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Utility-Uncertainty Trade-Off Structures*

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A qualitative axiomatization of a generalization of expected utility theory is given in which the expected value of simple gambles is not necessarily the product of subjective probability and utility. Representation and uniqueness theorems for these generalized structures are derived for both Archimedean and nonarchimedean cases. It is also shown that a simple condition called *distributivity* is necessary and sufficient in the case of simple gambles for one of these generalized expected utility structures to have simultaneously an additive subjective probability function and a multiplicative combining rule for expected values.

1. INTRODUCTION

Pascal in his *Pensées* makes the following remarkable observation:

> Our soul is tossed into the body where it finds number, time, dimensions. It argues about them, calls them nature or necessity, and cannot believe in anything else. ...
>
> Let us consider the point and say: 'Either God exists, or he does not exist.' But which of the alternatives shall we choose? Reason can determine nothing: there is an infinite chaos which divides us. A coin is being spun at the extreme point of this infinite distance which will turn up heads or tails. What is your bet? If you rely on reason you cannot settle for either, or defend either position.
>
> Do not therefore accuse those who have made their choice of falseness because you know nothing about it.
>
> 'No, I do not blame them for their choice, but for making a choice at all because he who calls heads and he who calls tails are guilty of the same mistake, they are both wrong: the right course is not to wager.' 'Yes, but we have to wager. You are not a free agent; you are committed. Which will you have then? Come on. Since you are obliged to choose, let us see which interests you least. You may lose two things: the true and the good; and there are two things that you stake: your reason and your will, your knowledge and your beatitude; and your nature has two things from which to escape: error and unhappiness. Your reason is not more deeply wounded by choosing one rather than the other because it is bound to choose. That disposes of one point. But what about your beatitude? Let us measure the gain and the loss by saying: "Heads God exists." Let us compare the two cases; if you win, you win everything; if you lose, you lose nothing. Don't hesitate then. Take a bet that he exists.'

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> 'That's fine. Yes, I must take a bet; but perhaps I am staking too much.'
>
> 'Come. Since there is an equal chance of gain and loss, if you were only to win two lives for one, you could still wager; but if there were three to be won, you would have to gamble (since you are bound to gamble), and it would be imprudent, when you are obliged to gamble, not to risk your life in order to win three lives at a game in which there is such a chance of loss and gain. But there is an eternity of life and happiness at stake. And since it is so, if there were an infinite number of chances of which only one was for you, you would still be right to risk one to win two; and you would be taking the wrong road if, being forced to gamble, you refuse to stake one life against three in a game in which, out of an infinite number of chances, one is for you, if the prize were in infinity of life which was infinitely happy. But in this game you can win eternal life which is eternally happy; you have one chance of winning against a finite number of chances of losing, and what you are staking is finite. That settles it: wherever there is infinity, and where there is not an infinity of chances of losing against the chance of winning, there is no room for hesitation: you must stake everything. And so, since you are forced to gamble, you must abandon reason in order to save your life, rather than risk it for the infinite gain which is just as likely to turn up as the loss of nothing.' Pascal [1670], pp. 200–203.

In modern terms, Pascal is presenting a version of qualitative expected utility. Basically, he is saying that a rational man M has a preference ordering $\succsim$ for certain types of gambles. These gambles are of the form $a\,A\,b$ where A is some chance event (e.g., heads occurring when a coin is flipped) and a, b are objects whose value to M are independent of the occurrence or nonoccurrence of A, and "$a\,A\,b$" means that M gets a if A occurs and b if A does not occur. Furthermore, it is implicit in the above passage that M has a probability function P on chance events and an utility function φ on objects of value such that

$$a\,A\,b \succsim c\,B\,d \quad \text{iff} \quad \varphi(a)P(A) + \varphi(b)(1 - P(A)) \geqslant \varphi(c)P(B) + \varphi(d)(1 - P(B)).$$

In the above passage Pascal allows some of the objects to have infinite utility and some of the chance events infinitesimal probability. In 17th Century mathematics (Pascal died in 1662), the use of infinities and infinitesimals in proofs of mathematical or physical propositions were common.

In Pascal's time (as well as today) measurement was not well understood and there was constant confusion between quantitative and qualitative concepts. For example, Pascal implicitly assumes that an infinity of happiness should (to a rational man) have infinite utility. This would be reasonable if the "additivity" of happiness corresponded to the additivity of the utility of happiness. If one were only considering happiness, then it is consistent to assume that such a correspondence always exists provided that the operation of "additivity" of happiness satisfies certain natural conditions. However, Pascal compares *chances* of happinesses. Probability is the measure of chance and chance has its own form of "additivity": if A and B are disjoint events then $A \cup B$ can be considered as the "sum" of A and B. Once again, if one were only considering chance by itself, then it is consistent to assume that the "additivity" of chance corresponds to the additivity of probabilities. However, if one has to simultaneously measure chance and happiness to measure the values of chances of happinesses, then only in special cases can this be done in a way such that the "additivity" of chance corresponds to the additivity of probabilities and the "additivity" of happiness corresponds to the additivity of the utility of happiness.

It turns out that a similar situation occurs in physics where there are natural ways of qualitatively defining an ordering relation $\succsim$ and an "addition" operation o on the set of speeds S so that $\langle S, \succsim, o\rangle$ is a closed extensive structure (see Definition 3.6) and thus has an additive numerical representation: i.e., there is a real valued function s defined on S such that for all x, y in S,

$$x \succsim y \quad \text{iff} \quad s(x) \geqslant s(y),$$

and

$$s(x \circ y) = s(x) + s(y).$$

Similarly, natural qualitative ordering relations $\succsim'$, $\succsim''$ and qualitative "addition" operations o', o'' can be defined on the set of lengths L and the set of times T so that $\langle L, \succsim', o'\rangle$ and $\langle T, \succsim'', o''\rangle$ are closed extensive structures and also have additive numerical representations. In relativistic physics there is a bit of a problem with the speed of light l. If s is an additive numerical representation for speed and x is a speed such that $l \succ x$, then in special relativity $l \succ x \circ x$. Thus $s(l) > s(x) + s(x) = 2s(x)$. Applying this reasoning to $x \circ x$, it follows that $s(l) > 4s(x)$. In general, for each positive integer n, $s(l) > 2^n s(x)$. Thus in relativity if s is an *additive* numerical representation for speed, then $s(l) = \infty$. Interestingly enough, if $l \succ x$ and x is qualitatively "added" to l then the result is qualitatively equivalent to l, i.e., $x \circ l \sim l$. Pascal in the above mediation gives his "infinities" a similar property:

> Unity joined to infinity does not add anything to it, any more than a foot to a measure which is infinite. The finite is annihilated in the presence of the infinite, and becomes pure nothingness.

Although the speed of light cannot be exceeded in relativistic physics, it is theoretically surpassable in classical physics. In classical physics there are additive numerical representations, d, t, s, for distance, time, and speed respectively such that the representation of the speed of a particle in uniform motion is the ratio of the representation of the length travelled to the representation of the time elapsed, i.e., $s = d/t$. In special relativity, one cannot simultaneously have additive numerical representations d, t, s for length, time and speed (even when s is only defined for speeds less than l) such that $s = d/t$. Therefore, in relativistic physics some nonadditive numerical representation must be given or law $s = d/t$ must be changed. Historically, physicists have chosen to preserve the law $s = d/t$ and give additive representations to length and time and a nonadditive representation to speed so that for all speeds x, y,

$$s(x \circ y) = \frac{s(x) + s(y)}{1 + s(x)\, s(y)}$$

where units are chosen so that $s(l) = 1$. If we let $\oplus$ be the binary operation defined on the positive real numbers by

$$u \oplus v = \frac{u + v}{1 + uv},$$

then in relativistic physics for all speeds x, y,

$$x \gtrsim y \quad \text{iff} \quad s(x) \geqslant s(y),$$

and

$$s(x \circ y) = s(x) \oplus s(y).$$

In this case s is a numerical representation for $\langle S, \gtrsim, o\rangle$ and the qualitative "addition" operation on speeds is interpreted quantitively as the numerical operation $\oplus$. (For qualitative treatments of relativistic velocities, see Luce and Narens [1975] and Narens and Luce [1975]).

In order to create an analogous situation for Pascal's example, let $\mathscr{E}$ be an algebra of chance events, $\mathscr{C}$ be a set of objects of value, and for each $a \in \mathscr{C}$ and $A \in \mathscr{E}$ let (a, A) mean that a rational man M will receive a if A occurs and receive nothing if A does not occur. Let $\gtrsim$ be M's preference ordering on $\mathscr{C} \times \mathscr{E}$. Call $\mathscr{R} = \langle \odot, \oplus, \varphi, P\rangle$ a *representation for* $\mathscr{U} = \langle \mathscr{C}, \mathscr{E}, \gtrsim\rangle$ if and only if for all $a, b \in \mathscr{C}$, all $A, B \in \mathscr{E}$, and all positive r, the following five conditions hold:

(1) $\odot$ and $\oplus$ are binary operations on the nonnegative reals such that $1 \odot r = r$;

(2) φ is a function from $\mathscr{C}$ into the positive reals and for some u, $\varphi(u) = 1$;

(3) P is a function from $\mathscr{E}$ into $[0, 1]$ such that $P(\varnothing) = 0$ and $P(X) = 1$ where $\varnothing$ is the null event in $\mathscr{E}$ and X is the sure event in $\mathscr{E}$;

(4) $(a, A) \gtrsim (b, B)$ iff $\varphi(a) \odot P(A) \geqslant \varphi(b) \odot P(B)$;

(5) if $A \cap B = \varnothing$ then $\varphi(a) \odot P(A \cup B) = \varphi(a) \odot (P(A) \oplus P(B))$.

Intuitively, φ is M's utility function for $\mathscr{C}$, P is M's subjective measure of the likelihood of the occurrence of members of $\mathscr{E}$, $\oplus$ is the quantitative interpretation of the qualitative "addition" operation for chance, and $\odot$ is the quantitative interpretation of M's expectation function. Note that if $\oplus$ is $+$ then from (1), (3), and (5) it follows that P is an (additive) probability function on $\mathscr{E}$. Thus if $\mathscr{U}$ has a representation where $\oplus$ is $+$ then the chance events in $\mathscr{E}$ have the *qualitative* structure of a probability space. If $\odot$ is multiplication, $\cdot$, then $\varphi(a) \cdot P(A)$ looks like the usual expectation for (a, A).

Let $\mathscr{R}$ be a representation for $\mathscr{U}$. If $\oplus$ is $+$ then $\mathscr{R}$ is said to be *additive*; if $\odot$ is $\cdot$ then $\mathscr{R}$ is said to be *multiplicative*; if $\mathscr{R}$ is both additive and multiplicative, then $\mathscr{R}$ is said to be *distributive*. Pascal as well as von Neuman and Morgenstern [1953] and others assume that $\mathscr{U}$ has a distributive representation. In Section 2, axioms are given that guarantee that $\mathscr{U}$ has a unique (up to a choice of units) distributive representation. There are, however, reasonable axioms for $\mathscr{U}$ that yield no distributive representations. In analogy with relativistic physics, in these axiomatizations either chance must be given a nonadditive representation or the usual law for computing expectations must be abandoned.

In our analogy, the speed of light corresponds to Pascal's infinity of happiness. In relativity, the speed of light, l, is assigned a finite numerical value. It is natural to ask what qualitative condition forces l to be assigned a *finite* value. The answer is that in relativity distance (length) is given an additive representation and because of the law $s = d/t$, speed can be measured in terms of the distance travelled in unit time. This allows the following qualitative boundedness principle for relativistic speed to be formulated:

> There is a speed x such that for all speeds y, twice the distance that a particle with speed x travels in one second is greater than the distance a particle with speed y travels in one second.

Naturally, the numerical value that is assigned to a speed z is the numerical value of the distance that a particle with speed z covers in one second. Note that the speed of light plays no essential role in the boundedness of speed since if one were to restrict relativity to speeds less than l then the above principle would still force the set of numerical values assigned to speeds to be a bounded set. If we consider chance acting like distance and utility like speed, then the following would be an analogous definition of boundedness for $\mathscr{U}$:

> There is an x in $\mathscr{C}$ such that for all y in $\mathscr{C}$, $(x, X) \succ (y, A)$ where X is the sure event, and A is some event such that $(y, A) \sim (y, X - A)$.

In Section 2, axioms are given that for bounded $\mathscr{U}$ yield an unique (up to a choice of units) additive representation. However, such a representation is not multiplicative unless a certain qualitative condition called *distributivity* is satisfied.

Another way of dealing with infinite quantities is to measure them in structures that are generalizations of the real number system. This is done in Section 2 where value and chance are measured in a generalization of the reals that look like a lexicographic ordering.

The proofs of the theorems in Section 2 are given in Section 3.

2. Utility-Uncertainty Trade-off Structures

Definition 2.1. $\langle X, \mathscr{E}, \mathscr{C}, \gtrsim\rangle$ is said to be a *utility-uncertainty trade-off structure* if and only if $\varnothing \notin \mathscr{E}$ and $\mathscr{E} \cup \{\varnothing\}$ is an algebra of subsets of some nonempty set X, $\mathscr{C}$ is a nonempty set, and $\gtrsim$ is a binary relation on $\mathscr{C} \times \mathscr{E}$ such that the following four axioms hold:

Ax.1. *weak order*: $\gtrsim$ is a weak order;

Ax.2. *independence*: (i) for all A, B in $\mathscr{E}$, if for some $x \in \mathscr{C}$, $(x, A) \gtrsim (x, B)$, then for all $y \in \mathscr{C}$, $(y, A) \gtrsim (y, B)$; and (ii) for all u, v in $\mathscr{C}$, if for some $E \in \mathscr{E}$ $(u, E) \gtrsim (v, E)$ then for all $F \in \mathscr{E}$ $(u, F) \gtrsim (v, F)$;

Ax.3. *trade-off*: (i) for all $x, y \in \mathscr{C}$, if $(x, X) \gtrsim (y, X)$ then for some $E \in \mathscr{E}$, $(x, E) \sim (y, X)$; and (ii) for all A in $\mathscr{E}$ and for all u in $\mathscr{C}$ there is a z in $\mathscr{C}$ such that $(u, A) \sim (z, X)$;

Ax.4. *uncertainty*: for some $x \in \mathscr{C}$ the following three conditions hold for all A, B, C, D in $\mathscr{E}$:

(i) if $A \cap B = A \cap C = \varnothing$ then: $(x, B) \gtrsim (x, C)$ iff $(x, A \cup B) \gtrsim (x, A \cup C)$;

(ii) if $A \cap B = \varnothing$, $(x, A) \succ (x, C)$, and $(x, B) \gtrsim (x, D)$, then there are E, C', D' in $\mathscr{E}$ such that $C' \cap D' = \varnothing$, $E \supseteq C' \cup D'$, $(x, E) \sim (x, A \cup B)$, $(x, C) \sim (x, C')$, *and* $(x, D) \sim (x, D')$;

(iii) if $(x, A) \succ (x, B)$ then for some E, F in $\mathscr{E}$, $(x, E) \sim (x, A)$, $(x, F) \sim (x, B)$, and $E \supseteq F$.

In Definition 2.1, the null event, $\varnothing$, is excluded from consideration for convenience. As before, (x, A) should be interpreted as receiving object x if event A occurs and receiving nothing if A does not occur. Ax.2 says that the values of objects are not influenced by the occurence of events and that the occurence of events are not influenced by the values of objects. Ax.4 are some of Luce's axioms for qualitative probability (Luce, 1967). Although Ax.3 and Ax.4 can be greatly weakened, I have decided to use them since they are easy to state and allow elementary proofs of most of the theorems that follow.

Definition 2.2. Let $\langle X, \mathscr{E}, \mathscr{C}, \gtrsim\rangle$ be a utility-uncertainty trade-off structure. A sequence $A_1, \ldots, A_n, \ldots$ of members of $\mathscr{E}$ is called a *standard sequence relative* to A if and only if for $i = 1, 2, \ldots$ there are B_i, C_i in $\mathscr{E}$ such that the following five conditions hold for some x in $\mathscr{C}$:

(1) $A_1 = B_1$ and $(x, B_1) \sim (x, A)$;
(2) $B_i \cap C_i = \varnothing$;
(3) $B_i \sim A_i$;
(4) $C_i \sim A$;
(5) $A_{i+1} = B_i \cup C_i$.

The following axiom prohibits the existence of events that have an infinitesimal chance of occuring.

DEFINITION 2.3. Let $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \succsim \rangle$ be a utility-uncertainty trade-off structure. $\mathscr{U}$ is said to be *Archimedean* if and only if the following axiom is satisfied:

A.5. *Archimedean*: For each A in $\mathscr{E}$ there is no infinite standard sequence relative to A.

DEFINITION 2.4. Let $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \succsim \rangle$ be a utility-uncertainty trade-off structure. $\mathscr{U}$ is said to be ***bounded*** if and only if the following two conditions hold:

(1) for each A, B in $\mathscr{E}$, if for some x in $\mathscr{C}$, $(x, A) \succ (x, B)$, then there exists y in $\mathscr{C}$ such that for all z in $\mathscr{C}$, $(y, A) \succ (z, B)$;

(2) for each u in $\mathscr{C}$ there exists C in $\mathscr{E}$ such that for all v in $\mathscr{C}$, $(u, X) \succsim (v, C)$.

Definition 2.4 could probably be simplified greatly. Its present form, however, greatly simplifies certain proofs.

DEFINITION 2.5. Let $\langle X, \mathscr{E}, \mathscr{C}, \succsim \rangle$ be a utility-uncertainty trade-off structure. x is said to be ***a maximal element of*** $\mathscr{C}$ if and only if x is in $\mathscr{C}$ and for all y in $\mathscr{C}$, $(x, X) \succsim (y, X)$. If x_i is an infinite sequence of members of $\mathscr{C}$, we say that $x_i \to \infty$ if and only if for all positive integers i, $(x_{i+1}, X) \succ (x_i, X)$ and for each $z \in \mathscr{C}$ there is a positive integer n such that $(x_n, X) \succ (z, X)$.

It is easy to show that if $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \succsim \rangle$ is an Archimedean utility-uncertainty trade-off structure such that $\mathscr{C}$ does not have a maximal element then there is a sequence of members of $\mathscr{C}$, x_i, such that $x_i \to \infty$.

DEFINITION 2.6. Let $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \succsim \rangle$ be an Archimedean utility-uncertainty trade-off structure and let z be a maximal element of $\mathscr{C}$. An ***additive representation*** for $\mathscr{U}$ is an ordered 3-tuple $\langle \odot, \varphi, P \rangle$ such that $\odot$ is a binary operation on $(0, 1]$, $\varphi: \mathscr{C} \to (0, 1]$, $P: \mathscr{E} \to (0, 1]$, and the following four conditions hold for all x, y in $\mathscr{C}$ and all A, B in $\mathscr{E}$:

(1) $P(X) = 1$ and $\varphi(z) = 1$;

(2) $\varphi(x) \odot P(X) = \varphi(x)$ and $\varphi(z) \odot P(A) = P(A)$;

(3) $(x, A) \succsim (y, B)$ iff $\varphi(x) \odot P(A) \geqslant \varphi(y) \odot P(B)$;

(4) if $A \cap B = \varnothing$ then $(\varphi(x) \odot P(A)) + (\varphi(x) \odot P(B)) = \varphi(x) \odot P(A \cup B)$.

DEFINITION 2.7. Let $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \succsim \rangle$ be a bounded, Archimedean utility-uncertainty trade-off structure and x_i be a sequence of elements of $\mathscr{C}$ such that $x_i \to \infty$. An ***additive representation*** for $\mathscr{U}$ is an ordered 3-tuple $\langle \odot, \varphi, P \rangle$ such that $\odot$ is a binary operation on (0, 1], $\varphi: \mathscr{C} \to (0, 1]$, $P: \mathscr{E} \to (0, 1]$, and the following four conditions hold for all x, y in $\mathscr{C}$ and all A, B in $\mathscr{E}$:

(1) $P(X) = 1$ and $\lim_{i\to\infty} \varphi(x_i) = 1$;

(2) $\varphi(x) \odot P(X) = \varphi(x)$ and $\lim_{i\to\infty}[\varphi(x_i) \odot P(A)] = P(A)$;

(3) $(x, A) \succsim (y, B)$ iff $\varphi(x) \odot P(A) \geqslant \varphi(y) \odot P(B)$;

(4) if $A \cap B = \varnothing$ then $\lim_{i\to\infty}[\varphi(x_i) \odot P(A)] + \lim_{i\to\infty}[\varphi(x_i) \odot P(B)] = \lim_{i\to\infty}[\varphi(x_i) \odot P(A \cup B)]$.

DEFINITION 2.8. Let $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \succsim \rangle$ be an Archimedean utility-uncertainty trade-off structure. Then $\mathscr{U}$ is said to have an *unique* additive representation if and only if $\mathscr{U}$ has an additive representation and for all additive representations $\langle \odot, \varphi, P\rangle$, $\langle \odot', \varphi', P'\rangle$ for $\mathscr{U}$, $\varphi = \varphi'$, $P = P'$ and for all $x \in \mathscr{C}$ and $A \in \mathscr{E}$, $\varphi(x) \odot P(A) = \varphi'(x) \odot' P'(A)$.

THEOREM 2.1. *Let $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \succsim \rangle$ be an Archimedean utility-uncertainty trade-off structure. Then the following two propositions are true*:

(1) *if $\mathscr{U}$ has a maximal element then $\mathscr{U}$ has an unique additive representation*;

(2) *if $\mathscr{U}$ is bounded and there exist $A_1, \ldots, A_i, \ldots$ in $\mathscr{E}$ such that $A_i \supset A_{i+1}$, then $\mathscr{U}$ has an unique additive representation.*

DEFINITION 2.9. Let $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \succsim \rangle$ be an utility-uncertainty trade-off structure. $\mathscr{U}$ is said to be *distributive* if and only if the following axiom is satisfied:

Ax.6. *distributivity*: for all x, y in $\mathscr{C}$ and all A, B, C, D in $\mathscr{E}$, if $A \cap B = C \cap D = \varnothing$, $(x, A) \sim (y, C)$, and $(x, B) \sim (y, D)$, then $(x, A \cup B) \sim (y, C \cup D)$.

DEFINITION 2.10. Let $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \succsim \rangle$ be an Archimedean, distributive utility-uncertainty trade-off structure. A *distributive representation* for $\mathscr{U}$ is an ordered 3-tuple $\langle \varphi, P, u\rangle$ such that φ is a function from $\mathscr{C}$ into the positive reals, $P: \mathscr{E} \to (0, 1]$, u is an element of $\mathscr{C}$, and the following three conditions hold for all x, y in $\mathscr{C}$ and all A, B in $\mathscr{E}$:

(1) $P(X) = \varphi(u) = 1$;

(2) $(x, A) \succsim (y, B)$ iff $\varphi(x) \cdot P(A) \geqslant \varphi(y) \cdot P(B)$;

(3) if $A \cap B = \varnothing$ then $P(A \cup B) = P(A) + P(B)$.

THEOREM 2.2. *Let $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \succsim \rangle$ be an Archimedean, distributive utility-uncertainty trade-off structure. Then for each u in $\mathscr{C}$ there is a distributive representation $\langle \varphi, P, u\rangle$ for $\mathscr{U}$. Furthermore, if $\langle \odot, \psi, Q, v\rangle$ is such that $\odot$ is a binary operation on the* *positive reals, ψ is a function from $\mathscr{C}$ into the positive reals, $\underset{\sim}{Q}$: $\mathscr{E} \to (0, 1]$, v is an element of $\mathscr{C}$, and for all x, y in $\mathscr{C}$ and all A, B in $\mathscr{E}$:*

(1) $\underset{\sim}{Q}(X) = \psi(v) = 1$,

(2) $\psi(x) \odot \underset{\sim}{Q}(X) = \psi(x), \psi(v) \odot \underset{\sim}{Q}(A) = \underset{\sim}{Q}(A)$,

(3) $(x, A) \gtrsim (y, B)$ iff $\psi(x) \odot \underset{\sim}{Q}(A) \geqslant \psi(y) \odot \underset{\sim}{Q}(B)$,

(4) *if* $A \cap B = \varnothing$ *then* $\psi(x) \odot (\underset{\sim}{Q}(A \cup B)) = (\psi(x) \odot \underset{\sim}{Q}(A)) + (\psi(x) \odot \underset{\sim}{Q}(B))$,

then $\underset{\sim}{Q} = P$, $\psi = \varphi/\varphi(v)$, *and for all z in $\mathscr{C}$ and all C in $\mathscr{E}$*, $\psi(z) \odot \underset{\sim}{Q}(C) = \psi(z) \cdot \underset{\sim}{Q}(C)$.

Let $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \gtrsim \rangle$ be a distributive utility-uncertainty trade-off structure that is not necessarily Archimedean. Intuitively, a representation for $\mathscr{U}$ is based upon a generalization of the idea of a lexicographic ordering. Basically, the elements of $\mathscr{C}$ are divided into dimensions or *commeasurability classes*. Two elements in the same commeasurability class can be measured with respect to one another, i.e., if some element e of a commeasurability class $\mathscr{K}$ is chosen as a unit, then there is essentially only one way of assigning real numbers to members of $\mathscr{K}$ so that e is assigned the number 1. Commeasurability classes are ordered as follows: $\mathscr{K}_1$ is greater than $\mathscr{K}_2$ if and only if for some x in $\mathscr{K}_1$ and some y in $\mathscr{K}_2$, $(x, X) \succ (y, X)$. As in lexicographic orderings, elements of different commeasurability classes are ordered only in terms of the commeasurability classes to which they belong and not by their position in their commeasurability class. The elements of $\mathscr{E}$ are divided into two sets: (i) those that can be measured with respect to X (*the noninfinitesimal elements*) and (ii) those that are too small to be measured with respect to X (*the infinitesimal elements*).

DEFINITION 2.11. Let $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \gtrsim \rangle$ be a distributive utility-uncertainty trade-off structure.

1. For all x, y in $\mathscr{C}$, x is said *to be commeasurable with y* (in symbols, $x \equiv y$) if and only if (1) for some A in $\mathscr{E}$, $(x, X) \sim (y, A)$ and there is no infinite fundamental sequence with respect to A, or (2) for some B in $\mathscr{E}$, $(y, X) \sim (x, B)$ and there is no infinite fundamental sequence with respect to B. It is easy to show that $\equiv$ is an equivalence relation on $\mathscr{C}$. Call each equivalence class determined by $\equiv$ a *commeasurability class.*

2. Let A be an arbitrary element of $\mathscr{E}$. A is said to be *infinitesimal* if and only if there is an infinite fundamental sequence with respect to A. A is said to be *noninfinitesimal* if and only if A is not infinitesimal.

It is easy to show that (i) X is noninfinitesimal, (ii) if A, B are infinitesimal then $A \cup B$ is infinitesimal, and (iii) if C is noninfinitesimal and D is an arbitrary member of $\mathscr{E}$ then $C \cup D$ is noninfinitesimal.

DEFINITION 2.12. Let $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \gtrsim \rangle$ be a distributive utility-uncertainty trade-off structure and let $\mathscr{L}$ be the set of noninfinitesimal members of $\mathscr{E}$. A *representa-*

tion for $\mathscr{U}$ is an ordered 3-tuple $\langle \varphi, P, \mathscr{F} \rangle$ such that φ is a function from $\mathscr{C}$ into the positive reals, P is a function from $\mathscr{L}$ into (0, 1], $\mathscr{F} \subseteq \mathscr{C}$, and for each x, y in $\mathscr{C}$ and each A, B in $\mathscr{L}$ the following eight conditions hold:

(1) for some u in $\mathscr{F}$, $u \equiv x$;

(2) if x, y are in $\mathscr{F}$ and $x \neq y$ then $x \not\equiv y$;

(3) if x is in $\mathscr{F}$ then $\varphi(x) = 1$;

(4) $P(X) = 1$;

(5) if $A \cap B = \varnothing$ then $P(A \cup B) = P(A) + P(B)$;

(6) if $x \equiv y$ and $(x, A) \succsim (y, B)$ then $\varphi(x) \cdot P(A) \geqslant \varphi(y) \cdot P(B)$;

(7) if $x \equiv y$ and $\varphi(x) \cdot P(A) > \varphi(y) \cdot P(B)$ then $(x, A) \succ (y, B)$;

(8) if $x \equiv y$ and $\varphi(x) \cdot P(A) = \varphi(y) \cdot P(B)$ then either $(x, A) \sim (y, B)$ or there exists an infinitesimal C in $\mathscr{E}$ such that either $(x, A - C) \sim (y, B)$ or $(x, A) \sim (y, B - C)$.

THEOREM 2.3. *Let* $\mathscr{U}$ *be a distributive utility-uncertainty trade-off structure. Let* $\mathscr{F}$ *be a set of elements of* $\mathscr{C}$ *such that* $\mathscr{F}$ *has exactly one member from each commeasurability class of* $\mathscr{C}$. *Then there is a representation* $\langle \varphi, P, \mathscr{F} \rangle$ *for* $\mathscr{U}$. *Futhermore, if* $\langle \psi, Q, \mathscr{G} \rangle$ *is another representation for* $\mathscr{U}$, *then* $P = Q$ *and for each* x *in* $\mathscr{F}$ *and each* y *in* $\mathscr{G}$ *if* $x \equiv y$ *then* $\varphi(x) = \varphi(y)\, \psi(x)$,

Notation. Let A be a nonempty set and x be an individual. Then, by definition, $f_A{}^x$ is the function from A into $\{x\}$. By convention, g_A, h_A, etc., will denote functions with domain A.

In what follows $f_A{}^x$ will denote the simple gamble of receiving x if the event A occurs and receiving nothing if A does not occur. That is, the notation $f_A{}^x$ will replace the previous notation (x, A). Consequently, previous definitions (e.g., fundamental sequence) and axioms (e.g., *Archimedean*, *independence*) equally well apply in this new notation.

DEFINITION 2.13. Let $\mathscr{C}$ and X be nonempty sets, $\mathscr{E}$ be a set such that $\varnothing \notin \mathscr{E}$ and $\mathscr{E} \cup \{\varnothing\}$ is an algebra of subsets of X, $\mathscr{G}$ be a set of functions from members of $\mathscr{E}$ into $\mathscr{C}$ such that each g in $\mathscr{G}$ takes on only finitely many values in $\mathscr{C}$, $x_1, \dots, x_n$, and for $i = 1, \dots, n$, $g^{-1}(x_i)$ is in $\mathscr{E}$, and let $\succsim'$ be a binary relation on $\mathscr{G}$. Then $\langle X, \mathscr{E}, \mathscr{C}, \mathscr{G}, \succsim' \rangle$ is said to be a *gambling structure* if and only if the following six axioms hold:

Ax.1′. *weak order′*: $\succsim'$ is a weak order;

Ax.2. *independence*: (see Definition 2.1);

Ax.3′. *trade-off′*: (i) for all x, y in $\mathscr{C}$, if $f_X{}^x \succsim' f_X{}^y$ then for some E in $\mathscr{E}$, $f_E{}^x \sim' f_X{}^y$; and (ii) for all g in $\mathscr{G}$ there exists x in $\mathscr{C}$ such that $g \sim' f_X{}^x$;

Ax.4′. *uncertainty′*: (i) there exists x in $\mathscr{C}$ such that for all A, B, C in $\mathscr{E}$, if $A \cap B = A \cap C = \varnothing$ then: $f_B{}^x \gtrsim' f_C{}^x$ iff $f^x_{A\cup B} \gtrsim' f^x_{A\cup C}$; and (ii) for each y in $\mathscr{C}$ and each D, E in $\mathscr{E}$, if $f_D{}^y \gtrsim' f_E{}^y$ then there exists F in $\mathscr{E}$ such that $D \supseteq F$ and $f_F{}^y \sim' f_E{}^y$;

Ax.5. *Archimedean*: (see Definition 2.3);

Ax.6′. *linearity* (*distributivity′*): for each A, B, C, D in $\mathscr{E}$, each x in $\mathscr{C}$, and each g_C, h_D in $\mathscr{G}$, if $A \cap B = C \cap D = \varnothing$, $f_A{}^x \sim' g_C$, $f_B{}^x \sim' h_D$ then $f^x_{A\cup B} \sim' g_C \cup h_D$.

It is easy to show that *trade-off′* implies *trade-off*, *uncertainty′* implies *uncertainty*, and *linearity* implies *distributivity*. Thus if we define $\gtrsim$ on $\mathscr{C} \times \mathscr{E}$ by

$$(x, A) \gtrsim (y, B) \qquad \text{iff} \quad f_A{}^x \gtrsim' f_B{}^y$$

then $\langle X, \mathscr{E}, \mathscr{C}, \gtrsim\rangle$ is an Archimedean, distributive utility-uncertainty trade-off structure.

DEFINITION 2.14. Let $\mathscr{E}$ be a set such that $\varnothing \notin \mathscr{E}$ and $\mathscr{E} \cup \{\varnothing\}$ is an algebra of subsets of some nonempty set X. Let P be an *additive* probability function on $\mathscr{E}$, i.e., P be a function from $\mathscr{E}$ into $(0, 1]$ such that $P(X) = 1$ and for all A, B in $\mathscr{E}$, if $A \cap B = \varnothing$ then $P(A \cup B) = P(A) + P(B)$. Let C be in $\mathscr{E}$ and let h be a function from C into the positive reals that has as values $r_1, \ldots, r_n$. For $i = 1, \ldots, n$, let $A_i = \{x \mid h(x) = r_i\}$. Assume A_i is in $\mathscr{E}$. Then *the expectation of h with respect to P*, $\mathbf{E}_P(h)$, is defined as follows:

$$\mathbf{E}_P(h) = \sum_{i=1}^{n} P(A_i) \cdot r_i .$$

THEOREM 2.4. *Let $\langle X, \mathscr{E}, \mathscr{C}, \mathscr{G}, \gtrsim'\rangle$ be a gambling structure and u, v be arbitrary elements of $\mathscr{C}$. Then there is an additive probability function P on $\mathscr{E}$ and a function φ from $\mathscr{C}$ into the positive reals such that $\varphi(u) = 1$ and for all f, g in $\mathscr{G}$,*

$$f \gtrsim' g \qquad \text{iff} \quad \mathbf{E}_P(\varphi(f)) \geqslant \mathbf{E}_P(\varphi(g)).$$

Furthermore, if Q is another additive probability function on $\mathscr{E}$ and ψ is another function from $\mathscr{C}$ into the positive reals such that $\psi(v) = 1$ and for all f, g in $\mathscr{G}$,

$$f \gtrsim' g \qquad \text{iff} \quad \mathbf{E}_Q(\psi(f)) \geqslant \mathbf{E}_Q(\psi(g)),$$

then $P = Q$ and $\varphi = \varphi(v)\psi$.

The above axiom systems can be modified in a very natural way to include the case of utilities having nonpositive values.

3. Proofs

Convention. Throughout this section let $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \succsim \rangle$ be a fixed utility-uncertainty trade-off structure.

Definition 3.1. Let $\succsim_1$ be the binary relation on $\mathscr{C}$ defined by: for all x, y in $\mathscr{C}$,

$$x \succsim_1 y \quad \text{iff} \quad (x, X) \succsim (y, X).$$

Let $\succsim_2$ be the binary relation on $\mathscr{E}$ defined by: for all A, B in $\mathscr{E}$, $A \succsim_2 B$ iff for some z in C, $(z, A) \succsim (z, B)$.

Lemma 3.1. *$\succsim_1$ is a weak order on $\mathscr{C}$ and $\succsim_2$ is a weak order on $\mathscr{E}$.*

Proof. Follows immediately from *weak order* and *independence.* ▌

Lemma 3.2. *The following six propositions are true for all A, B, C, D in $\mathscr{E}$:*

(i) *Suppose $A \cap B = C \cap D = \varnothing$. Then: if $A \succsim_2 C$ and $B \succsim_2 D$ then $A \cup B \succsim_2 C \cup D$, and if $A \succ_2 C$ and $B \succsim_2 D$ then $A \cup B \succ_2 C \cup D$.*

(ii) *If $A \cap B = C \cap D = \varnothing$, $A \sim_2 C$, and $B \sim_2 D$, then $A \cup B \sim_2 C \cup D$.*

(iii) *If $A \supseteq B$ then $A \succsim_2 B$.*

(iv) *$X \succsim_2 A$.*

(v) *If $A \supseteq B$ then: $A \succ_2 B$ iff $A - B$ is in $\mathscr{E}$.*

(vi) *If $A \neq X$ and $B \neq X$ and $A \succsim_2 B$ then $X - B \succsim_2 X - A$.*

Proof. Left to reader. ▌

Definition 3.2. For each x in $\mathscr{C}$:

(1) let $\mathscr{C}_x = \{y \in \mathscr{C} \mid x \succsim_1 y\}$;

(2) let $\succsim_x$ be the restriction of $\succsim_1$ to $\mathscr{C}_x$;

(3) let $\oplus_x$ be the partial relation on $\mathscr{C}_x$ defined by: for all u, y, z in $\mathscr{C}_x$, $u \oplus_x y \sim_x z$ iff for some A, B in $\mathscr{E}$, $A \cap B = \varnothing$, $(x, A) \sim (u, X)$, $(x, B) \sim (y, X)$, and $(x, A \cup B) \sim (z, X)$.

Lemma 3.3. *For each x in $\mathscr{C}$, $\oplus_x$ is a binary partial operation on $\mathscr{C}_x$, i.e., for all u, v, y, z in $\mathscr{C}_x$, if $u \oplus_x v \sim_x z$ and $u \oplus_x v \sim_x y$ then $z \sim_x y$.*

Proof. Suppose that $u \oplus_x v \sim_x z$ and $u \oplus_x v \sim_x y$. Let A, B, C, D be elements of $\mathscr{E}$ such that $A \cap B = C \cap D = \varnothing$, $(x, A) \sim (u, X)$, $(x, B) \sim (v, X)$, $(x, A \cup B) \sim (z, X)$, $(x, C) \sim (u, X)$, $(x, D) \sim (v, X)$, and $(x, C \cup D) \sim (y, X)$. By *independence* and Lemma 3.1 it follows that $A \sim_2 C$ and $B \sim_2 D$. Thus by Lemma 3.2(ii), $A \cup B \sim_1 C \cup D$. By *independence* and Definition 3.1, $(x, A \cup B) \sim (x, C \cup D)$. Thus $(z, X) \sim (y, X)$. *By independence*, $x \sim_x z$. ▮

(Propertly speaking, one might call $\oplus_x$ a "multivalued operation." For convenience, we will consider $u \oplus_x v$ to be an element of $\mathscr{C}_x$ although it is really an equivalence class of members of $\mathscr{C}_x$ determined by the equivalence relation $\sim_x$.)

LEMMA 3.4. *For each x in $\mathscr{C}$, $\succsim_x$ is a weak order on $\mathscr{C}_x$.*

Proof. Left to reader. ▮

LEMMA 3.5. *For each x in $\mathscr{C}$ and each u, v, y in $\mathscr{C}_x$, if $(u \oplus_x v) \oplus_x y$ is defined then $u \oplus_x (v \oplus_x y)$ is defined and $(u \oplus_x v) \oplus_x y \sim_x u \oplus_x (v \oplus_x y)$.*

Proof. Suppose that $(u \oplus_x v) \oplus_x y$ is defined. Let A, B, C be elements of $\mathscr{E}$ such that $A \cap B = \varnothing$, $(x, C) \sim (y, X)$, $(A \cup B) \cap C = \varnothing$, $(x, A) \sim (u, X)$, $(x, B) \sim (v, X)$, $(x, A \cup B) \sim (u \oplus_x v, X)$, and $(x, A \cup B \cup C) \sim ((u \oplus_x v) \oplus_x y, X)$. Since $B \cap C = \varnothing$, $(x, B \cup C) \sim (v \oplus_x y, X)$. Since $A \cap (B \cup C) = \varnothing$,

$$(x, A \cup B \cup C) \sim (u \oplus_x (v \oplus_x y), X) \sim (u \oplus_x (v \oplus_x y), X). \quad ▮$$

LEMMA 3.6. *For each x in $\mathscr{C}$ and each u, v, y in $\mathscr{C}_x$, if $u \oplus_x v$ is defined and $u \succsim_x y$ then $v \oplus_x y$ is defined and $u \oplus_x v \succsim_x v \succsim_x y$.*

Proof. Suppose that $u \oplus_x v$ is defined and $u \succsim_x y$. Let A, B, C be elements of $\mathscr{E}$ such that $A \cap B = \varnothing$, $(x, A) \sim (u, X)$, $(x, B) \sim (v, X)$, and $(x, C) \sim (y, X)$. Since $u \succsim_x y$, $(x, A) \sim (u, X) \succsim (y, X) \sim (x, C)$. Thus by *independence*, $A \succsim_2 C$. Since $A \cap B = \varnothing$, $A \succsim_2 C$ and $B \succsim_2 B$, by *independence* and *uncertainty* ir follows that there are D, E, F in $\mathscr{E}$ such that $D \sim_2 A \cup B$, $E \cap F = \varnothing$, $D \supseteq E \cup F$, $C \sim_2 E$, and $B \sim_2 F$. By *independence*, $(u, X) \sim (x, A) \succsim (x, C) \sim (x, E) \sim (y, X)$ and $(v, X) \sim (x, B) \sim (x, F)$. Since $A \cap B = E \cap F = \varnothing$, $(x, A \cup B) \sim (u \oplus_x v, X)$ and $(x, E \cup F) \sim (y \oplus_x v, X)$. Since $A \cup B \sim_2 D \supseteq E \cup F$, from Lemma 3.2(iii) and *independence* it follows that $(x, A \cup B) \succsim (x, E \cup F)$; that is, $(u \oplus_x v, X) \succsim (y \oplus_x v, X)$. Thus, $u \oplus_x v \succsim_x y \oplus_x v$. ▮

LEMMA 3.7. *For all x in $\mathscr{C}$ and all u, v in $\mathscr{C}_x$, if $u \succ_x v$ then for some y in $\mathscr{C}_x$, $u \succsim_x v \oplus_x y$.*

Proof. Suppose that $u \succ_x v$. By trade-off let A, B be such that $(x, A) \sim (u, X)$ and $(x, B) \sim (v, X)$. Since $(u, X) \succ (v, X)$, $(x, A) \succ (x, B)$. By *independence* it follows that $A \succ_2 B$. By *uncertainty*, let E, F be such that $A \sim_2 E$, $B \sim_2 F$ and $E \supseteq F$. By Lemma 3.2(v), $E - F$ is in $\mathscr{E}$. By *trade-off*, let y be such that $(x, E - F) \sim (y, X)$. Then y is in $\mathscr{C}_x$. Thus $(u, X) \sim (x, A) \sim (x, E) \sim (x, F \cup (E - F))$. Since $v, X) \sim (x, F)$ and $(y, X) \sim (x, E - F)$, $(v \oplus_x y, X) \sim (x, E)$. Thus $u \sim_x v \oplus_x y$. ▮

LEMMA 3.8. *For each x in $\mathscr{C}$ and each u, v in $\mathscr{C}_x$, if $u \oplus_x v$ is defined then $u \oplus_x v \succ_x u$.*

Proof. Suppose that $u \oplus_x v$ is defined. Let A, B be such that $A \cap B = \varnothing$, $(x, A) \sim (u, X)$, $(x, B) \sim (v, X)$, and $(x, A \cup B) \sim (u \oplus_x v, X)$. Since $B = (A \cup B) - A$ is in $\mathscr{E}$, it follows from Lemma 3.2(v) that $A \cup B \succ_2 A$. By independence, $(x, A \cup B) \succ (x, A)$, i.e., $u \oplus_x v \succ u$. ▌

LEMMA 3.9. *If $\mathscr{U}$ is Archimedean then for each x in $\mathscr{C}$ there is no infinite sequence of members of $\mathscr{C}_x$, $y_1, y_2, \ldots$, such that for each positive integer i, $y_{i+1} \sim_x y_i \oplus_x y_1$.*

Proof. Suppose that $y_1, y_2, \ldots$ is an infinite sequence of members of $\mathscr{C}_x$ and for each positive integer i, $y_{i+1} \sim_x y_i \oplus_x y_1$. A contradiction will be shown. Let $B_1, B_2, \ldots$ be a sequence of members of $\mathscr{E}$ such that for all positive integers $i \neq j$, $B_i \cap B_j = \varnothing$ and $(x, B_i) \sim (y, X)$. Let $A_1 = B_1$ and for each positive integer i, $A_{i+1} = A_i \cup B_{i+1}$. This contradicts *Archimedean* (Ax. 5). ▌

Lemmas 3.3 to 3.9 establish that for each x in $\mathscr{C}$, $\langle \mathscr{C}_x, \succsim_x, \oplus_x \rangle$ is an (Archimedean) extensive structure as defined in Krantz, *et al* [1971], page 84. Thus by Theorem 3 on page 85 of Krantz, *et al* [1971], the following Lemma is true:

LEMMA 3.10. *For each x in $\mathscr{C}$ there is an unique function φ_x from $\mathscr{C}_x$ into* (0, 1] *such that for all u, v in $\mathscr{C}_x$ the following three conditions hold:*

(1) $\varphi_x(x) = 1$;

(2) $u \succsim_x v$ *iff* $\varphi_x(u) \geqslant \varphi_x(v)$;

(3) $\varphi_x(u \oplus_x v) = \varphi_x(u) + \varphi_x(v)$.

DEFINITION 3.3. φ_x in Lemma 3.10 is said to be *the unique extensive representation for* $\langle \mathscr{C}_x, \succsim_x, \oplus_x \rangle$.

DEFINITION 3.4. P is said to be a *probability representation for* $\langle X, \mathscr{E}, \succsim_2 \rangle$ if and only if P is a function from $\mathscr{E}$ into (0, 1] such that the following three conditions are satisfied for all A, B in $\mathscr{E}$:

(1) $P(X) = 1$;

(2) $A \succsim_2 B$ iff $P(A) \geqslant P(B)$;

(3) if $A \cap B = \varnothing$ then $P(A \cup B) = P(A) + P(B)$.

Convention. For each x in $\mathscr{C}$ and each A in $\mathscr{E}$ let $\pi(x, A)$ be an element y in $\mathscr{C}$ such that $(x, A) \sim (y, X)$. (By *trade-off* π exists.)

LEMMA 3.11. *If $\mathscr{U}$ is Archimedean then there is an unique probability representation for* $\langle X, \mathscr{E}, \succsim_2 \rangle$.

Proof. Let x be a fixed element of $\mathscr{C}$. By Lemma 3.10, let φ be the unique extensive representation for $\langle \mathscr{C}_x, \gtrsim_x \gtrsim_x \rangle$. Define P on $\mathscr{E}$ as follows: for each A in $\mathscr{E}$, $P(A) = \varphi(\pi(x, A))$. Then it is easy to verify that P is a probability representation for $\langle X, \mathscr{E}, \gtrsim_2 \rangle$.

Suppose that P, Q are two probability representations for $\langle X, \mathscr{E}, \gtrsim_2 \rangle$. Define ψ, ξ on $\mathscr{C}_x$ as follows: for each y in $\mathscr{C}_x$, $\psi(y) = P(A)$, $\xi(y) = Q(A)$ where A is such that $(y, X) \sim (x, A)$. It follows that $\psi(x) = \xi(x) = 1$. Suppose that $u \gtrsim_x v$. Let C, D be such that $(x, C) \sim (u, X)$ and $(x, D) \sim (v, X)$. Then $(x, C) \gtrsim (x, D)$ and by *independence* $C \gtrsim_2 D$. Thus

$$\psi(u) = P(C) \geqslant P(D) = \psi(v)$$

and

$$\xi(u) = Q(C) \geqslant Q(D) = \xi(v).$$

Suppose that $w \oplus_x z$ is defined, $(w, X) \sim (x, E)$, $(z, X) \sim (x, F)$, and $E \cap F = \varnothing$. Then $(w \oplus_x z, X) \sim (x, E \cup F)$. Thus

$$\psi(w \oplus_x z) = P(E \cup F) = P(E) + P(F) = \psi(w) + \psi(z),$$

and similarly

$$\xi(w \oplus_x z) = \xi(w) + \xi(z).$$

Thus by Lemma 3.10, $\psi = \xi$. ∎

LEMMA 3.12. *If $\mathscr{U}$ is Archimedean and x is a maximal element of $\mathscr{C}$, then $\mathscr{U}$ has an unique additive representation.*

Proof. Existence. By Lemma 3.10 let φ be the unique extensive representation for $\langle \mathscr{C}_x, \gtrsim_x, \oplus_x \rangle$ and P be the unique probability representation for $\langle X, \mathscr{E}, \gtrsim_2 \rangle$. Define the binary operation $\odot$ on $(0, 1] \times (0, 1]$ as follows: if for some y in $\mathscr{C}$ and some A in $\mathscr{E}$ $r = \varphi(y)$ and $s = P(A)$ then $r \odot s = \varphi(\pi(y, A))$, otherwise $r \odot s$ is some arbitrary member of $(0, 1]$. To show that $\langle \odot, \varphi, P \rangle$ is an additive representation for $\mathscr{U}$ it is only necessary to verify conditions 1 to 4 of Definition 2.6. Let u, v be arbitrary members of $\mathscr{C}$ and B, C be arbitrary members of $\mathscr{E}$. Then:

1. $P(X) = 1$ since P is a probability representation for $\langle X, \mathscr{E}, \gtrsim_2 \rangle$. If z is a maximal element of $\mathscr{C}$, then $z \sim_x x$ and thus $\varphi(z) = \varphi(x) = 1$.

2. Since $\pi(u, X) \sim_x u$,

$$\varphi(\pi(u, X)) = \varphi(u) \odot P(X) = \varphi(u).$$

If z is a maximal element of $\mathscr{C}$ then $z \sim_2 x$ and thus

$$\varphi(\pi(z, B)) = \varphi(\pi(x, B)) = P(B).$$

3. Since

$$(u, B) \gtrsim (v, C) \quad \text{iff} \quad \pi(u, B) \gtrsim \pi(v, C),$$
$$(u, B) \gtrsim (v, C) \quad \text{iff} \quad \varphi(\pi(u, B)) \geqslant \varphi(\pi(v, C)) \text{ iff } \varphi(u) \odot P(B) \geqslant \varphi(v) \odot P(C).$$

4. If $B \cap C = \varnothing$ then

$$\pi(u, B) \oplus_x \pi(u, C) \sim_x \pi(u, B \cup C).$$

Since

$$\varphi(\pi(u, B) \oplus_x \pi(u, C)) = \varphi(\pi(u, B)) + \varphi(\pi(u, C)),$$

it follows that

$$(\varphi(u) \odot P(B)) + (\varphi(u) \odot P(C)) = \varphi(u) \odot (P(B) + P(C)).$$

Uniqueness. Suppose that $\langle \odot, \varphi, P \rangle$ and $\langle \odot', \psi, Q \rangle$ are two additive representations for $\mathscr{U}$. Since x is a maximal element of $\mathscr{C}$, $\varphi(x) \odot P(A) = P(A)$ for each A in $\mathscr{E}$. Since for each A, B in $\mathscr{E}$ if $A \cap B = \varnothing$ then

$$(\varphi(x) \odot P(A)) + (\varphi(x) \odot P(B)) = \varphi(x) \odot P(A \cup B),$$

it follows that $P(A) + P(B) = P(A \cup B)$. Suppose that C, D are arbitrary members of $\mathscr{E}$ and $C \gtrsim_2 D$. By *independence*, $(x, C) \gtrsim (x, D)$ and thus

$$P(C) = \varphi(x) \odot P(C) \geqslant \varphi(x) \odot P(D) = P(D).$$

Therefore, P is a probability representation for $\langle X, \mathscr{E}, \gtrsim_2 \rangle$. Similarly, Q is also a probability representation for $\langle X, \mathscr{E}, \gtrsim_2 \rangle$. Since by Lemma 3.11 $\langle X, \mathscr{E}, \gtrsim_2 \rangle$ has only one probability representation, $P = Q$.

Let u be an arbitrary element of $\mathscr{C}$. By *trade-off*, let E be such that $(x, E) \sim (u, X)$. Then $\varphi(u) = \varphi(u) \odot P(X) = \varphi(x) \odot P(E) = P(E)$. Similarly, $\psi(u) = Q(E)$. Since $P = Q$, $\varphi = \psi$.

Let v be an arbitrary member of $\mathscr{C}$ and F be an arbitrary member of $\mathscr{E}$. By *trade-off* let G be such that $(v, F) \sim (x, G)$. Then

$$\varphi(v) \odot P(F) = \varphi(x) \odot P(G) = P(G).$$

Similarly

$$\psi(v) \odot' Q(F) = Q(G).$$

Since $P(G) = Q(G)$, $\varphi(v) \odot P(F) = \psi(u) \odot' Q(F)$. ∎

LEMMA 3.13. *Suppose that P is a probability representation for $\langle X, \mathscr{E}, \gtrsim_2 \rangle$ and $A_1, \ldots, A_i, \ldots$ is a sequence of members of $\mathscr{E}$ such that for each i, $A_i \supset A_{i+1}$. Then for each A in $\mathscr{E}$, $P(A) = \sup_{A \succ_2 B} P(B)$.*

Proof. We will first show that $\lim_{i\to\infty} P(A_i - A_{i+1}) = 0$. Assume that for some positive real r there are infinitely many i such that $P(A_i - A_{i+1}) \geqslant r$. Since

$$P(A_1) > P(A_1 - A_n) = P((A_1 - A_2) \cup (A_2 - A_3) \cup \cdots \cup (A_{n-1} - A_n))$$
$$= P(A_1 - A_2) + P(A_2 - A_3) + \cdots + P(A_{n-1} - A_n),$$

it is easy to show that $P(A_1) \geqslant mr$ for each positive integer m. This contradicts that $P(A_1) \leqslant 1$. Let A be an arbitrary element of $\mathscr{E}$. Since $\lim_{i\to\infty} P(A_i - A_{i+1}) = 0$, we may suppose without loss of generality that for each i, $A \succ_2 A_i - A_{i+1}$. By *uncertainty*, for each i let E_i, F_i be such that $E_i \sim_2 A$, $F_i \sim_2 A_i - A_{i+1}$, and $E_i \supset F_i$. Then

$$P(E_i) = P((E_i - F_i) \cup F_i)$$
$$= P(E_i - F_i) + P(F_i).$$

Thus

$$P(A) = \lim_{i\to\infty} P(E_i)$$
$$= \lim_{i\to\infty} P(E_i - F_i) + \lim_{i\to\infty} P(F_i)$$
$$= \lim_{i\to\infty} P(E_i - F_i).$$

Since for each i

$$A \succ_2 E_i - F_i, \qquad A = \sup_{A \succ_2 B} P(B). \quad \blacksquare$$

LEMMA 3.14. *Suppose that $\mathscr{U}$ is bounded and Archimedean, x_i is a sequence of members of $\mathscr{C}$ such that $x_i \to \infty$, and A_i is a sequence of members of $\mathscr{E}$ such that $A_i \supset A_{i+1}$. Then there is an unique additive representation for $\mathscr{U}$.*

Proof. Existence. By Lemma 3.11, let P be the unique probability representation for $\langle X, \mathscr{E}, \succsim_2\rangle$. By Lemma 3.12, for each positive integer i, let $\langle \bigcirc_i, \varphi_i, P_i\rangle$ be the unique representation for $\langle X, \mathscr{E}, \mathscr{C}_{x_i}, \succsim^i\rangle$ where $\succsim^i$ is the restriction of $\succsim$ to $\mathscr{C}_{x_i} \times \mathscr{E}$. Since for each i $\succsim_2^i = \succsim_2$, it follows that $P_i = P$. Let x be an arbitrary member of $\mathscr{C}$. Since $x_i \to \infty$, $x_i \succ_1 x$ for all but finitely many positive integers i. Without loss of generality, suppose that for all i, $x_i \succ_1 x$. We will first show that $\lim_{i\to\infty} \varphi_i(x)$ exists and is positive. By boundedness, let A in $\mathscr{E}$ be such that for all y in $\mathscr{C}$, $(x, X) \succsim (y, A)$. By *trade-off*, let A_i be such that $(x_i, A_i) \sim (x, X)$. Since $x_i \to \infty$ and $x_i \succ_1 x$, by *trade-off and independence*, $A_i \succ_2 A_{i+1} \succ_2 A$. Therefore the sequence $P(A_i)$ is monotonically decreasing and bounded from below by $P(A)$. Thus $\lim_{i\to\infty} P(A_i)$ exists and is positive. But since $\varphi_i(x) = \varphi_i(\pi(x_i, A_i)) = P_i(A_i) = P(A_i)$, it follows that $\lim_{i\to\infty} \varphi_i(x) = \lim_{i\to\infty} P(A_i) > 0$. Thus for each x in $\mathscr{C}$ let $\varphi(x) = \lim_{i\to\infty} \varphi_i(x)$.

1. Since P is a probability representation for $\langle X, \mathscr{E}, \gtrsim_2 \rangle$, $P(X) = 1$. Suppose that $y_i \to \infty$. Then for each positive integer i, there are j_i, k_i such that $x_{j_i} \succ_1 y_{k_i} \succ_1 x_i$. Thus $\lim_{i\to\infty} \varphi(x_i) = \lim_{i\to\infty} \varphi(y_i)$. Since $\varphi_i(x_j) \leqslant 1$ for all $i \geqslant j$,

$$\varphi(x_j) = \lim_{i\to\infty} \varphi_i(x_j) \leqslant 1.$$

Since the sequence $\varphi(x_j)$ is increasing and bounded by 1, $\lim_{j\to\infty} \varphi(x_j)$ exists and is $\leqslant 1$. We will show that $\lim_{i\to\infty} \varphi(x_i) = 1$. Let B be an arbitrary member of $\mathscr{E}$ such that $X \succ_2 B$. By boundedness, let n be a positive integer such that for all y in $\mathscr{C}$ $(x_n, X) \gtrsim (y, B)$. Thus for each positive integer $j \geqslant n$, $\varphi_j(x_n) \geqslant \varphi_j(\pi(x_j, B)) = P(B)$. Thus,

$$\varphi(x_n) = \lim_{j\to\infty} \varphi_j(x_n) \geqslant P(B).$$

Therefore,

$$\lim_{i\to\infty} \varphi(x_i) \geqslant \varphi(x_n) \geqslant P(B).$$

Since B is an arbitrary member of $\mathscr{E}$ such that $X \succ_2 B$, by Lemma 3.13,

$$\lim_{i\to\infty} \varphi(x_i) \geqslant \sup_{X \succ_2 B} P(B) = P(X) = 1.$$

Therefore,

$$\lim_{i\to\infty} \varphi(x_i) = 1.$$

2. Let $\odot$ be a binary operation on $(0, 1]$ defined by: for each x in $\mathscr{C}$ and each A in $\mathscr{E}$, if $r = \varphi(x)$ and $s = P(A)$ then $r \odot s = \varphi(\pi(x, A))$, otherwise let $r \odot s$ be an arbitrary member of $(0, 1]$. Then for each x in $\mathscr{C}$, $\varphi(x) \odot P(X) = \varphi(\pi(x, X)) = \varphi(x)$. Suppose that y_i is a sequence of members of $\mathscr{C}$ such that $y_i \to \infty$. Let C be an arbitrary element of $\mathscr{E}$. Then $\lim_{i\to\infty}[\varphi(y_i) \odot P(C)] = \lim_{i\to\infty}[\varphi(x_i) \odot P(C)] = \lim_{i\to\infty} \varphi(\pi(x_i, C))$. Therefore we need only to show that $\lim_{i\to\infty} \varphi(\pi(x_i, C)) = P(C)$. Let D be an arbitrary element of $\mathscr{E}$ such that $C \succ_2 D$. By boundedness, let z be such that for all y in $\mathscr{C}$, $(z, C) \gtrsim (y, D)$. Let m be such that $x_m \gtrsim_1 z$. Then for all $j \geqslant m$, $(x_m, C) \gtrsim (x_j, D)$. Therefore, for $j \geqslant m$,

$$P(C) = \varphi_j(\pi(x_j, C)) \geqslant \varphi_j(\pi(x_m, X)) \geqslant \varphi_j(\pi(x_j, D)) = P(D).$$

Thus

$$P(C) \geqslant \lim_{j\to\infty} \varphi_j(\pi(x_m, C)) = \varphi(\pi(x_m, C)) \geqslant P(D).$$

Letting $m \to \infty$ we get

$$P(C) \geqslant \lim_{i\to\infty} \varphi(\pi(x_i, C)) \geqslant P(D).$$

Since D is an arbitrary element of $\mathscr{E}$ such that $C \succ_2 D$, by Lemma 3.13,

$$P(C) \geqslant \lim_{i\to\infty} \varphi(\pi(x_i, C)) \geqslant \sup_{C \succ_2 D} P(D) = P(C).$$

3. Let $\odot$ be as defined in part 2 of this proof. For each x, y in $\mathscr{C}$ and each A, B in $\mathscr{E}$, $(x, A) \succsim (y, B)$ iff $\pi(x, A) \succsim_1 \pi(y, B)$ iff $\varphi(\pi(x, A)) \geqslant \varphi(\pi(y, B))$ iff $\varphi(x) \odot P(A) \geqslant \varphi(y) \odot P(B)$.

4. Let $\odot$ be as defined in part 2 of this proof. Let E, F be arbitrary members of $\mathscr{E}$ such that $E \cap F = \varnothing$ and let y_i be an arbitrary sequence of members of $\mathscr{E}$ such that $y_i \to \infty$. Then by part 2 of this proof, $\lim_{i\to\infty}[\varphi(y_i) \odot P(E)] = P(E)$,

$$\lim_{i\to\infty}[\varphi(y_i) \odot P(F)] = P(F), \qquad \text{and} \qquad \lim_{i\to\infty}[\varphi(y_i) \odot P(E \cup F)] = P(E \cup F).$$

Since P is a probability representation for $\langle X, \mathscr{E}, \succsim_2\rangle$, $P(E) + P(F) = P(E \cup F)$. Thus

$$\lim_{i\to\infty}[\varphi(y_i) \odot P(E)] + \lim_{i\to\infty}[\varphi(y_i) \odot P(F)] = \lim_{i\to\infty}[\varphi(y_i) \odot P(E \cup F)].$$

Uniqueness. Let $\langle \odot, \varphi, P\rangle$ and $\langle \odot', \psi, Q\rangle$ be additive representations for $\mathscr{U}$. Then for each A in $\mathscr{E}$, $\lim_{i\to\infty}[\varphi(x_i) \odot P(A)] = P(A)$. Also, for each A, B in $\mathscr{E}$, if $A \cap B = \varnothing$ then

$$\lim_{i\to\infty}[\varphi(x_i) \odot P(A)] + \lim_{i\to\infty}[\varphi(x_i) \odot P(B)] = \lim_{i\to\infty}[\varphi(x_i) \odot P(A \cup B)],$$

i.e., if $A \cap B = \varnothing$ then

$$P(A) + P(B) = P(A \cup B).$$

Also for each A, B in $\mathscr{E}$, if $A \succsim_2 B$ then for each positive integer i, $(x_i, A) \succsim (x_i, B)$ and thus $\varphi(x_i) \odot P(A) \geqslant \varphi(x_i) \odot P(B)$ from which follows

$$P(A) = \lim_{i\to\infty}[\varphi(x_i) \odot P(A)] \geqslant \lim_{i\to\infty}[\varphi(x_i) \odot P(B)] = P(B).$$

Thus P is a probability representation for $\langle X, \mathscr{E}, \succsim_2\rangle$. Similarly, Q is a probability representation for $\langle X, \mathscr{E}, \succsim_2\rangle$. By Lemma 3.11, $P = Q$.

Let z be an arbitrary member of $\mathscr{C}$. Without loss of generality, assume that $x_i \succ_1 z$ for each i. For each positive integer i, let A_i be such that $(z, X) \sim (x_i, A_i)$. Then for each $j \geqslant i$, $(x_j, A_i) \succsim (x_i, A_i) \sim (z, X)$. Thus for each i,

$$P(A_i) = \lim_{j\to\infty}[\varphi(x_j) \odot P(A_i)] \geqslant \varphi(z) \odot P(X) = \varphi(z).$$

Let $r = \lim_{i\to\infty} P(A_i)$. Note that for each i, $P(A_i) \geqslant r$. Then $r \geqslant \varphi(z)$. We will show that for all B in $\mathscr{E}$, if $r > P(B)$ then for all y in $\mathscr{C}$, $(z, X) \succsim (y, B)$. Suppose that B

in $\mathscr{E}$ and y in $\mathscr{C}$ are such that $r > P(B)$ and $(y, B) \succ (z, X)$. A contradiction will be shown. Let k be such that $x_k \succ_1 y$. Then $(y, B) \succ (x_k, A_k) \sim (z, X)$. This implies that $B \succ_2 A_k$. Since P is a probability representation for $\langle X, \mathscr{E}, \succsim_2 \rangle$, $P(B) > P(A_k)$. This is a contradiction. Thus for all B in $\mathscr{E}$, if $r > P(B)$ then for all y in $\mathscr{C}$, $(z, X) \succsim (y, B)$. We will now show that $r = \varphi(z)$. Suppose that $r > \varphi(z)$. A contradiction will be shown. By the proof of Lemma 3.13, let D be an element of $\mathscr{E}$ such that $r - \varphi(z) > P(D)$. Let n be a positive integer such that $P(D) > P(A_n) - r$. Thus

$$r > P(A_n) - P(D) > r - P(D) > \varphi(z).$$

Since P is a probability representation for $\langle X, \mathscr{E}, \succsim_2 \rangle$ and $P(A_n) \geqslant r > P(D)$, $A_n \succ_2 D$. By *uncertainty* part (iii) we may assume that $A_n \supseteq D$. By Lemma 3.2(v), $A_n - D$ is in $\mathscr{E}$. Since P is a probability representation, $P(A_n) - P(D) = P(A_n - D)$. Thus $r > P(A_n - D) > \varphi(z)$. Thus

$$\lim_{i\to\infty}[\varphi(x_i) \odot P(A_n - D)] = P(A_n - D) > \varphi(z) = \varphi(z) \odot P(X).$$

Therefore let m be such that

$$\varphi(x_m) \odot P(A_n - D) > \varphi(z) \odot P(X).$$

It follows that $(x_m, A_n - D) \succ (z, X)$. This contradicts the previously established result that for all B in $\mathscr{E}$, if $r > P(B)$ then for all y in $\mathscr{C}$, $(z, X) \succsim (y, B)$. Thus we have shown that $\lim_{i\to\infty} P(A_i) = r = \varphi(z)$. Similarly, $\lim_{i\to\infty} Q(A_i) = \psi(z)$. Since $P = Q$, $\varphi(z) = \psi(z)$. Since z is an arbitrary element of $\mathscr{C}$, $\varphi = \psi$.

Let u be an arbitrary element of $\mathscr{C}$ and E be an arbitrary element of $\mathscr{E}$. By *trade-off* let v be such that $(u, E) \sim (v, X)$. Then $\varphi(u) \odot P(E) = \varphi(v) \odot P(X) = \varphi(v)$. Similarly, $\psi(u) \odot' Q(E) = \psi(v)$. Since $P = Q$ and $\varphi = \psi$,

$$\varphi(u) \odot P(E) = \psi(u) \odot' Q(E). \quad \blacksquare$$

THEOREM 2.1. *If $\mathscr{U}$ is Archimedean then the following two propositions are ture*:

(1) *if $\mathscr{U}$ has a maximal element then $\mathscr{U}$ has an unique additive representation*;

(2) *if $\mathscr{U}$ is bounded and there exist $A_1, \ldots, A_i, \ldots$ in $\mathscr{E}$ such that $A_i \supset A_{i+1}$, then $\mathscr{U}$ has an unique additive representation.*

Proof. Lemmas 3.12 and 3.14. $\blacksquare$

DEFINITION 3.5. Let $\mathscr{U}$ be distributive. Then define the binary operation $\oplus$ on $\mathscr{C}$ as follows:

$x \oplus y \sim z$ iff x, y, z are in $\mathscr{C}$ and for some A, B in $\mathscr{E}$ such that $A \cap B = \varnothing$ and for some u in $\mathscr{C}$, $(u, A) \sim (x, X)$, $(u, B) \sim (y, X)$, and $(u, A \cup B) \sim (z, X)$.

(Properly speaking, one might call $\textcircled{1}$ a "multivalued operation." As before, for convenience we will consider $x \textcircled{1} y$ to be an element of $\mathscr{C}$ although it is really an equivalence class of members of $\mathscr{C}$ determined by the equivalence relation $\sim$.)

LEMMA 3.15. *Suppose $\mathscr{U}$ is distributive. Then for all x, y, z, w in $\mathscr{C}$, if $x \textcircled{1} y \sim_1 z$ and $x \textcircled{1} y \sim_1 w$ then $z \sim_1 w$.*

Proof. Left to reader. ∎

LEMMA 3.16. *Suppose that $\mathscr{U}$ is distributive and x, y, z, v, w are arbitrary members of $\mathscr{C}$. Then the following two propositions hold:*

(i) $x \textcircled{1} y \sim_1 z$ iff *for some u in $\mathscr{C}$, $x \textcircled{1}_u y \sim_1 z$;*

(ii) *if $v \gtrsim_1 w$ and $x \textcircled{1}_w y \sim_1 z$, then $x \textcircled{1}_v y$ is defined and $x \textcircled{1}_v y \sim_1 z$.*

Proof. (i) follows immediately from Definitions 3.5 and 3.2. To show (ii) assume that $v \gtrsim_1 w$ and $x \textcircled{1}_w y \sim_1 z$. Let A, B be elements of $\mathscr{E}$ such that $A \cap B = \varnothing$, $(w, A) \sim (x, X)$, $(w, B) \sim (y, X)$, and $(w, A \cup B) \sim (z, X)$. Since $v \gtrsim_1 w$, by *trade-off* let C, D be such that $(v, C) \sim (x, X)$ and $(v, D) \sim (y, X)$. Since $v \gtrsim_1 w$, $A \gtrsim_2 C$ and $B \gtrsim_2 D$. By *uncertainty*, let E, F be such that $C \sim_2 E$, $D \sim_2 F$, and $E \cap F = \varnothing$. Then $(w, A) \sim (v, E)$ and $(w, B) \sim (v, F)$. By *distributivity* $(w, A \cup B) \sim (v, E \cup F)$. Thus $z \sim_1 x \textcircled{1}_w y \sim_1 x \textcircled{1}_v y$. ∎

By using Lemma 3.16 and Lemmas 3.3 to 3.9, it is easy to establish that $\langle \mathscr{C}, \gtrsim_1, \textcircled{1} \rangle$ is an (Archimedean) extensive structure as defined in Krantz, *et al* [1971] page 84. Thus by Theorem 3 on page 85 of Krantz, *et al* [1971], the following Lemma is true:

LEMMA 3.17. *Suppose that $\mathscr{U}$ is distributive and Archimedean and that u, v are elements of $\mathscr{U}$. Then there is a function φ from $\mathscr{C}$ into the positive reals such that the following three conditions hold:*

(1) $\varphi(u) = 1$;

(2) *for all x, y in $\mathscr{C}$, $x \gtrsim_1 y$ iff $\varphi(x) \geqslant \varphi(y)$;*

(3) *for all x, y in $\mathscr{C}$, $\varphi(x \textcircled{1} y) = \varphi(x) + \varphi(y)$.*

Furthermore, if ψ is another function from $\mathscr{C}$ into the positive reals such that $\psi(v) = 1$ and ψ satisfies (2) *and* (3) *above, then $\psi = \varphi/\varphi(v)$.*

THEOREM 2.2. *Let $\mathscr{U}$ be Archimedean and distributive. Then for each u in $\mathscr{C}$ there is a distributive representation $\langle \varphi, P, u \rangle$ for $\mathscr{U}$. Furthermore, if $\langle \odot, \psi, Q, v \rangle$ is such that $\odot$ is a binary operation on the positive reals,* Re^+, $\psi: \mathscr{C} \to \mathrm{Re}^+$, $Q: \mathscr{E} \to (0, 1]$, $v \in \mathscr{C}$, *and for all x, y in $\mathscr{C}$ and all A, B in $\mathscr{E}$:*

(1) $Q(X) = \psi(v) = 1$,

(2) $\psi(x) \odot Q(X) = \psi(x), \psi(v) \odot Q(A) = Q(A)$,

(3) $(x, A) \gtrsim (y, B)$ *iff* $\psi(x) \odot Q(A) \geqslant \psi(y) \odot Q(B)$,

(4) *if* $A \cap B = \varnothing$ *then* $\psi(x) \odot (Q(A \cup B)) = (\psi(x) \odot Q(A)) + (\psi(x) \odot Q(B))$,

then $Q = P, \psi = \varphi/\varphi(v)$, *and for all* z *in* $\mathscr{C}$ *and all* C *in* $\mathscr{E}$, $\varphi(z) \odot P(C) = \psi(z) \odot Q(C)$.

Existence. Let u be an arbitrary element of $\mathscr{C}$. By Lemma 3.17, let φ be a function from $\mathscr{C}$ into Re^+ such that (i) $\varphi(u) = 1$, (ii) for all x, y in $\mathscr{C}$, $x \gtrsim_1 y$ iff $\varphi(x) \geqslant \varphi(y)$, and (iii) for all x, y in $\mathscr{C}$, $\varphi(x \oplus y) = \varphi(x) + \varphi(y)$. Define the function P from $\mathscr{E}$ into $(0, 1]$ as follows: for each A in $\mathscr{E}$, $P(A) = \varphi(\pi(u, A))$. Then it is easy to verify that P is a probability representation for $\langle X, \mathscr{E}, \gtrsim_2\rangle$. For each x in $\mathscr{C}$, let φ_x be the restriction of $\varphi/\varphi(x)$ to $\mathscr{C}_x$. Then φ_x is the unique extensive representation for $\langle \mathscr{C}_x, \gtrsim_x, \oplus_x\rangle$. Let y be an arbitrary element of $\mathscr{C}$. Then by the construction in the proof of Lemma 3.12, there is an unique additive representation for $\mathscr{U}_y = \langle X, \mathscr{E}, \mathscr{C}_y, \gtrsim_y\rangle$ of the form $\langle \odot_y, \varphi_y, P_y\rangle$. Since P_y is a probability representation for $\langle X, \mathscr{E}, \gtrsim_2\rangle$, by Lemma 3.11, $P = P_y$. Note that the proof of Lemma 3.12 also establishes that for each s in $\mathscr{C}$ and each A in $\mathscr{E}$, $\varphi_y(\pi(s, A)) = \varphi_y(s) \odot_y P(A)$. Suppose that $y \gtrsim_1 p \gtrsim_1 q$. Then $\varphi_y(q) = \varphi_y(p) \cdot \varphi_p(q)$ since

$$[\varphi(q)/\varphi(y)] = [\varphi(p)/\varphi(y)] \cdot [\varphi(q)/\varphi(p)].$$

Thus for each A in $\mathscr{E}$,

$$\begin{aligned} \varphi_y(p) \odot_y P(A) &= \varphi_y(\pi(p, A)) \odot_y P(X) \\ &= \varphi_y(\pi(p, A)) \\ &= \varphi_y(p) \cdot \varphi_p(\pi(p, A)) \\ &= \varphi_y(p) \cdot (\varphi_p(p) \odot_p P(A)) \\ &= \varphi_y(p) \cdot P(A). \end{aligned}$$

Therefore, for each t, w in $\mathscr{C}$ and each A, B in $\mathscr{E}$, there exists y in $\mathscr{C}$ such that $y \gtrsim_1 t$, $y \gtrsim_1 w$, and

$$\begin{aligned} (t, A) \gtrsim (w, B) \quad &\text{iff} \quad \varphi_y(t) \odot_y P(A) \geqslant \varphi_y(w) \odot_y P(B) \\ &\text{iff} \quad \varphi_y(t) \cdot P(A) \geqslant \varphi_y(w) \cdot P(B) \\ &\text{iff} \quad [\varphi(t)/\varphi(y)] \cdot P(A) \geqslant [\varphi(w)/\varphi(y)] \cdot P(B) \\ &\text{iff} \quad \varphi(t) \cdot P(A) \geqslant \varphi(w) \cdot P(B). \end{aligned}$$

Thus for each u in $\mathscr{C}$, $\langle \varphi, P, u\rangle$ is a distributive representation for $\mathscr{U}$.

Uniqueness. Suppose that u is an arbitrary element of $\mathscr{C}$ and $\langle \varphi, P, u \rangle$ is as in the existence part of this proof and $\langle \odot, \psi, Q\, v \rangle$ is as in the statement of Theorem 2.2. Let ψ_v be the restriction of ψ to $\mathscr{C}_v$ and φ_v be the restriction of $\varphi/\varphi(v)$ to $\mathscr{C}_v$. Then $\langle \odot, \psi_v, Q \rangle$ and $\langle \cdot, \varphi_v, P \rangle$ are additive representations for $\langle X, \mathscr{E}, \mathscr{C}_v, \gtrsim_v \rangle$. Thus by Lemma 3.12, (i) $P = Q$, (ii) for all z in $\mathscr{C}_v$,

$$\varphi(z)/\varphi(v) = \varphi_v(z) = \psi_v(z) = \psi(z),$$

and (iii) for all z in $\mathscr{C}_v$ and all D in $\mathscr{E}$,

$$\psi(z) \odot Q(D) = \psi(z) \cdot Q(D).$$

Suppose that $w \succ_1 v$. Let $1v = v$, and for each positive integer m, if $(mv) \oplus v$ is defined, let $(m + 1)v = (mv) \oplus v$. Since $\mathscr{U}$ is Archimedean, let n be the largest integer such that nv is defined and $w \gtrsim_1 nv$. Let $B_1, \ldots, B_n$ be such that for all $i, j \leqslant n$, (i) if $i \neq j$ then $B_i \cap B_j = \varnothing$, and (ii) $(w, B_i) \sim (v, X)$. Let $B = \bigcup_{i=1}^{n} B_i$. Then either $X - B = \varnothing$ or $B_1 \succ_2 X - B$. (If $X - B$ were $\gtrsim_2 B_1$, then by ***uncertainty*** we could find B_{n+1} such that $B_{n+1} \lesssim_2 X - B$ and $B_{n+1} \sim_2 B_1$ and thus show that $(n + 1)v$ is defined and that $w \gtrsim_1 (n + 1)v$, which contradicts the defining properties of n). For convenience we will assume that $X - B \neq \varnothing$. The case where $X - B = \varnothing$ will follow by a similar argument. Thus

$$\begin{aligned}
\psi(w) &= \psi(w) \odot Q(X) \\
&= \psi(w) \odot Q((X - B) \cup B) \\
&= (\psi(w) \odot Q(X - B)) + (\psi(w) \odot Q(B)) \\
&= (\psi(w) \odot Q(X - B)) + \sum_{i=1}^{n} \psi(w) \odot Q(B_i).
\end{aligned}$$

Since $B_1 \succ_2 X - B$, let $z \in \mathscr{C}_v$ be such that $(w, X - B) \sim (z, X)$. Then $\psi(w) \odot Q(X - B) = \psi(z) \odot Q(X) = \psi(z)$. For each $i \leqslant n$, $\psi(w) \odot Q(B_i) = \psi(v) \odot Q(X) = \psi(v)$. Thus $\psi(w) = \psi(z) + n\psi(v) = \psi(z) + n$. Similarly,

$$\varphi(w)/\varphi(v) = [\varphi(z)/\varphi(v)] + n[\varphi(v)/\varphi(v)] = [\varphi(z)/\varphi(v)] + n.$$

Since z is in $\mathscr{C}_v$, we have already shown that $\psi(z) = \varphi(z)/\varphi(v)$. Thus $\psi(w) = \varphi(w)/\varphi(v)$. Therefore we have shown that $\psi = \varphi/\varphi(v)$.

Let s be an arbitrary element of $\mathscr{C}$ and D be an arbitrary element of $\mathscr{E}$. Let t in $\mathscr{C}$ be such that $(s, D) \sim (t, X)$. Then since $P = Q$ and $\psi = \varphi/\varphi(v)$,

$$\begin{aligned}\psi(s) \odot Q(D) &= \psi(t) \odot P(X)\\ &= \psi(t)\\ &= \varphi(t)/\varphi(v)\\ &= [\varphi(t)/\varphi(v)] \cdot P(X)\\ &= [\varphi(s)/\varphi(v)] \cdot P(D)\\ &= \psi(s) \cdot Q(D). \quad \blacksquare\end{aligned}$$

THEOREM 2.3. *Suppose that $\mathscr{U}$ is distributive and $\mathscr{F}$ is a subset of $\mathscr{C}$ such that $\mathscr{F}$ has exactly one member from each commeasurability class of $\mathscr{C}$. Then there is a representation for $\mathscr{U}$ of the form $\langle \varphi, P, \mathscr{F} \rangle$. Furthermore, if $\langle \psi, Q, \mathscr{G} \rangle$ is another representation for $\mathscr{U}$, then $P = Q$ and for each x in $\mathscr{F}$ and each y in $\mathscr{G}$ if $x \equiv y$ then $\varphi(x) = \varphi(y)\,\psi(x)$.*

Outline of proof. Define the relation $\simeq$ on $\mathscr{C}$ as follows: $x \simeq y$ iff $x, y \in \mathscr{C}$ and either $(x, X) \sim (y, X)$ or for some infinitesimal A in $\mathscr{E}$, either $(x, X - A) \sim (y, X)$ or $(x, X) \sim (y, X - A)$. Then it is easy to show that $\simeq$ is an equivalence relation on $\mathscr{C}$.

Let $\mathscr{L}$ be the set of noninfinitesimal elements of $\mathscr{E}$. Define the relation $\cong$ on $\mathscr{L}$ as follows: $A \cong B$ iff $A, B \in \mathscr{L}$ and either $A \sim_2 B$ or for some infinitesimal C in $\mathscr{E}$, either $A - C \sim_2 B$ or $A \sim_2 B - C$. Then it is easy to show that $\cong$ is an equivalence relation on $\mathscr{L}$.

Let $\mathscr{L}'$ be the set of equivalence classes of $\mathscr{L}$ determined by the $\cong$ equivalence relation. Let $\mathscr{C}'$ be the set of $\simeq$ equivalence classes of $\mathscr{C}$.

Let $\mathbf{M}$ be an arbitrary commeasurability class of $\mathscr{C}$. Let $\mathbf{M}' = \{a \in \mathscr{C}' \mid a \cap \mathbf{M} \neq \varnothing\}$. Define $\succsim_{\mathbf{M}}$ on $\mathbf{M}' \times \mathscr{L}'$ as follows:

> $(a, R) \succsim_{\mathbf{M}} (b, S)$ iff $a, b \in \mathbf{M}'$ and $R, S \in \mathscr{L}'$ and for some x in a, y in b, A in R, B in S, $(x, A) \succsim (y, B)$.

Let X' be the element R of $\mathscr{L}'$ such that $X \in R$. Define the (partial) operations $\boldsymbol{\cap}, \boldsymbol{\cup}, \boldsymbol{\sim}$, on $\mathscr{L}'$ as follows: For each R, S, T in $\mathscr{L}'$,

(1) $R \boldsymbol{\cup} S = T$ iff for some $A \in R$, $B \in S$, $C \in T$, $A \cup B \sim C$;

(2) $R \boldsymbol{\cap} S = T$ iff for some $A \in R$, $B \in S$, $C \in T$, $A \cap B \sim C$;

(3) $R^{\boldsymbol{\sim}} = S$ iff for some $A \in R$, $B \in S$, $A^{\sim} \sim B$.

Define the relation $R \boldsymbol{\cap} S = \varnothing$ by: $R \boldsymbol{\cap} S = \varnothing$ iff $R, S \in \mathscr{L}'$ and for some $A \in R$, $B \in S$, $A \cap B = \varnothing$. Then in a natural way, $\mathscr{L}'$ with the operations $\boldsymbol{\cup}$, $\boldsymbol{\cap}$, $\boldsymbol{\sim}$ looks formally like an algebra of subsets of X' minus the empty set.

Under the above interpretations one can verify that for each commeasurability class $\mathbf{M}$ of $\mathscr{C}$, $\langle X', \mathscr{L}', \mathbf{M}', \succsim_{\mathbf{M}} \rangle$ is an Archimedean, distributive utility-uncertainty trade-off structure. Since $\mathscr{F}$ contains exactly one member from each commeasurability class $\mathbf{M}$ of $\mathscr{C}$, let $u_{\mathbf{M}}$ be the single member of $\mathbf{M} \cap \mathscr{F}$. Then by Theorem 2.2, for each commeasurability class $\mathbf{M}$ of $\mathscr{C}$, let $\langle \varphi_{\mathbf{M}}, P_{\mathbf{M}}, u_{\mathbf{M}} \rangle$ be a distributive representation for $\langle X', \mathscr{L}', \mathbf{M}', \gtrsim_{\mathbf{M}} \rangle$. Then as in the proof of Lemma 3.11, it is easy to show that if $\mathbf{M}, \mathbf{N}$ are commeasurability classes of $\mathscr{C}$, then $P_{\mathbf{M}} = P_{\mathbf{N}}$. Thus let $P = P_{\mathbf{M}}$ for some commeasurability class $\mathbf{M}$ of $\mathscr{C}$. Define φ on $\mathscr{C}$ as follows: if x is in the commeasurability class $\mathbf{M}$ of $\mathscr{C}$ then $\varphi(x) = \varphi_{\mathbf{M}}(a)$ where a is the $\simeq$ equivalence class such that $x \in a$. Then it can be shown that $\langle \varphi, P, \mathscr{F} \rangle$ is a representation for $\mathscr{U}$. If $\langle \psi, Q, \mathscr{G} \rangle$ is another representation for $\mathscr{U}$, then by using Theorem 2.2 it can be shown that $P = Q$ and for all x in $\mathscr{F}$ and all y in $\mathscr{G}$, if $x \equiv y$ then $\varphi(x) = \varphi(y) \cdot \psi(x)$. ▮

THEOREM 2.4. *Let $\langle X, \mathscr{E}, \mathscr{C}, \mathscr{G}, \gtrsim' \rangle$ be a gambling structure and u, v be arbitrary elements of $\mathscr{C}$. Then there is an additive probability function P on $\mathscr{E}$ and a function φ from $\mathscr{C}$ into the positive reals such that $\varphi(u) = 1$ and for all f, g in $\mathscr{G}$,*

$$f \gtrsim' g \quad \text{iff} \quad \mathbf{E}_P(\varphi(f)) \geqslant \mathbf{E}_P(\varphi(g)).$$

Furthermore, if Q is another additive probability function on $\mathscr{E}$ and ψ is another function from $\mathscr{C}$ into the positive reals such that $\psi(v) = 1$ and for all f, g in $\mathscr{G}$,

$$f \gtrsim' g \quad \text{iff} \quad \mathbf{E}_Q(\psi(f)) \geqslant \mathbf{E}_Q(\psi(g)),$$

then $P = Q$ and $\varphi = \varphi(v)\psi$.

Proof. Define $\gtrsim$ on $\mathscr{C} \times \mathscr{E}$ as follows: for each $x, y \in \mathscr{C}$ and each $A, B \in \mathscr{E}$,

$$(x, A) \gtrsim (y, B) \quad \text{iff} \quad f_A{}^x \gtrsim' f_B{}^y.$$

Then it is easy to show that $\mathscr{U} = \langle X, \mathscr{E}, \mathscr{C}, \gtrsim \rangle$ is an Archimedean, distributive utility-uncertainty trade-off structure. By Theorem 2.2 let $\langle \varphi, P, u \rangle$ be a distributive representation for $\mathscr{U}$. Then for each x, y in $\mathscr{C}$ and each A, B in $\mathscr{E}$,

$$\begin{aligned} f_A{}^x \gtrsim' f_B{}^y \quad & \text{iff} \quad (x, A) \gtrsim (y, B) \\ & \text{iff} \quad \varphi(x) \cdot P(A) \geqslant \varphi(y) \cdot P(B) \\ & \text{iff} \quad \mathbf{E}_P(\varphi(f_A{}^x)) \geqslant \mathbf{E}_P(\varphi(f_B{}^y)). \end{aligned}$$

Thus we have shown that for all $x, y \in \mathscr{C}$ and all $A, B \in \mathscr{E}$,

(1) $f_A{}^x \gtrsim' f_B{}^y$ iff $\mathbf{E}_P(\varphi(f_A{}^x)) \geqslant \mathbf{E}_P(\varphi(f_B{}^y))$.

Let g be an arbitrary element of $\mathscr{G}$ and $x_1, \ldots, x_n, A_1, \ldots, A_n$ be such that

$$g = f_{A_1}^{x_1} \cup \cdots \cup f_{A_n}^{x_n}$$

and for $i, j \leqslant n$, if $i \neq j$ then $x_i \neq x_j$ and $A_i \cap A_j = \varnothing$. Let x be the maximal element of $\{x_1, ..., x_n\}$ with respect to the $\gtrsim'$ ordering, i.e., $x = x_j$ for some $j \leqslant n$ and $f_X{}^x \gtrsim' f_X^{x_i}$ for all $i \leqslant n$. By *trade-off'*, let $B_1, ..., B_n$ be such that $f^x_{B_i} \sim' f^{x_i}_{A_i}$. Then for $i = 1, ..., n$, $f^x_{A_i} \gtrsim f^x_{B_i}$. By *uncertainty'*, we may assume that $A_i \supseteq B_i$. Then for $i, j \leqslant n$, if $i \neq j$ then $B_i \cap B_j = \varnothing$. Let $B = \bigcup_{i=1}^n B_i$. By *linearity*, $f_B{}^x \sim' g$.

$$\begin{aligned}
\mathbf{E}_P(\varphi(f_B{}^x)) &= \varphi(x) \cdot P(B) \\
&= \varphi(x) \cdot (P(B_1) + \cdots + P(B_n)) \\
&= \varphi(x) \cdot P(B_1) + \cdots + \varphi(x) \cdot P(B_n) \\
&= \mathbf{E}_P(\varphi(f^x_{B_1})) + \cdots + \mathbf{E}_P(\varphi(f^x_{B_n})) \\
&= \mathbf{E}_P(\varphi(f^x_{A_1})) + \cdots + \mathbf{E}_P(\varphi(f^x_{A_n})) \text{ (by (1))} \\
&= \varphi(x_1) \cdot P(A_1) + \cdots + \varphi(x_n) \cdot p(A_n) \\
&= \mathbf{E}_P(\varphi(g)).
\end{aligned}$$

Thus we have shown that

(2) for each $g \in \mathscr{G}$ there are x and B such that $g \sim' f_B{}^x$ and $\mathbf{E}_P(\varphi(g)) = \mathbf{E}_P(\varphi(f_B{}^x))$.

Let d, h be arbitrary members of $\mathscr{G}$ such that $d \gtrsim' h$. By (2) let x, y, C, D be such that $d \sim' f_C{}^x$, $h \sim' f_D{}^y$, $\mathbf{E}_P(\varphi(d)) = \mathbf{E}_P(\varphi(f_C{}^x))$, and $\mathbf{E}_P(\varphi(h)) = \mathbf{E}_P(\varphi(f_D{}^y))$. Then by (1), $\mathbf{E}_P(\varphi(f_C{}^x)) \geqslant \mathbf{E}_P(\varphi(f_D{}^y))$. Thus $\mathbf{E}_P(\varphi(d)) \geqslant \mathbf{E}_P(\varphi(h))$.

Suppose that Q is another additive probability function on $\mathscr{E}$ and ψ is another function from $\mathscr{C}$ into the positive reals such that $\psi(v) = 1$ and for all f', g' in $\mathscr{G}$,

$$f' \gtrsim' g' \quad \text{iff} \quad \mathbf{E}_Q(\psi(f')) \geqslant \mathbf{E}_Q(\psi(g')).$$

Then $\langle \psi. Q, v \rangle$ is a distributive representation for $\mathscr{U}$. Thus by Theorem 2.2, $\varphi = \varphi(v)\psi$. ▌

DEFINITION 3.6. Let Y be a nonempty set, $\gtrsim^*$ a binary operation on Y and $\circ$ be a closed operation on Y, i.e., for all x, y in Y, $x \circ y$ is in Y. Then $\langle Y, \gtrsim^*, \circ \rangle$ is said to be a *closed extensive structure* if and only if the following five conditions hold for all x, y, z, w in Y

(1) $\gtrsim^*$ is a weak order;

(2) $x \circ (y \circ z) \sim^* (x \circ y) \circ z$;

(3) $x \gtrsim^* y$ iff $x \circ z \gtrsim^* y \circ z$ iff $z \circ x \gtrsim^* z \circ y$;

(4) if $x \succ^* y$ then there exists a positive integer n such that $nx \circ z \gtrsim^* ny \circ w$ where mx is define inductively as: $1x = x$, $(m + 1)x = (mx) \circ x$;

(5) $x \circ y \succ^* x$.

4. Historical Note

An axiomatization of expected utility with numerical probabilities occuring in the axioms is given in von Neumann and Morgenstern [1953]. A purely qualitative treatment is given in Savage [1954]. These axiomatizations yield representations similar to those for gambling structures (Theorem 2.4).

Structures similar to utility-uncertainty trade-off structures have been experimentally studied in Tversky [1967].

Lexicographic orderings and commeasurability classes have been used in Narens [1974a] to give representations for non-Archimedean extensive structures. A representation theorem for qualitative probability without an Archimedean axiom is given in Narens [1974b].

The concept of distributivity presented here seems to play a very important role in measurement theory. This will be more deeply investigated in Narens and Luce [1975].

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