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QUALITATIVE INDEPENDENCE IN PROBABILITY THEORY [1]

Probability theory is measure theory specialized by assumptions having to do with stochastic independence. Delete from probability and statistics those theorems that explicitly or implicitly (e.g., by postulating a random sample) invoke independence, and relatively little remains. Or attempt to estimate probabilities from data without assuming that at least certain observations are independent, and little results. Everyone who has worked with or applied probability is keenly aware of the importance of stochastic independence; experimenters go to some effort to ensure, and to check, that repeated observations are independent.

Kolmogorov (1933, 1950) wrote:

> "The concept of mutual *independence* of two or more experiments holds, in a certain sense, a central position in the theory of probability" (p. 8).
>
> "In consequence, one of the most important problems in the philosophy of the natural sciences is – in addition to the well known one regarding the essence of the concept of probability itself – to make precise the premises which would make it possible to regard any given real events as independent" (p. 9).

Despite these views, his classical axiomatization of numerical probability brings in stochastic independence not as a primitive, but as a defined quantity. If $\langle X, \xi, P\rangle$ is a finitely or countably additive probability space, then he defines two events A, B in ξ to be (stochastically) *independent* if and only if

$$P(A \cap B) = P(A)P(B).$$

The same is true of most presentations of qualitative probability, such as Savage (1954), in which sufficient axiomatic structure is introduced on $\langle X, \xi, \succsim\rangle$, where $\succsim$ is a binary relation of qualitative probability on ξ, so as to be able to construct a finitely additive probability representation in terms of which independence is defined in the usual way. An exception is the work of Domotor (1970) who combines axioms involving qualitative probability and independence to construct a finitely additive probability representation.

Since it is easy to give examples of qualitative structures for which the representation P is not unique, it is clear that stochastic independence cannot

Theory and Decision 9 (1978) 225–239.

be defined in terms of $\succsim$. (And even if P is unique, it appears that by using metamathematical results one can show the impossibility of giving a general definition of independence in terms of the underlying order.) In this connection, two quotations from Fine (1973) are relevant:

"...we must be cognizant of the fact that invocations of [stochastic independence] are usually not founded upon empirical or objective knowledge of probabilities. Quite the contrary. Independence is adduced to permit us to simplify and reduce the family of possible probabilistic descriptions for a given experiment" (p. 78).

"Given our views as to the problems encountered in assessing probability, we do not favor a purely probability-based definition [of independence]" (p. 141).

Once it is accepted that independence should not be treated as a defined concept, but rather should be in some sense a primitive which must, however, be consistent with the numerical probabilities, then another distinction looms important. One way to suggest it is to ask whether the author speaks of events or experiments as being independent. This distinction seems to lurk below the surface of many discussions, but it is difficult to cite brief quotations to bring it out, and in many cases (e.g., Kolmogorov above) authors treat events and experiments interchangeably.

In the case of events, one may think of independence as a binary relation between events, written $\perp$, and one is interested in its being consistent with numerical probability in the weak sense that for all A, B in ξ,

(1) $A \perp B$ implies $P(A \cap B) = P(A)P(B)$,

or in the strong sense that

(2) $A \perp B$ iff $P(A \cap B) = P(A)P(B)$.

In the case of independent experiments, one usually finds the discussion cast in terms of two or more independent random variables, as in a random sample. The usual probabilistic approach assumes that the experiments underlying the two random variables are each run, that a joint probability distribution exists over the various pairs of events, and that the relevant pairs induced by the random variables are stochastically independent as events. In practice, however, what one does is attempt to devise experimental realizations for which there are ample structural reasons for believing the two experiments are independent of each other in the sense that knowledge of the one does not affect the other. This concept of independence is discussed at length by Keynes (1962), see especially Ch. XVI.

Our aim here is to axiomatize qualitatively the concept of independent experiments and then we show how it relates to stochastic independence of events. The outline of the paper is, first, to discuss the qualitative idea of independent events and to show the conditions under which the usual multiplicative representation is forced. This leads naturally to the question of qualitative conditional probability and how its induced notion of independent events relates to the given primitive. Next we formulate the idea of independent experiments, show why that definition leads naturally to multiplication of probabilities and how such a structure can be embedded in a probability space in which independence of experiments is reflected as stochastic independence of events. Finally, we discuss how all of this relates to the usual normalization of probability, to meaningfulness of probabilistic statements, and to dimensional analysis. Proofs of all theorems are relegated to an appendix.

QUALITATIVE INDEPENDENCE OF EVENTS

The goal of this section is to understand what properties a qualitative relation of independence between events must satisfy in order for it to be represented weakly (Eq. 1) or strongly (Eq. 2) as stochastic independence.

To this end, let us begin with a qualitative probability structure $\langle X, \xi, \succsim\rangle$, where X is a set, ξ an algebra of subsets of X, and $\succsim$ a weak ordering (transitive and connected) of 'at least as possible as' on ξ. Let us suppose there is a unique finitely additive probability measure P that represents $\succsim$. (See Krantz *et al.*, 1971, Ch. 5 for various sets of sufficient conditions for P to exist.) And finally let $\perp$ be a binary relation over ξ, which is to be interpreted as qualitative independence of events. If $\mathfrak{F}$ is a subalgebra of ξ, it is convenient to abbreviate statements of the form $A \perp B$ for all B in $\mathfrak{F}$ by $A \perp \mathfrak{F}$.

To insure that $\perp$ is a sufficiently rich relation, we impose the following structural conditions.

AXIOM 1. $\langle X, \xi, \succsim, \perp\rangle$ *is a qualitative probability structure with* $\perp$ *a binary relation on* ξ *such that there exists a set* ξ' *of subsets of* X *with the following four properties:*

(i) ξ' *is an algebra of sets.*

(ii) $\xi' \subseteq \xi$,

(iii) *for all* A *in* ξ, *there exists* A' *in* ξ' *such that* $A' \sim A$,

(iv) *for all* A' *in* ξ', *there exists* A *in* ξ *such that* $A \sim A'$ *and* $A \perp \xi'$.

Next we formulate a necessary interlock between $\perp$, $\succsim$, and $\cap$ if $\perp$ is to be represented as in Eq. 1.

AXIOM 2. *For all A, B, C, D in ξ, suppose $A \perp B$, $C \perp D$, and $A \sim C$. Then $B \succsim D$ iff $A \cap B \succsim C \cap D$.*

THEOREM 1. *Suppose $\langle X, \xi, \succsim, \perp\rangle$ satisfies Axioms 1 and 2 and that $\langle X, \xi, \succsim\rangle$ has a unique probability representation $\langle X, \xi, P\rangle$. Then for all A, B in ξ,*

$A \perp B$ implies $P(A \cap B) = P(A)P(B)$.

To achieve the equivalence of $\perp$ and stochastic independence we need something close to Axiom 2 but with a statement about $\perp$ as the consequence. The following necessary condition seems appropriate.

AXIOM 3. *For all A, B, C, D in ξ, if $A \perp B$, $A \sim C$, $B \sim D$, and $A \cap B \sim C \cap D$, then $C \perp D$.*

THEOREM 2. *Suppose $\langle X, \xi, \succsim, \perp\rangle$ satisfies Axioms* 1, 2, and 3 *$\langle X, \xi, \succsim\rangle$ has a unique probability representation $\langle X, \xi, P\rangle$. Then for all A, B in ξ,*

$A \perp B$ iff $P(A \cap B) = P(A)P(B)$.

The proofs of these theorems are given in the appendix; however, it is important to understand the basic nature of the proofs. The key idea is to study the property

$$A \succsim B \text{ iff } A \cap Y \succsim B \cap Y$$

for some fixed Y in ξ and all A, B in the subalgebra ξ'. The mapping $F(A) = A \cap Y$ is shown to establish an isomorphism between the structure restricted to ξ' and that restricted to $F(\xi')$. And because of the uniqueness of the postulated probability measure, this isomorphism forces the multiplicative property $P(A \cap Y) = P(A)P(Y)$. In our opinion, the basic feature of independence is the above monotonicity of $\succsim$ under the operation $\cap$; the multiplicative representation of independence results from the isomorphism induced by monotonicity. All of this is captured in Lemmas 1-3 of the Appendix.

QUALITATIVE CONDITIONAL PROBABILITY AND INDEPENDENCE

Given the structure $\langle X, \xi, \succsim, \perp\rangle$, one can introduce the following concept of qualitative conditional probability. *Define the ternary relation* $|$ on ξ *by*:

for A, B, C in ξ, $A|B \sim C$ iff there exist B', C' in ξ such that $B' \sim B$, $C' \sim C$, $B' \perp C'$, and $B' \cap C' \sim A \cap B$.

In terms of this concept, another notion of independence is available, namely, $A|B \sim A$. We first show that in the presence of all three axioms, the two concepts agree.

THEOREM 3. *Suppose $\langle X, \xi, \succsim, \perp \rangle$ is a qualitative probability structure with a binary relation $\perp$. For all A, B in ξ, if the structure satisfies Axiom 3, $A|B \sim A$ implies $A \perp B$; if the structure satisfies Axioms 1 and 2, $A \perp B$ implies $A|B \sim A$.*

Furthermore, the probability representation of $\perp$ is as one would expect.

THEOREM 4. *Suppose $\langle X, \xi, \succsim, \perp \rangle$ satisfies Axioms 1 and 2 and has a unique probability representation P. Then for all A, B, C, in ξ,*

$$A|B \sim C \text{ iff } P(C) = P(A \cap B)/P(B).$$

INDEPENDENCE OF EXPERIMENTS

The idea of the same experiment being repeated twice (the generalizations to any finite number of replications are obvious) can be captured by considering $\xi \times \xi$, where ξ is an algebra of subsets of a set X. To be strictly correct, we should distinguish the two occurrences of ξ, treating them as isomorphic copies, but such an abuse of notation does not seem to lead to problems. The notion of qualitative probability corresponding to the idea of a joint distribution is a binary relation $\succsim$ on $\xi \times \xi$. Our problem is to formulate axioms on $\succsim$ sufficient to represent it as a joint probability measure that is a product of a measure on ξ. Roughly, the axioms will formulate the idea that the experiments are independent and will also reduce the construction of the probability measure to a known result.

DEFINITION 1. *Suppose X is a set, ξ an algebra of subsets of X, and $\succsim$ a binary relation on $\xi \times \xi$. The structure $\langle X, \xi \times \xi, \succsim \rangle$ is called an* independent joint qualitative probability structure *if and only if the following axioms hold for all A, B, C, D, E, F in ξ:*

AXIOM 1. *$\succsim$ is a weak ordering.*

AXIOM 2. (Independence) *$(A, C) \succsim (B, C)$ iff $(A, D) \succsim (B, D)$.*

AXIOM 3. (Symmetry) *$(A, B) \sim (B, A)$.*

AXIOM 4. (Distribution) *If* $A \cap B = C \cap D = \emptyset$, $(A, E) \succsim (C, F)$, *and* $(B, E) \succsim (D, F)$, *then* $(A \cup B, E) \succsim (C \cup D, F)$. *Moreover, the conclusion is* $\succ$ *if either hypothesis is* $\succ$.

Note: By Axioms 2 and 3, a unique ordering $\succsim_1$ is induced on ξ which, by Axiom 1, is a weak order.

AXIOM 5. $X \succ_1 \emptyset; A \succsim_1 \emptyset$.

AXIOM 6. (Archimedean) *Every sequence of the following form is finite:* $A_1 \succ_1 \emptyset, B \succ_1 C \succ_1 \emptyset$, *and* $(A_{i+1}, C) \sim (A_i, B)$.

The remaining axioms are structural.

AXIOM 7. *There exists* U *in* ξ *such that* $(U, X) \sim (A, B)$.

AXIOM 8. *If* $A \cap B = \emptyset$ *and* $(A, B) \succsim (C, D)$, *then there exist* C', D' *in* ξ *such that* $C' \succsim_1 C, D' \succsim_1 D$, *and* $C' \cap D' = \emptyset$.

AXIOM 9. *If* $A \succsim_1 B$, *then there exists* A' *in* ξ *such that* $A' \sim_1 A$ *and* $A' \supseteq B$.

Comment: If one knows the theory of additive conjoint measurement, it is not surprising that we invoked Axiom 2. One might also have expected us to postulate the Thomsen condition which is so essential to a product representation. However, as has been demonstrated in slightly different contexts (Narens, 1976; Narens & Luce, 1976), when one has a conjoint structure with an operation on a component that has an additive representation and is related to the conjoint structure by a distribution law (Axiom 4), then the multiplicative representation follows without the Thomsen condition.

The three structural axioms are strong – in essence they imply an atomless structure with many events equivalent in probability. It would be nice to weaken them.

THEOREM 5. *Suppose* $\langle X, \xi \times \xi, \succsim \rangle$ *is an independent joint qualitative probability structure. Then there exists a real-valued function* P *on* ξ *such that*

1. $\langle X, \xi, P \rangle$ *is a finitely additive probability space, and*
2. *for all* A, B, C, D, *in* ξ,
   $(A, B) \succsim (C, D)$ *iff* $P(A)P(B) \geqslant P(C)P(D)$.

The next obvious question is whether the conjoint independence captured in part 2 of Theorem 5 is also reasonably interpreted as stochastic independence of events, as is usual in probability theory. Put another way, can one imbed the structure $\langle X, \xi \times \xi, P \rangle$ into a probability space. The answer is Yes.

Let $(\xi \times \xi)^*$ consist of all subsets of $X \times X$ which are unions of finitely many disjoint subsets from $\xi \times \xi$. For E in $(\xi \times \xi)^*$ with

$$E = \bigcup_{i=1}^{n} (A_i, B_i),$$

where for $i, j = 1, 2, \ldots, n$, $i \neq j$, (A_i, B_i) and (A_j, B_j) in $\xi \times \xi$ and $(A_i, B_i) \cap (A_j, B_j) = \emptyset$, define

$$P^*(E) = \sum_{i=1}^{n} P(A_i)P(B_i).$$

THEOREM 6. *Suppose ξ is an algebra of subsets of X, then*

(i) *$(\xi \times \xi)^*$ is an algebra of subsets of $X \times X$; and*

(ii) *$(\xi \times \xi)^*$ is the minimal algebra of $X \times X$ that contains $\xi \times \xi$.*

If $\langle X, \xi, P\rangle$ is a finitely additive probability space, then

(iii) *$\langle X \times X, (\xi \times \xi)^*, P^*\rangle$ is a finitely additive probability space;*

(iv) *it is the unique space with the property that for (A, B) in $\xi \times \xi$, $P^*(A, B) = P(A)P(B)$; and*

(v) *$P^*[(A, X) \cap (X, B)] = P^*(A, X)P^*(X, B)$.*

UNIQUENESS OF PROBABILITY REPRESENTATIONS

The literature on numerical representations of qualitative probability structures is not really satisfactory for the following reason. In order for the representation to be unique (with $P(X) = 1$), one is forced to postulate very strong solvability conditions. Without such conditions, such as in the finite case, not only do several representations exist but their relations to one another are difficult to characterize. But the real issue centers on nonuniqueness which is clearly incompatible with independence being formulated as $P(A \cap B) = P(A)P(B)$. This has led many of us to believe that were we to introduce an appropriate qualitative notion of independence, it could be used to force uniqueness of the representation. So far, however, this has not proved successful.

The discussion of the preceding section suggests an alternative route, namely, to require that the measure represent repeated, independent experiments. When we do this, the representation is unique for any X.

THEOREM 7. *Suppose $\langle X, \xi^n \succsim_n\rangle$, $n = 1, 2, \ldots$, are ordered structures that are represented by a probability space $\langle X, \xi, P\rangle$ in the sense that for every n and all $A_1, \ldots, A_n, B_1, \ldots, B_n$ in ξ.*

$$(A_1, \ldots, A_n) \succsim_n (B_1, \ldots, B_n) \text{ iff } \prod_{i=1}^{n} P(A_i) \geq \prod_{i=1}^{n} P(B_i).$$

Then P is the unique representation.

DISCUSSION

Among the various numerical measures used in the physical sciences, probability seems to have the unique status of being unique, of having a natural unit as well as zero. This does not follow simply because the measure is bounded from above, for velocity has that property and no one claims that the velocity of light must be 1. Rather, it is a claim that the numerical measure admits no transformations, that the statement $P(X) = 1$ is not purely conventional. Where does this added constraint come from?

It arises from two facts. The first is the definition of stochastic independence, and the second is that X is independent of every event in ξ. From these two we see

$$P(A) = P(X \cap A) = P(X)P(A),$$

whence

$$P(X) = 1.$$

This uniqueness of probability is not without some philosophic and, perhaps, practical disadvantages. These disadvantages arise when we consider the possibility that some statements which can be formulated in terms of numbers between 0 and 1 are not really meaningful when the numbers are treated as probabilities of events. More generally, if one looks at the overall structure of physical dimensions, which presumably should include probability as well as length, mass, etc , there again is a distinction between meaningful and meaningless statements, the meaningful ones corresponding to what physicists call dimensionally invariant laws (Krantz *et al.*, 1971, Ch. 10; Luce, 1978). The definition of meaningful qualitative relations which has evolved is that the relation should be invariant under the group of transformations which take one representation into another – the uniqueness of the representation. Thus, if we take seriously the absolute uniqueness of probability, there are no transformations other than the identity and so all quantitative relations which can be formulated are meaningful. Such a conclusion strongly suggests that either something is wrong with our concept of meaningfulness or we are wrong about the absolute uniqueness of probability.

If one examines carefully the proofs given in the Appendix, homomorphisms of qualitative probability sub-structures play an essential role in our analysis of independence, and so it is very difficult to believe that this natural quantitative interpretation – a scale change in probability – should be wiped out by the definition of stochastic independence. Indeed, it is quite clear that the concept of independence does not need to restrict the representation of probability beyond that of a ratio scale (multiplication by a positive constant). This is transparent in our analysis of independent experiments, where the multiplicative structure arises from conjoint measurement, but it is equally true for events. Turn to the proof of Lemma 2, where we invoke the uniqueness of the additive representation, and assume P is a ratio scale rather than an absolute one, then we see that the assertion is altered to

$$P(A \cap Y)P(X) = P(A)P(Y).$$

In the usual case, $P(X) = 1$ and it is suppressed. This change then leads to the following modified representation of independence

$$A \perp B \text{ iff } P(A \cap B) = P(A)P(B)/P(X).$$

With that slight change, probability is then a ratio scale, just like the other basic extensive measures of physics. In particular, probability can be incorporated into the dimensional structure of physics (if it is found to interrelate with other dimensions via distribution laws) and meaningfulness can be studied as with other dimensions.

APPENDIX. PROOFS OF THEOREMS

General notation and terminology: ξ and ξ' are algebras of subsets of X (closed under unions and complementation), $\bar{A}$ denotes the complement of A, $\succsim$ is a weak order of ξ, and if $\xi' \subseteq \xi$, $\succsim_{\xi'}$ is the restriction of $\succsim$ to ξ'. If $\langle X, \xi, P\rangle$ is a probability space and P preserves the order $\succsim$, then we say P represents $\langle X, \xi, \succsim\rangle$. Throughout, we will deal with the equivalence classes of $\sim$.

LEMMA 1. *Suppose $\langle X, \xi, \succsim\rangle$ is a weakly ordered algebra, $\xi' \subseteq \xi$, and there is a Y in ξ such that for all A, B in ξ',*

$$A \succsim B \textit{ iff } A \cap Y \succsim B \cap Y.$$

Define $F(A) = A \cap Y$ *and let* $\xi'' = F(\xi') = \{F(A) | A \text{ in } \xi'\}$. *Then F is a qualitative probability isomorphism of* $\langle X, \xi', \succsim_{\xi'}\rangle$ and $\langle Y, \xi'', \succsim_{\xi''}\rangle$.

Proof. By definition of ξ'', F is onto ξ''. To show F is one to one, suppose $F(A) = F(B)$, i.e., $A \cap Y = B \cap Y$. Thus, $(A - B) \cap Y = \emptyset = (B - A) \cap Y$. If $A = B$, we are done. If not, then either $A - B \neq \emptyset$ or $B - A \neq \emptyset$, so without loss of generality assume $A - B \succ \emptyset$. Since $A - B$, $\emptyset$ are in ξ', $(A - B) \cap Y \succ \emptyset \cap Y = \emptyset$, a contradiction. So F is 1:1.

Next we show F is a Boolean isomorphism.

$$F(A \cup B) = (A \cup B) \cap Y = (A \cap Y) \cup (B \cap Y) = F(A) \cup F(B)$$

$$F(\bar{A}) = \bar{A} \cap Y = (X \cap Y) - (A \cap Y) = Y - F(A).$$

Finally, F is an order isomorphism since for all A, B in ξ',

$$A \succsim_{\xi'} B \text{ iff } A \cap Y \succsim B \cap Y \text{ iff } F(A) \succsim_{\xi'} F(B).$$

LEMMA 2. *In addition to the hypothesis of Lemma* 1, *suppose* P *and* $P' = P \mid \xi'$ *the unique probability representations of* ξ *and* ξ' *respectively, Then for each* A *in* ξ',

$$P(A \cap Y) = P(A)P(Y).$$

Proof. Let $Q = P|\xi''$ and $Q' = Q/P(Y)$. Clearly, Q' is a probability representation for $\langle Y, \xi'', \succsim_{\xi''}\rangle$. But by the isomosphism of Lemma 1 and the fact that $\langle X, \xi', \succsim_{\xi'}\rangle$ has a unique probability representation, we must have $P'(A) = Q'[F(A)]$. Therefore, for each A in ξ',

$$P(A) = P'(A) = Q'[F(A)] = \frac{Q'(A \cap Y)}{P(Y)} = \frac{P(A \cap Y)}{P(Y)}.$$

LEMMA 3. *Suppose in addition to the hypothesis of Lemma* 2, *there is a binary relation* $\perp$ *with the property that if* $A \perp B$ *and* $A \perp C$ *then* $B \succsim C$ *iff* $A \cap B \succsim A \cap C$. *Suppose* $Y \perp \xi'$. *Then for all* A *in* ξ',

$$P(A \cap Y) = P(A)P(Y).$$

Proof. Immediate from Lemma 2.

Proof of Theorem 1. Suppose $A \perp B$. By Axiom 2(iii) let A', B' in ξ' be such that $A' \sim A$, $B' \sim B$. By Axiom 2(iv) let A'' in ξ be such that $A'' \sim A$ and $A'' \perp \xi'$. Since Axiom 2 (with $A = C$) yields the hypothesis of Lemma 3, we have $P(A'' \cap B') = P(A'')P(B')$. But Axiom 2 applied to $A \perp B$, $A'' \perp B'$, $A \sim A''$, and $B \sim B'$ implies $A \cap B \sim A'' \cap B'$, and so the result follows.

Proof of Theorem 2. By Theorem 1, we know the implication goes in one direction. So suppose $P(A \cap B) = P(A)P(B)$. According to Axiom 2(iii) and (iv), there exist A'' in ξ and B' in ξ' such that $A'' \sim A$, $B' \sim B$, $A'' \perp \xi'$. By Lemma 3,

$$P(A'' \cap B') = P(A'')P(B') = P(A)P(B) = P(A \cap B),$$

so $A'' \cap B' \sim A \cap B$. Since $A'' \perp B'$, Axiom 3 implies $A \perp B$.

Proof of Theorem 3. Suppose $A|B \sim A$, i.e., there exist B', A' in ξ such that $B' \sim B, A' \sim A, B' \perp A'$, and $A' \cap B' \sim A \cap B$. By Axiom 3, $A \perp B$.

Conversely, suppose $A \perp B$. By Axiom 1(iii) and (iv) there exist A', B' in ξ' such that $A' \sim A$, $B' \sim B$, and there exists A'' in ξ such that $A'' \sim A'$ and $A'' \perp \xi'$. In particular, $A'' \perp B'$. So by Axiom 2, $A'' \cap B' \sim A \cap B$. Thus, by definition, $A|B \sim A$.

Proof of Theorem 4. Suppose $A|B \sim C$, i.e., there exist B', C' in ξ such that $B' \sim B, C' \sim C, B' \perp C'$, and $B' \cap C' \sim A \cap B$. By Theorem 1,

$$P(A \cap B) = P(B' \cap C') = P(B')P(C') = P(B)P(C).$$

Conversely, suppose $P(C) = P(A \cap B)|P(B)$. By Axiom 1(iii) and (iv), there exist B', C' such that $B' \sim B$, $C' \sim C, B' \perp C'$. By Theorem 1,

$$P(B' \cap C') = P(B')P(C') = P(B)P(C) = P(A \cap B),$$

so $B' \cap C' \sim A \cap$ B. By definition, $C \sim A|B$.

Proof of Theorem 5. All references to Axioms are those of Definition 1.

LEMMA 4. *Suppose $B \succ_1 \emptyset$ and $A \cap B = \emptyset$, then $A \cup B \succ_1 A$.*
Proof. By $B \succ_1 \emptyset$ and Axiom 2, $(B, X) \succ (\emptyset, X)$. By Axiom 1, $(A,X) \sim (A,X)$, so by Axiom 4, $(A \cup B, X) \succ (A,X)$.

LEMMA 5. (i) *$A \succsim_1 B$ iff $\bar{B} \succsim_1 \bar{A}$.*
(ii) *If $A \succsim_1 B$, there exists B' in ξ such that $B' \subseteq A$ and $B' \sim {}_1A$.*
Proof. (i) Suppose both $A \succsim_1 B$ and $\bar{A} \succ {}_1\bar{B}$, then using Axiom 2, $(A,X) \succsim (B,X)$ and $(\bar{A},X) \succ (\bar{B},X)$, whence by Axiom 4, $(X,X) \succ (X,X)$, contrary to Axiom 1.

(ii) If $A \succsim_1 B$, then by part (i) $\bar{B} \succsim_1 \bar{A}$, whence by Axiom 9, there is C in ξ such that $C \sim \bar{B}$ and $C \supseteq \bar{A}$. Thus, $B' = \bar{C}$ has the two properties.

LEMMA 6. $\langle X, \xi, \succsim_1 \rangle$ *is an Archimedean structure of qualitative probability* (*Def.* 5.4, Krantz *et al.*, 1971) *for which Axiom* 5.5 (p. 207, Krantz *et al.*, 1971) *holds.*

Proof. Axiom 5.4.1. holds by Axioms 1 and 2.

Axiom 5.4.2. is the same as Axiom 5.

Axiom 5.4.3. Suppose $A \cap B = A \cap C = \emptyset$ and, without loss of generality, $B \succsim_1 C$. Using Axioms 1 and 2, $(A,X) \sim (A,X)$ and $(B,X) \succsim (C,X)$, so by Axiom 4 $(A \cup B,X) \succsim (A \cup C,X)$, so $A \cup B \succsim_1 A \cup C$. The converse holds because Axiom 4 holds for strict inequalities.

Axiom 5.4.4. Suppose $\{A_i\}$ is a standard sequence relative to $A \succ_1 \emptyset$ (Def. 5.3, Krantz *et al.*, 1971). Using Axiom 7, let U_i solve $(U_i,X) \sim (A_k,A)$. Observe that by the definition of A_i,

$$
\begin{aligned}
(A_{i+1},A) &\sim (B_i \cup C_i, A) && \text{(Def. 5.3, Krantz } et\ al.\text{, 1971)}\\
&\succ (B_i, A) && (C_i \sim {}_1 A \succ_1 \emptyset\text{, Lemma 1)}\\
&\sim (A_i, A),
\end{aligned}
$$

so $U_{i+1} \succ_1 U_i$. The sequence $\{U_i\}$ is bounded from above by X and from below by $\emptyset$ (Axiom 5 and Lemma 5), so by Axiom 6 it is finite. Therefore, $\{A_i\}$ is finite.

Axiom 5.4.5. Suppose $A \cap B = \emptyset, A \succ_1 C$, and $B \succ_1 D$. By Axiom 2, $(A,B) \succ (C,D)$. By Axiom 8, there exist C', D' in ξ such that $C' \sim {}_1 C, D' \sim {}_1 D$, and $C' \cap D' = \emptyset$. By Axiom 9, there exists E in ξ such that $E \sim {}_1 A \cup B$ and $E \supset C' \cup D'$.

Part 1 of Theorem 5 follows from Lemma 6 and Theorem 5.2 of Krantz *et al.* (1971).

To prove part 2, let the solution U of $(U,X) \sim (A,B)$ (Axiom 7) be denoted $A * B$. By Axioms 1, 2, and 3, $A * B \sim {}_1 B * A$. If $A * B \succ_1 B$, then by Axioms 2 and 5 and Lemma 5(i),

$$(A * B, X) \succ (B,X) \succsim (B,A) \sim (A,B),$$

contrary to definition of $A * B$. So $B \succsim_1 A * B$. By Lemma 5(ii), there exists $(A * B)' \sim_1 A * B$ with $(A * B)' \subseteq B$. So by Axiom 2, $(A * B)'$ is an isomorphism from ξ into

$$\xi_B = \{C | C \text{ in } \xi \text{ and } C \subseteq B\}.$$

By the uniqueness of extensive structures, there exists a constant ϕ_B such that

$$P(A * B) = P[(A * B)'] = \phi_B P(A).$$

Setting $A = X$,

$$P(A * B) = P(B) = \phi_B P(X) = \phi_B,$$

so

$$P(A * B) = P(A)P(B).$$

The conclusion follows since by Axiom 2,

$$(A * B, X) \sim (A, B) \succsim (C, D) \sim (C * D, X).$$

Proof of Theorem 6.

(i) Note that $(A,B) \cap (C,D) = (A \cap C, B \cap D)$. Thus if $E = \bigcup_{i=1}^{n} (A_i, B_i)$ and $F = \bigcup_{i=1}^{m} (C_i, D_i)$, where $(A_i,B_i) \cap (A_j,B_j) = \emptyset$ for $1 \leqslant i, j \leqslant n, i \neq j$, and $(C_i, D_i) \cap (C_j,D_j) = \emptyset$ for $1 \leqslant i, j \leqslant m, i \neq j$, then $E \cap F = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_i \cap C_j, B_i \cap D_j)$, where

$$(A_i \cap C_j, B_i \cap D_j) \cap (A_{i'} \cap C_{j'}, B_{i'} \cap D_{j'}) = \emptyset$$

for $1 \leqslant i, i' \leqslant n$, $1 \leqslant j, j' \leqslant m$, $ij \neq i'j'$. Thus $(\xi \times \xi)^*$ is closed under intersections. To show it is also closed under complementations, assume E is above and let $A'_1 = A_1$, and for $1 < i < n$ let $A'_{i+1} = A_{i+1} - \bigcup_{k=1}^{i} A'_k$, and let $A_{n+1} = X - \bigcup_{k=1}^{n} A'_k$. B'_i are similarly defined. Then A'_i and B'_i are partitions of X and A_k, B_k for $k = 1, \ldots, n + 1$ are definable as finite unions of elements of the partitions A'_i, B'_i, respectively. Thus (A_k, B_k) are definable as finite unions of the partition (A'_i, B'_j), $1 \leqslant i, j \leqslant n + 1$, and therefore E and hence $\bar{E}$ are definable as finite unions of elements of this partition. Since unions are definable in terms of intersections and complements, $(\xi \times \xi)^*$ is closed under unions.

(ii) Since an algebra is closed under finite unions, $(\xi \times \xi)^*$ is clearly the minimal one containing $\xi \times \xi$.

(iii) By construction, P^* is finitely additive, and $P^*(X \times X) = P(X)P(X) = 1$.

(iv) This follows from (ii) and the construction of P^*.

(v) $$\begin{aligned} P^*[(A,X) \cap (X,B)] &= P^*[(A,B)] \\ &= P(A)P(B) \\ &= P^*[(A,X)]P^*[(X,B)]. \end{aligned}$$

Proof of Theorem 7. The proof of the result is trivial for $\xi = \{X, \emptyset\}$, so we assume there is an A in ξ for which $1/2 \leqslant p = P(A) < 1$. Clearly, it suffices to work with the subalgebra $\{X, A, \bar{A}, \phi\}$ and to show p is unique. For each positive integer n there exists a unique positive integer $m(n)$ such that

$$p^{m(n)} \geq (1-p)^n > p^{m(n)+1}.$$

since $p \geq 1 - p$, $m(n) + 1 \geq n$. This means that there are events $A_i = A$, $\bar{A}_i = \bar{A}$, and $X_i = X$ such that

$$(A_1, A_2, \dots, A_{m(n)}, X_1) \succsim_{m(n)+1} (\bar{A}_1, \bar{A}_2, \dots, \bar{A}_n, X_1, \dots, X_{m(n)+1-n}) \succ_{m(n)+1} (A_1, A_2, \dots, A_{m(n)+1}).$$

Therefore, if $\langle x, \xi, Q\rangle$ is another probability representation of $\langle X, \xi^n, \succ_n\rangle$ for $n = 1, 2, \dots$, and if $Q(A) = q$, then

$$q^{m(n)} \geq (1-q)^n > q^{m(n)+1}.$$

Observe that

$$\frac{m(n)}{n} \geq \frac{\log(1-p)}{\log p} > \frac{m(n)+1}{n}.$$

Thus, $h = \lim\limits_{n \to \infty} \dfrac{m(n)}{n}$ exists and

$$h = \frac{\log(1-p)}{\log p} = \frac{\log(1-q)}{\log q}.$$

Thus,

$$q^h + q - 1 = p^h + p - 1.$$

Since $f(p) = p^h + p - 1$ is monotonically increasing for $p \geq 0$ and $f(0) < 0$ and $f(1) > 0$, there is a unique p such that $f(p) = 0$. So $p = q$.

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