)$ is differentiable in c since by assumption $(\partial/\partial z)\, H_y(z, p)$ exists and is continuous and since by Lemma 4.1, $(\partial/\partial z)\, G(z, p)$ exists and is continuous. Thus by holding Q constant and applying the chain rule,

$$\frac{\partial E}{\partial c} = \frac{-\alpha(Q)}{[H_y(c, G(c, Q))]^2} (H_{xy}[c, G(c, Q)] + H_{yy}[c, G(c, Q)]\, G_x(c, Q)). \tag{4.12}$$

Thus, since E_c is formed by taking compositions, sums, and products of continuous functions, E_c itself is continuous. Combining the above results we get

$$\begin{aligned} \frac{dW(x)}{dx} &= \frac{\partial L}{\partial a}\frac{\partial U}{\partial x} + \frac{\partial L}{\partial b}\frac{\partial B}{\partial x} + \frac{\partial L}{\partial c}\frac{\partial C}{\partial x} \\ &= \frac{\alpha[H(x, v)]}{H_y(x, v)} H_x(x, v) - \frac{\alpha[H(x, u)]}{H_y(x, u)} H_x(x, u) \\ &\quad + \int_{H(x,u)}^{H(x,v)} \frac{\partial E}{\partial c}\, dQ, \end{aligned} \tag{4.13}$$

from which it immediately follows that $W'(x)$ is continuous. ∎

THEOREM 4.4. *Suppose $\mathscr{X}$ is smooth and α is an automorphism of $\mathscr{X}$. Then α is continuously differentiable on* Re$^+$.

Proof. Since α is an automorphism of $\mathscr{X}$, the functional equation

$$\alpha[H(x, y)] = H[\alpha(x), \alpha(y)] \tag{4.14}$$

holds. Letting $u < v$ and integrating both sides of Eq. (4.14) with respect to y, we get

$$\int_u^v \alpha H(x, y)\, dy = \int_u^v H[\alpha(x), \alpha(y)]\, dy. \tag{4.15}$$

However, by Lemma 4.2 we know that the left side of Eq. (4.15) is a continuously differentiable function, $W(x)$, of x. Thus

$$W(x) = \int_u^v H(\alpha(x), \alpha(y)] \, dy \tag{4.16}$$

and $W'(x)$ exists and is continuous for each x in X. Let P be the function from Re^+ into Re^+ defined by

$$P(z) = \int_u^v H[z, \alpha(y)] \, dy. \tag{4.17}$$

Since α is monotonic and onto X, α is continuous. Also, H is continuous and monotonic increasing. Therefore P is continuous and monotonic increasing. Since H is continuously differentiable, it follows that both $H[z, \alpha(y)]$ and $H_x[z, \alpha(y)]$ are continuous on $(z - \epsilon, z + \epsilon) \times [u, v]$ for some $\epsilon > 0$. Thus by the differentiation of integrals theorem, P is continuously differentiable, and since P is monotonic increasing, $P'(z) > 0$ for all z in X. Thus by the inverse function theorem, P has a differentiable inverse function, P^{-1}. Applying P^{-1} to both sides of Eq. (4.16), we get

$$P^{-1}[W(x)] = P^{-1} \int_u^v H[\alpha(x), \alpha(y)] \, dy = \alpha(x).$$

Thus by the chain rule,

$$\alpha'(x) = (P^{-1}[W(x)])' \cdot W'(x). \quad \blacksquare$$

EXAMPLE 4.1. Let $\mathscr{X} = \langle \mathrm{Re}^+, \geqslant, \bigcirc \rangle$, where

$$x \bigcirc y = x + y + xy.$$

Then $\mathscr{X}$ is a positive concatenation structure and $\bigcirc$ is smooth and smooth at zero. Since $H_y(x, 0) = 1 + x$ and H is smooth, $g(x) = 1 + x$ and therefore,

$$J(z) = \int_0^z \frac{dt}{1+t} = \log(1 + z).$$

Thus

$$\begin{aligned} x \odot y &= J[J^{-1}(x) \bigcirc J^{-1}(y)] \\ &= \log[1 + (J^{-1}(x) \bigcirc J^{-1}(y))] \\ &= \log[1 + (e^x - 1 \bigcirc e^y - 1)] \\ &= \log[1 + e^x - 1 + e^y - 1 + (e^x - 1)(e^y - 1)] \\ &= \log[e^x e^y] \\ &= x + y. \end{aligned}$$

Thus $\langle \mathrm{Re}^+, \geqslant, \bigcirc \rangle$ is isomorphic to $\langle \mathrm{Re}^+, \geqslant, + \rangle$ and $J(z) = \log(1 + z)$ is the isomorphism.

EXAMPLE 4.2. (An example of a Dedekind-complete positive concatenation structure with a closed commutative operation that has only the trivial automorphism.) Let $\mathscr{X} = \langle \mathrm{Re}^+, \geqslant, \bigcirc \rangle$, where

$$x \bigcirc y = x + y + x^2 y^2.$$

Since $H_y(x, 0) = 1$, $g(x) = 1$ and therefore

$$J(z) = \int_0^z dt = z,$$

and thus it follows that $\odot$ is $\bigcirc$, where $\odot$ is defined by

$$x \odot y = J[J^{-1}(x) \bigcirc J^{-1}(y)].$$

Since by Theorem 4.5 all automorphisms of $\mathscr{X}$ are continuously differentiable and by Theorem 4.3 all continuously differentiable automorphisms of $\langle \mathrm{Re}^+, \geqslant, \odot \rangle$ are multiplications of reals, it follows from $\odot = \bigcirc$ that all automorphisms of $\mathscr{X}$ are multiplications of reals. However,

$$\begin{aligned} r(1 \bigcirc 2) = (r \cdot 1) \bigcirc (r \cdot 2) &\text{ iff } r \cdot 7 = 3r + 4r^4 \\ &\text{ iff } 7 = 3 + 4r^3 \\ &\text{ iff } r = 1. \end{aligned}$$

Thus $\mathscr{X}$ has only the trivial automorphism.

5. PRE-UNIT STRUCTURES

Although the axioms for fundamental unit structures are straightforward and simple, they are stronger than what one would like for measurement theory. In its present state, measurement theory is concerned with idealizations of empirical processes. By their nature, empirical processes are finite, and therefore their idealizations should be potentially infinite, or at most denumberably infinite. Dedekind completeness forces fundamental unit structures to have much higher cardinality. The homogeneity condition of fundamental unit structures (like Dedekind completeness) is not formulatable in first-order languages; and the assumption of closed operations of fundamental unit structures excludes many natural measurement applications. In this section we will avoid these difficulties by axiomatizing a weaker structure that is imbeddable in a fundamental unit structure.

CONVENTION. In this section we will often write $x = (1/m)y$ or $y/m = x$ for $y = mx$ when $x, y \in X$ and $m \in I^+$.

DEFINITION 5.1. $\mathscr{X}$ is said to be a *pre-unit structure* if and only if the following two conditions hold for all x, y in X and all m in I^+.

(i) *Partial homogeneity*: If either $m(x \bigcirc y)$ or $mx \bigcirc my$ are defined, then

$$m(x \bigcirc y) = (mx) \bigcirc (my);$$

(ii) *Partial divisibility*: There exists z in X such that $mz = x$.

Let $\mathscr{X}$ be a pre-unit structure. Then the partial operation $\bigcirc$ need be not closed. However, as the following theorem shows, $\mathscr{X}$ can be imbedded in a pre-unit structure with a closed operation. The proof is straightforward but long, and we shall omit it.

THEOREM 5.1. *Suppose $\mathscr{X}$ is a pre-unit structure. Then $\mathscr{X}$ is isomorphically imbeddable in a pre-unit structure with a closed operation.*

Basically, partial homogeneity and partial definability is one way of saying that $\mathscr{X}$ has a dense set of "local" automorphisms. More precisely, a *local automorphism* of $\mathscr{X}$ is a function α defined on some nonempty subset Y of X such that for all x, y, $x \bigcirc y$ in Y and z in X:

(i) If $x \succ z$, then $z \in Y$;

(ii) if $\alpha(x) \succ z$, then $z \in \alpha(Y)$;

(iii) $\alpha(x \bigcirc y) = \alpha(x) \bigcirc \alpha(y)$;

(iv) $x \succsim y$ iff $\alpha(x) \succsim \alpha(y)$.

A development similar to the one in Section 2 for automorphisms can be given for partial automorphisms, and a result similar to Theorem 5.1 can be shown: namely, if $\mathscr{X}$ has a dense set of local automorphisms, then $\mathscr{X}$ can be imbedded in a positive concatenation structure with a dense automorphism group. However, we will not proceed further with this topic in this paper.

Suppose $\mathscr{X} = \langle X, \succsim, \bigcirc \rangle$ is a pre-unit structure and $\bigcirc$ is a closed operation. Then nx is defined for all n in I^+ and all x in X. Thus by partial homogeneity and monotonicity, for each x, y in X and each n in I^+, $n(x \bigcirc y) = nx \bigcirc ny$ and $x \succsim y$ iff $nx \succsim ny$, i.e., nx is an automorphism of $\mathscr{X}$. By Lemma 3.3, $\mathscr{X}$ has a dense group of automorphisms. Observe that x/n is the inverse of the automorphism nx and is itself an automorphism of X. We will often write

$$m\frac{z}{n} \quad \text{for} \quad m\left(\frac{z}{n}\right)$$

and note that since the automorphisms of $\mathscr{X}$ are commutative,

$$m\frac{z}{n} = \frac{mz}{n}.$$

We state two definitions and prove a lemma to be used in subsequent development.

DEFINITION 5.2. x_i is said to be a *standard sequence* in $\mathscr{X}$ iff there are u, v in X, $u \succ v$ and either

(i) $x_i \bigcirc v \succsim x_{i-1} \bigcirc u$,

(ii) $v \bigcirc x_i \succsim u \bigcirc x_{i-1}$.

CONVENTION. $\bar{U}$ denotes the complement of U.

DEFINITION 5.3. $\mathscr{X}$ is said to be *strongly Archimedean* if and only if every strictly bounded standard sequence is finite.

LEMMA 5.1. *Suppose $\mathscr{X}$ is a pre-unit structure with a closed operation. Then $\mathscr{X}$ is strongly Archimedean.*

Proof. Suppose $\mathscr{X}$ is not strongly Archimedean. A contradiction will be shown. Let x_i be an infinite standard sequence in $\mathscr{X}$ that is bounded by a in X. Let

$$\mathscr{C} = \{z \mid z \text{ is in } X \text{ and for some } i \text{ in } I^+,\ x_i \succ z\}.$$

Then $\mathscr{C}$ is clearly a bounded set. Let w be an element of $\mathscr{C}$, and let

$$\mathscr{H} = \{\sigma \mid \sigma \text{ is in } \mathscr{G} \text{ and } \sigma(w) \in \mathscr{C}\}.$$

We will show the following three propositions:

(i) $\mathscr{H}$ and $\bar{\mathscr{H}}$ are nonempty;

(ii) $\mathscr{H}$ contains positive elements;

(iii) for each σ, η in $\mathscr{G}$, if $\sigma \in \mathscr{H}$ and $\sigma \succ \eta$, then η is in $\mathscr{H}$.

Proposition (iii) immediately follows from the definitions of $\mathscr{C}$ and $\mathscr{H}$ and the definition of $\succ$ on $\mathscr{G}$. To show $\mathscr{H}$ is nonempty, observe that $x_i \succ w$ for some i in I^+. Therefore since $\mathscr{G}$ is dense, by Lemma 2.4 there exists α in $\mathscr{G}$ such that $x_i \succ \alpha(w) \succ w$ for all i in I^+, which also shows that $\mathscr{H}$ contains positive elements. By the proof that $\mathscr{G}$ is Archimedean (Theorem 2.1), there exists t in X such that $\alpha^n(w) \succ nt$ for all n in I^+. By choosing n such that $nt \succ a$, it follows that α^n is in $\bar{\mathscr{H}}$. Thus we have shown propositions (i), (ii), and (iii).

Now since x_i is a standard sequence, without loss of generality, let u, v be elements of X such that $u \succ v$ and $x_i \bigcirc v \succ x_{i-1} \bigcirc u$. We will show that there exists σ in $\mathscr{H}$ and $\bar{\sigma}$ in $\bar{\mathscr{H}}$ such that $\sigma(u) \succ \bar{\sigma}(v)$. Since $\mathscr{G}$ is dense, by Lemma 2.4 there exists ξ in $\mathscr{G}$ such that $u \succ \xi(v) \succ v$. Let $k = \max\{m \mid \xi^m \in \mathscr{H}\}$. By the above argument, k exists. Then $\xi^k(u) \succ \xi^{k+1}(v)$, ξ^k is in $\mathscr{H}$, ξ^{k+1} is in $\bar{H}$. Thus letting $\sigma = \xi^k$ and $\bar{\sigma} = \xi^{k+1}$, we get the desired result.

Now to complete the proof of this lemma, let $\bar{\sigma}$ in $\bar{\mathscr{H}}$ σ in $\mathscr{H}$ be such that

$$\bar{\sigma}^{-1}(u) \succ \sigma^{-1}(v).$$

By monotonicity in $\mathscr{X}$,

$$w \bigcirc \bar{\sigma}^{-1}(u) \succ w \bigcirc \sigma^{-1}(v).$$

Again by Lemma 2.4, let η in $\mathscr{G}$ be such that

$$w \bigcirc \bar{\sigma}^{-1}(u) \succ \eta(w) \bigcirc \eta\sigma^{-1}(v) \succ w \bigcirc \sigma^{-1}(v).$$

Let $p = \max\{m \mid \eta^m \in \mathscr{H}\}$. Then

$$\eta^p(w) \bigcirc \eta^p\bar{\sigma}^{-1}(u) \succ \eta^{p+1}(w) \bigcirc \eta^{p+1}\sigma^{-1}(v).$$

But by the definition of p, σ, $\bar{\sigma}$,

$$\eta^p\bar{\sigma}^{-1} \prec \iota,$$
$$\eta^{p+1}\sigma^{-1} \succ \iota,$$
$$\eta^p \in \mathscr{C},$$

and

$$\eta^{p+1}(w) \in \bar{\mathscr{C}}.$$

But then

$$\eta^p(w) \bigcirc u \succ \eta^{p+1}(w) \bigcirc v.$$

Since $\eta^{p+1}(w) \in \bar{\mathscr{C}}$, $\eta^{p+1}(w) \succ x_i$ for $i \in I^+$. However, since $\eta^p(w) \in \mathscr{C}$, there exists j such that $x_j \succ \eta^p(w)$. Therefore by monotonicity in $\mathscr{X}$,

$$x_j \bigcirc u \succ \eta^{p+1}(w) \bigcirc v \succ x_{j+1} \bigcirc v \succsim x_j \bigcirc u,$$

and this is a contradiction. ∎

Suppose $\mathscr{X} = \langle X, \succcurlyeq, \bigcirc \rangle$ is a pre-unit structure and $\bigcirc$ is a closed operation. Let $\mathbf{X}$ consist of all subsets Y of X for which the following three conditions hold:

(i) Y and $X - Y$ are nonempty.

(ii) For each x, y in Y, if $x \succcurlyeq y$ and $x \in Y$, then $y \in Y$.

(iii) Y does not have a maximal element. For each x in X, let

$$\mathbf{x} = \{y \mid y \in X \text{ and } x \succ y\}.$$

Let $X^* = \{\mathbf{x} \mid x \in X\}$. Then it immediately follows that $X^* \subseteq \mathbf{X}$. Define $\boldsymbol{\succcurlyeq}$ on X as follows: For each Y, Z in $\mathbf{X}$,

$$Y \boldsymbol{\succcurlyeq} Z \text{ iff } Y \supseteq Z.$$

Define $\mathbf{O}$ on $\mathbf{X}$ as follows: For each Y and Z in $\mathbf{X}$,

$$Y \mathbf{O} Z = \{x \in X \mid \text{there exist } y \in Y, z \in Z \text{ such that } y \bigcirc z \succ x\}.$$

Then the following lemma follows from the proof of Theorem 7.4, especially Lemmas 7.3, 7.4, 7.6, and 7.8 of Narens and Luce (1976). (Although the proof of Lemma 7.8 of Narens and Luce (1976) uses an additional assumption called "interval solvability," another proof can easily be given that does not use this assumption.)

LEMMA 5.2. *$\mathscr{X} = \langle \mathbf{X}, \succcurlyeq, \mathbf{O} \rangle$ satisfies all the axioms for a totally ordered, positive concatenation structure (Definition 2.1) except possibly for restricted solvability, monotonicity, and positivity. Furthermore, $\mathbf{O}$ is a closed operation, $\langle \mathbf{X}, \succcurlyeq \rangle$ is Dedekind complete, X^* is an order dense subset of $\langle \mathbf{X}, \succcurlyeq \rangle$, and $\mathscr{X}^* = \langle X^*, \succcurlyeq', \mathbf{O}' \rangle$ and $\langle X, \succcurlyeq, \bigcirc \rangle$ are isomorphic, where $\succcurlyeq'$ and $\mathbf{O}'$ are the restrictions of $\succcurlyeq$* and $\mathbf{O}$ to X^*.

LEMMA 5.3. *Let Y, Z be arbitrary elements of $\mathbf{X}$. Then the following two statements* are true:

(i) $Y \mathbf{O} Z \succ Y$;
(ii) $Y \mathbf{O} Z \succ Z$.

Proof. (i) Let Y, Z be in $\mathbf{X}$ and let $Y \mathbf{O} Z$ be given. Then by definition and by positivity in $\mathscr{X}$, $Y \bigcirc Z \succcurlyeq Y$. Suppose that $Y \mathbf{O} Z = Y$. Then there is a sequence y_i such that y_i is in Y and

$$y_{i+1} \succ y_i \bigcirc z$$

for a fixed z in Z. Since for arbitrary $u \prec z$, $y_{i+1} \bigcirc u \succ y_i \bigcirc z$, the sequence y_i is a standard sequence. Since by Lemma 5.1 $\mathscr{X}$ is strongly Archimedean, the sequence y_i is unbounded. However by definition of Y, there is a z in X such that $z \succ y$ for each y in Y. In particular, $z \succ y_i$ for each i, and this contradicts unboundedness of the sequence. The proof of (ii) is entirely similar. ▮

LEMMA 5.4. *Let W, Y, Z be arbitrary elements of $\mathbf{X}$. Then the following two statements are true:*

(i) $W \mathbf{O} Z \succcurlyeq W \mathbf{O} Y$ iff $Z \succcurlyeq Y$;
(ii) $Z \mathbf{O} W \succcurlyeq Z \mathbf{O} Y$ iff $Z \succcurlyeq Y$.

Proof. (i) Let $Z \succ Y$. It suffices to show that $W \mathbf{O} Z \succ W \mathbf{O} Y$. By definition of $\mathbf{O}$ and monotonicity in $\mathscr{X}$, $W \mathbf{O} Z \succcurlyeq W \mathbf{O} Y$. Suppose $W \mathbf{O} Z = W \mathbf{O} Y$, a contradiction will be shown. Since $Z \succcurlyeq Y$ and Z has no maxima, there are u, v such that $u \succ v$ and u, v are in $Z - Y$. We will now show that there is a sequence of w_i in W such that

$$W_{i+1} \bigcirc v \succcurlyeq w_i \bigcirc u.$$

We define the sequence by induction. Let w_0 be an arbitrary element of W and suppose w_i is defined. Since $w_i \bigcirc u$ is in $W \bigcirc Z$, it is in $W \bigcirc Y$, and so there exists y in Y such that $w_{i+1} \bigcirc y \succ w_i \bigcirc u$. But by definition of v, $v \succ y$ for each y in Y, and from monotonicity in $\mathscr{X}$ it follows that

$$w_{i+1} \bigcirc v \succ w_{i+1} \bigcirc y \succcurlyeq w_i \bigcirc u.$$

The sequence w_i is a standard sequence, and since $\mathscr{X}$ is strongly Archimedean, it is unbounded. However, by the definition of W there is a z in X such that $z \succ w$ for each in W. In particular $z \succ w_i$ for each i, contradicting the unboundedness of W. ▮

LEMMA 5.5. *$\mathscr{X}$ satisfies restricted solvability.*

Proof. Let Y, W be arbitrary elements of $\mathscr{X}$ and suppose $Y \succ W$. Since X^* is order dense in $\langle \mathbf{X}, \succcurlyeq \rangle$, let u, v in X be such that

$$Y \succ \mathrm{u} \succ \mathrm{v} \succ W.$$

Then $u \succ v$ by the isomorphic imbedding of $\mathscr{X}$ into $\mathscr{X}$. By restricted solvability in $\mathscr{X}$, let z in X be such that $u \succ v \succ \bigcirc z \succ v$. Then $\mathbf{u} \succ \mathbf{v} \bigcirc \mathbf{z} \succ \mathbf{v}$. Thus by positivity and monotonicity in $\mathscr{X}$,

$$Y \succ \mathrm{u} \succ \mathrm{v} \bigcirc \mathrm{z} \succ W \bigcirc \mathrm{z} \succ W,$$

i.e., $Y \succ W \bigcirc \mathrm{z} \succ W$. ▮

LEMMA 5.6. *For each Y, Z in* $\mathbf{X}$,

$$Y \bigcirc Z = \text{l.u.b.}\{\mathbf{x} \bigcirc \mathbf{w} \mid Y \succcurlyeq \mathbf{x} \text{ and } Z \succcurlyeq \mathbf{w}\}.$$

Proof. Let Y, Z be arbitrary elements of $\mathbf{X}$ and let

$$S = \text{l.u.b.}\{\mathbf{x} \bigcirc \mathbf{w} \mid Y \succcurlyeq \mathbf{x} \text{ and } Z \succcurlyeq \mathbf{w}\}.$$

It immediately follows from the definition of $\bigcirc$ and $\succcurlyeq$ that $Y \bigcirc Z \supseteq S$. Thus we need only show that $S \supseteq Y \bigcirc Z$. Let $x \in Y \bigcirc Z$. Let $y \in Y$ and $z \in Z$ be such that $y \bigcirc z \succ x$. Since $\mathscr{X}$ is a pre-unit structure, let α be an automorphism of $\mathscr{X}$ such that

$$y \bigcirc z \succ \alpha(y \bigcirc z) = \alpha(y) \bigcirc \alpha(z) \succ x.$$

Then it follows that x is in $\mathbf{y} \bigcirc \mathbf{z}$. Since $y \in Y$ and $z \in Z$, $Y \succcurlyeq \mathbf{y}$ and $Z \succcurlyeq z$. Thus $x \in \mathbf{y} \bigcirc \mathbf{z} \in S$. ▮

The following lemma summarizes the results of Lemmas 5.2–5.6 in a more convenient notation.

LEMMA 5.7. *Let $\mathscr{X} = \langle X, \succcurlyeq, \bigcirc \rangle$ be a pre-unit structure with a closed operation $\bigcirc$. Then there exists an extension of $\mathscr{X}$, $\mathscr{X}_1 = \langle X_1, \succcurlyeq, \bigcirc_1 \rangle$, with a closed operation $\bigcirc_1$ such that X is an order dense subset of $\langle X_1, \succcurlyeq_1 \rangle$, $\langle X_1, \succcurlyeq \rangle$ is Dedekind complete, and $\langle X_1, \succcurlyeq, \bigcirc_1 \rangle$ is a positive concatenation structure. Furthermore for each x, y in X_1,*

$$x \bigcirc_1 y = \text{l.u.b.}\{ p \bigcirc_1 q \mid p, q \in X, x \succcurlyeq_1 p, y \succcurlyeq_1 q \}.$$

LEMMA 5.8. *Let $\mathscr{X}$, $\mathscr{X}_1$ be as in Lemma 5.7. Then y/n exists for each y in X_1 and each n in I^+.*

Proof. Let $y \in X_1$ and $n \in I^+$. Let

$$\alpha_{1/n}(y) = \text{l.u.b.} \left\{ \frac{x}{n} \,\middle|\, x \in X \text{ and } y \succcurlyeq_1 x \right\}. \tag{5.1}$$

Then

$$\begin{aligned} n\alpha_{1/n}(y) &= n \left[\text{l.u.b.} \left\{ \frac{x}{n} \,\middle|\, x \in X \text{ and } y \succcurlyeq_1 x \right\} \right. \\ &\succcurlyeq \text{l.u.b.} \left\{ n \frac{x}{n} \,\middle|\, x \in X \text{ and } y \succcurlyeq_1 x \right\} \\ &= y. \end{aligned}$$

Thus to show that $n\alpha_{1/n}(y) = y$, we need only show that $n\alpha_{1/n}(y) \succ_1 y$ leads to a contradiction. Suppose $n\alpha_{1/n}(y) \succ_1 y$. Since X is order dense in $\langle X_1, \succcurlyeq_1 \rangle$, let u in X be such that $n\alpha_{1/n}(y) \succ_1 u_1 \succ y$. Since $\mathscr{X}$ is a pre-unit structure, $u = n(u/n)$. Thus $n\alpha_{1/n}(y) \succ_1 n(u/n)$, which by monotonicity of 0_1 yields $\alpha_{1/n}(y) \succ u/1n$. Thus by Eq. (5.1), let z in X be such that $\alpha_{1/n}(y) \succcurlyeq z/1n \succ u/1n$ and $y \succ_1 z$. Then we have $u \succ_1 y \succ_1 z$ and $z/n \succ u/1n$ which is impossible by monotonicity in $\mathscr{X}_1$.

LEMMA 5.9. *Let $\mathscr{X}$ and $\mathscr{X}_1$ be as in Lemma 5.7. Suppose x, y are in X_1 and n is in I^+. Then*

$$n(x \bigcirc_1 y) = nx \bigcirc_1 ny.$$

Proof. We need only show that both $n(x \bigcirc_1 y) \succ_1 nx \bigcirc_1 ny$ and $nx \bigcirc_1 ny \succ_1 n(x \bigcirc y)$ lead to contradiction.

Suppose $n(x \bigcirc_1 y) \succ_1 nx \bigcirc_1 ny$. Then by monotonicity of $\bigcirc_1$, $x \bigcirc_1 y \succ (1/1n)(nx \bigcirc_1 ny)$. By Lemma 5.8, let u, v in X be such that $x \succcurlyeq_1 u$, $y \succcurlyeq_1 v$, and

$$x \bigcirc_1 y \succ_1 u \bigcirc_1 v \succ_1 \frac{1}{n}(nx \bigcirc_1 ny).$$

Then since $\mathscr{X}$ is a pre-unit structure and by Lemma 5.7,

$$n(x \bigcirc_1 y) \succ_1 n(u \bigcirc_1 v) = nu \bigcirc_1 nv \succ_1 nx \bigcirc_1 ny. \tag{5.2}$$

However, $nx \succcurlyeq_1 nu$ since $x \succcurlyeq_1 u$. Similarly $ny \succcurlyeq_1 nv$. Thus by monotonicity,

$$nx \bigcirc_1 ny \succcurlyeq_1 nu \bigcirc_1 nv,$$

and this contradicts Eq. (5.2).

Suppose $nx \bigcirc_1 ny \succ_1 n(x \bigcirc_1 y)$. By Lemma 5.7, let z and w in X be such that $nx \succcurlyeq_1 z$, $ny \succcurlyeq_1 w$, and

$$nx \bigcirc_1 ny \succ_1 z \bigcirc_1 w \succ_1 n(x \bigcirc_1 y).$$

Since $z = n(z/n)$, $w = n(x/n)$, and $\mathscr{X}$ is a pre-unit structure, it follows that

$$nx \bigcirc_1 ny \succ_1 n\frac{z}{n} \bigcirc_1 n\frac{w}{n} = n\left(\frac{z}{n} \bigcirc_1 \frac{w}{n}\right) \succ_1 n(x \bigcirc_1 y). \tag{5.3}$$

However, $x \succcurlyeq_1 z/n$ and $y \succcurlyeq_1 w/n$ by monotonicity of $\bigcirc_1$. Thus by monotonicity, $x \bigcirc_1 y \succcurlyeq z/n \bigcirc_1 w/n$, which by monotonicity of $\mathbf{O}_1$ yields

$$n(x \bigcirc_1 y) \succcurlyeq_1 n\left(\frac{z}{n} \bigcirc_1 \frac{w}{n}\right),$$

and this contradicts Eq. (5.3). ∎

THEOREM 5.2. *Let $\mathscr{X}$, $\mathscr{X}_1$ be as in Lemma 5.7. Then $\mathscr{X}_1$ is a fundamental unit structure.*

Proof. By Lemma 5.7, $\mathscr{X}_1$ is a Dedekind complete, positive concatenation structure. By Lemma 5.8, nx is an automorphism of $\mathscr{X}_1$ for each n in I^+. Thus by Theorem 3.1, $\mathscr{X}_1$ is a fundamental unit structure. ∎

THEOREM 5.3. *Suppose $\mathscr{X}$ is a pre-unit structure. Then $\mathscr{X}$ is isomorphically imbeddable in a fundamental unit structure.*

Proof. Immediately follows from Theorems 5.1 and 5.2. ∎

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