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THE ALGEBRA OF MEASUREMENT

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1. Introduction

Theories of measurement – at least those for classical physics, probability, and the behavioral and social sciences – study ordered algebraic structures that fulfill two conditions. First, at least one empirical interpretation of the primitives exists for which the axioms appear to be either approximately true laws or plausible, if untestable, conditions. Second, a homomorphism into some numerical structure can be established in which the order maps into ordinary inequality and which is essentially unique in the sense that its value at one (or two) points determines it. (When one value is sufficient, the measurement literature refers to the homomorphism as a ratio scale, and when two are needed, as some species of interval scale.)

Most of the literature has focussed on structures which either have or induce operations (either closed or partial) † that are associative (see, for example, Krantz et al. [4] and Pfanzagl [13]). Such structures have numerical homomorphisms with the operations mapping into +. Aside from some work on bisymmetric intensive structures – those with the intensive property that if $x \succsim y$, then $x \succsim x \circ y \succsim y$ and the bisymmetry condition $(x \circ y) \circ (u \circ v) \sim (x \circ u) \circ (y \circ v)$ – very little has been done on non-associative measurement structures. Our purpose here is to work out some of the basic features of such structures. In doing so we adhere strictly to the demand of essentially unique homomorphisms, but relax considerably the requirement of citing existing empirical interpretations. One consequence of this program is to enhance our understanding of the interconnections between positive concatenation structures (ordered partial operations with the property that $x \circ y \succ x, y$), general intensive structures, and general conjoint ones (orderings of Cartesian products).

The paper has the following structure. The next section is devoted to positive concatenation structures which meet the structural condition of having half elements, i.e., a function θ from X into X such that for all x in X, $\theta(x) \circ \theta(x) = x$. The resulting homomorphism is into $\langle \mathrm{Re}, \geq, \circledcirc \rangle$, where $\circledcirc$ is a partial, binary, numerical operation. Section 3 takes up intensive structures and conditions under which it is possible to

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convert one into a positive concatenation structure with half elements and conversely. The topic of Section 4 is conjoint structures $\langle X \times P, \succsim \rangle$ for which the representation involves functions ψ_X and ψ_P from X and P, respectively, into the reals and a real function F of two real variables such that for all x, y in X and p, q in P,

$$xp \succsim yq \quad \text{iff} \quad F[\psi_X(x), \psi_P(p)] \geq F[\psi_X(y), \psi_P(q)] \ .$$

It is shown that under reasonable conditions this problem can be solved by inducing on one of the components a partial operation of the sort studied in Section 2, and its representation is used to construct one for the conjoint structure. In Section 5 we turn to conjoint structures which also possess a positive partial operation on one of the sets X, P, or $X \times P$. A concept of distributivity is introduced, and it is shown that in its presence the operation has an additive representation if and only if the conjoint structure has a multiplicative one. Although this subsumes much of the measurement structure of classical physics, it does omit such important cases as relativistic velocity. That case is taken up in Section 6, where a qualitative assumption is shown to be equivalent to the usual relativistic "addition" law for velocities. Finally, Section 7 returns to positive concatenation structures but with a focus on the question of algebraic conditions under which it is plausible to suppose that the homomorphism is into a continuous, strictly increasing operation on the positive reals. We provide purely algebraic conditions under which it is possible to construct a Dedekind completion of the given structure, and so a representation onto the positive reals. These conditions, which appear both to be new and interesting, avoid the usual mixture of algebraic and topological assumptions, and are necessary conditions in a Dedekind complete structure with a closed operation. Some open problems are cited in the end.

Throughout this paper, the following conventions and definitions will be observed.

Re will denote the set of reals, Re^+ the positive reals, I the integers, and I^+ the positive integers. Elements of Cartesian products $X \times Y$ will be written as (x, y) or xy.

We say $\circ$ is a *partial operation* on X if and only if for some nonempty subset A of $X \times X$, $\circ$ is a function from A into X.

Let $\circ$ be a partial operation on X, A be the domain of $\circ$, and x, y be arbitrary elements of X. $x \circ y$ is said to be *defined* if and only if (x, y) is in A. $\circ$ is said to be a *closed operation* if and only if $x \circ y$ is defined for each x, y in X, i.e., if and only if $\circ$ is an operation. For each n in I^+, nx is inductively defined as follows:

(i) $1x = x$;

(ii) if $n > 1$ and $[(n-1)x] \circ x$ is defined, then $nx = [(n-1)x] \circ x$;

(iii) if $n > 1$ and $[(n-1)x] \circ x$ is not defined, then nx is not defined.

Let Y be a subset of X. The *closure of Y (with respect to $\circ$)* is the smallest set Z such that $Y \subseteq Z$ and for each x, y in Z, if $x \circ y$ is defined then $x \circ y$ is in Z.

θ is said to be a *half element function* on X if and only if θ is a function on X such that for each x in X, $\theta(x) \circ \theta(x) = x$.

$\succsim$ is said to be a *weak ordering* (or *weak order*) on X if and only if X is a nonempty set and $\succsim$ is a transitive and connected binary relation on X.

Let $\succsim$ be a weak ordering on X and u, v be arbitrary elements of X. Then $u \sim v$

denotes $u \gtrsim v$ and $v \gtrsim u$. It is easy to show that $\sim$ is an equivalence relation on X and that $\gtrsim/\sim$ is a total ordering on $X/\sim$. $u \succ v$ denotes $u \gtrsim v$ and not $v \gtrsim u$, and $u \prec v$ denotes $v \succ u$. $\langle X, \gtrsim \rangle$ is said to have a *countable dense subset* if and only if there exists a countable subset Y of X such that for each x, z in X, if $x \succ z$ then for some y in Y, $x \gtrsim y \gtrsim z$. $\langle X, \gtrsim \rangle$ is said to be *Dedekind complete* if and only if each nonempty bounded subset Y of X has a least upper bound (l.u.b.) in X. φ is said to be an *order homomorphism* of $\langle X, \gtrsim \rangle$ into Re (respectively, Re^+) if and only if φ is a function from X into Re (respectively, Re^+) such that for each x, y in X,

$$x \gtrsim y \quad \text{iff} \quad \varphi(x) \geqslant \varphi(y) .$$

By well-known theorems of Cantor, $\langle X, \gtrsim \rangle$ has a countable dense subset if and only if order homomorphisms into Re and Re^+ exist; and if $\langle X, \gtrsim \rangle$ has a countable dense subset, has no minimal or maximal elements, is Dedekind complete, and is such that for each x, y in X if $x \succ y$, then for some z, $x \succ z \succ y$, then there exists an order homomorphism that is onto Re^+.

Let $\gtrsim$ be a weak ordering on X. Throughout this paper, we will often treat multi-valued functions whose values are unique up to $\sim$ as functions, e.g., if φ is an order homomorphism of $\langle X, \gtrsim \rangle$ into Re then φ^{-1} will often be treated as a function.

Let $R \subseteq \text{Re}^+$. For convenience, instead of forming a new relation that is the restriction of $\geq$ to R, we will often consider $\geq$ to be a relation on R.

2. Positive concatenation structures

Definition 2.1. Let X be a nonempty set, $\gtrsim$ a binary relation on X, and $\circ$ a partial binary operation on X. The structure $\mathcal{X} = \langle X, \gtrsim, \circ \rangle$ is a *positive concatenation structure* if and only if the following seven axioms hold for all w, x, y, z in X:

Axiom 1. *Weak ordering*: $\gtrsim$ is connected and transitive.

Axiom 2. *Nontriviality*: there exist u, v in X such that $u \succ v$.

Axiom 3. *Local definability*: if $x \circ y$ is defined, $x \gtrsim w$, and $y \gtrsim z$, then $w \circ z$ is defined.

Axiom 4. *Monotonicity*: (i) if $x \circ z$ and $y \circ z$ are defined, then

$$x \gtrsim y \quad \text{iff} \quad x \circ z \gtrsim y \circ z ,$$

and (ii) if $z \circ x$ and $z \circ y$ are defined, then

$$x \gtrsim y \quad \text{iff} \quad z \circ x \gtrsim z \circ y .$$

Axiom 5. *Restricted solvability*: if $x \succ y$, then there exists u such that $x \succ y \circ u$.

Axiom 6. *Positivity*: if $x \circ y$ is defined, then $x \circ y \succ x$ and $x \circ y \succ y$.

Axiom 7. *Archimedean*: there exists $n \in I^+$ such that either nx is not defined or $nx \gtrsim y$. □

Definition 2.2. Let $\mathcal{X} = \langle X, \gtrsim, \circ \rangle$ be a positive concatenation structure. φ is said to be a $\circledcirc$*-representation* for $\mathcal{X}$ if and only if φ is a function from X onto some subset R

of Re^+ such that $\langle R, \geq, \circledcirc \rangle$ is a positive concatenation structure and the following two conditions are true for each x, y in X:

(i) $x \succsim y$ iff $\varphi(x) \geq \varphi(y)$;

(ii) if $x \circ y$ is defined, then $\varphi(x) \circledcirc \varphi(y)$ is defined and $\varphi(x \circ y) = \varphi(x) \circledcirc \varphi(y)$.

If $\circledcirc$ is $+$, then φ is said to be an *additive representation* for $\mathcal{X}$. □

Lemma 2.1. *Let $\langle X, \succsim, \circ \rangle$ be a positive concatenation structure and x be an arbitrary element of X. Then the following three statements are true:*

(i) *There exists y in X such that $x \succ y$.*

(ii) *There exists y in X such that $y \circ y$ is defined and $x \succ y \circ y$.*

(iii) *There exists a sequence of elements of X, $x_1, x_2, \ldots$ such that* (1) *for each $i \in I^+$, $x_{i+1} \circ x_{i+1}$ is defined and $x_i \succ x_{i+1} \circ x_{i+1}$, and* (2) *for each z in X there exists $j \in I^+$ such that $z \succ x_j$.*

Proof. Left to reader. □

Lemma 2.2. *If $\langle X, \succsim, \circ \rangle$ is a positive concatenation structure, then $\langle X, \succsim \rangle$ has a countable dense subset.*

Proof. By Lemma 2.1, let $x_1, x_2, \ldots, x_i, \ldots$ be a sequence of members of X such that for each $z \in X$ there exists m such that $z \succ x_m$. Let Y be the closure of $\{x_1, x_2, \ldots, x_i, \ldots\}$ with respect to $\circ$. Then Y is a countable set. Suppose that $u \succ v$. By restricted solvability, let w be such that $u \succ v \circ w$. Let n be such that $w \succ x_n$ and $v \succ x_n$. We will first show that for some positive integer k,

$$(2.1) \qquad v \succsim kx_n \quad \text{and} \quad (k+1)x_n \succ v.$$

If (2.1) does not hold for some k, then from Archimedean it follows that it must be the case that for some positive integer p,

$$v \succsim px_n \quad \text{and} \quad (p+1)x_n \quad \text{is not defined.}$$

But since $v \succsim px_n$, $w \succ x_n$, and $v \circ w$ is defined, it follows from local definability that $(px_n) \circ x_n = (p+1)x_n$ is defined – a contradiction. Therefore, let k be such that $v \succsim kx_n$ and $(k+1)x_n \succ v$. Since $v \succsim kx_n$ and $w \succsim x_n$,

$$u \succ v \circ w \succ (kx_n) \circ x_n = (k+1)x_n \succ v.$$

Since Y is the closure of $\{x_1, x_2, \ldots, x_i, \ldots\}$ with respect to $\circ$, $(k+1)x_n$ is in Y. □

Theorem 2.1. *Let $\mathcal{X} = \langle X, \succsim, \circ \rangle$ be a positive concatenation structure. Then there exists φ and $\circledcirc$ such that φ is a $\circledcirc$-representation for $\mathcal{X}$.*

Proof. Since by Lemma 2.2 $\langle X, \succsim \rangle$ has a countable dense subset, by a well known theorem of set theory, let φ be an order homomorphism from $\langle X, \succsim \rangle$ into Re^+. Let

$R = \{\varphi(x) \mid x \in X\}$ and for each $r \in R$ let $\varphi^{-1}(r)$ be an element x of X such that $\varphi(x) = r$. Let $\circledcirc$ be the partial binary operation on R such that for each r, s in R,

$$r \circledcirc s = \varphi(\varphi^{-1}(r) \circ \varphi^{-1}(s)) \quad \text{iff} \; \varphi^{-1}(r) \circ \varphi^{-1}(s) \text{ is defined.}$$

Then φ is a $\circledcirc$-representation for $\mathfrak{X}$. □

Definition 2.3. A positive concatenation structure $\mathfrak{X} = \langle X, \succsim, \circ \rangle$ is said to be *with half elements* if and only if $\mathfrak{X}$ satisfies the following axiom:

Half elements: for each x in X there exists u such that $u \circ u \sim x$. □

Theorem 2.2. *Let $\mathfrak{X} = \langle X, \succsim, \circ \rangle$ be a positive concatenation structure with half elements and φ, ψ be $\circledcirc$-representations for $\mathfrak{X}$ such that for some u, $\varphi(u) = \psi(u)$. Then $\varphi = \psi$.*

Proof. Suppose that φ, ψ are $\circledcirc$-representations for $\mathfrak{X}$ and $\varphi(u) = \psi(u)$. Assume that $\varphi \neq \psi$. A contradiction will be shown. Without loss of generality assume that v is such that $\varphi(v) > \psi(v)$. Let $u_1 = u$, and by half-elements, for each $n \in I^+$ let u_{n+1} be such that $u_n \sim u_{n+1} \circ u_{n+1}$. For each $n \in I^+$, let $\alpha_n = \varphi(u_n)$. Then

$$\alpha_2 \circledcirc \alpha_2 = \varphi(u_2) \circledcirc \varphi(u_2) = \varphi(u_2 \circ u_2) = \varphi(u_1) = \alpha_1 \;,$$

and by induction, for each $n \in I^+$,

$$\alpha_{n+1} \circledcirc \alpha_{n+1} = \alpha_n .$$

Since $\langle R, \geq, \circledcirc \rangle$ is a positive concatenation structure for some $R \subseteq \mathrm{Re}^+$, by the proof of Lemma 2.2, let $p, m \in I^+$ be such that

$$\varphi(v) > p\alpha_m > \psi(v) \,,$$

where of course $p\alpha_m$ stands for $[(p-1)\alpha_m] \circledcirc \alpha_m$. Since $u_1 \sim u_2 \circ u_2$,

$$\alpha_2 \circledcirc \alpha_2 = \alpha_1 = \varphi(u_1) = \psi(u_1) = \psi(u_2 \circ u_2) = \psi(u_2) \circledcirc \psi(u_2) \,.$$

Since $\circledcirc$ is monotonic this means that $\psi(u_2) = \alpha_2$. By induction, for each $n \in I^+$,

$$\psi(u_n) = \alpha_n \;.$$

Thus,

$$\varphi(v) > p\alpha_m = \varphi(pu_m) = \psi(pu_m) > \psi(v) \,.$$

Since $\varphi(v) > \varphi(pu_m)$, $v \succ pu_m$; since $\psi(pu_m) > \psi(v)$, $pu_m \succ v$. This is a contradiction. □

Definition 2.4. $\mathfrak{X} = \langle X, \succsim, \circ \rangle$ is said to be an *extensive structure* if and only if $\mathfrak{X}$ is a positive concatenation structure that satisfies the following axioms:

Associativity: for each x, y, z in X, if $x \circ (y \circ z)$ and $(x \circ y) \circ z$ are defined, then $x \circ (y \circ z) \sim (x \circ y) \circ z$.

Unboundedness: for each x, there exists y such that $y \succ x$. □

The following theorem due to Krantz et al. [4] is a generalization of a classic theorem of Hölder [2].

Theorem 2.3. *If $\mathfrak{X}$ is an extensive structure, then there exists an additive representation for $\mathfrak{X}$. Furthermore, if φ and ψ are additive representations for $\mathfrak{X}$, then for some $r \in \mathrm{Re}^+$, $\varphi = r\psi$.*

Proof. Theorem 2.4 of Krantz et al. [4]. □

Sometimes it is convenient to consider additive structures with maximal elements. $\mathfrak{X} = \langle X, \succsim, \circ \rangle$ is said to be an *extensive structure with a maximal element* if and only if $\mathfrak{X}$ satisfies all the conditions of Definition 2.4 except for *unboundedness* and there exist u, v in X such that $u \circ v$ is defined and for each x in X, $u \circ v \succsim x$. Then it is easy to show that Theorem 2.3 remains valid if "extensive structure" is replaced by "extensive structure with a maximal element".

3. Intensive concatenation structures

Definition 3.1. Let X be a nonempty set, $\succsim$ a binary relation on X, and $*$ a (partial) binary operation on X. The structure $\mathfrak{X} = \langle X, \succsim, * \rangle$ is an *intensive concatenation structure* iff for every x, y, z in X the following five axioms hold:

Axiom 1. *Weak order*: $\succsim$ is transitive and connected.

Axiom 2. *Nontriviality*: there exist u, v in X such that $u \succ v$.

Axiom 3. *Local definability*: If $x * y$ is defined and $x \succsim w$ and $y \succsim z$, then $w * z$, is defined.

Axiom 4. *Monotonicity*: (i) if $x * z$ is defined, then $x \succsim y$ iff $x * z \succsim y * z$; and (ii) if $z * x$ is defined, then $x \succsim y$ iff $z * x \succsim z * y$.

Axiom 5. *Intern*: if $x \sim y$, then $x * y$, $y * x$ are defined and $x \sim x * y \sim y * x$. If $x \succ y$, then $x \succ x * y \succ y$ and $x \succ y * x \succ y$. ($x * y$ and $y * x$ are defined by $x \sim x$ and Axiom 3.) □

Definition 3.2. Suppose $\mathfrak{X} = \langle X, \succsim, * \rangle$ is an intensive concatenation structure and δ is a function from $A \subseteq X$ into X. δ is a *doubling function* iff for every x, y in X:

(i) δ is strictly monotonic increasing.

(ii) If $x \succsim y$ and x is in A, then y is in A.

(iii) If $x \succ y$, then there is u in X such that $y * u$ is in A and $x \succ \delta(y * u)$.

(iv) If $x * y$ is in A, then $\delta(x * y) \succ x, y$.

(v) Let x_n, $n = 1, 2, \ldots$, be such that $x_1 \sim x$ and if x_{n-1} is in A, then $x_n \sim \delta(x_{n-1}) * x$. Either there exists $n \in I^+$ such that x_n is not defined or $x_n \succsim y$. Such a sequence is called a standard sequence of δ. □

Intensive concatenation structures resemble positive concatenation ones in that the first four axioms – weak order, nontriviality, local definability, and monotonicity – are the same. They differ sharply in that $*$ is not assumed to be positive, but rather intern. Nevertheless, as we show in Theorem 3.1, the two kinds of structures are closely related provided they are sufficiently rich. For positive structures, it is sufficient to postulate the existence of half elements. For intensive structures, the concept of a doubling function appears to be needed. The reason for the name will become apparent. But whereas the concept of a half element in a positive structure is unique up to $\sim$, that of a double element in an intensive one is not – it matters what one adjoins as zero in the positive structure. The non-uniqueness is partially discussed in Theorems 3.2 and 3.3.

Theorem 3.1. *Suppose X is a nonempty set, $\succsim$ a binary relation on X, $*$ and $\circ$ (partial) binary operations defined for the same pairs from X, and θ a function from X into X such that for all x,y in X for which $x \circ y$ and $x * y$ are defined $\theta(x \circ y) \sim x * y$. Then, $\langle X, \succsim, \circ\rangle$ is a positive concatenation structure with half element function θ* (i.e., *$x \sim \theta(x) \circ \theta(x)$ for each $x \in X$) iff $\langle X, \succsim, *\rangle$ is an intensive concatenation structure with θ^{-1} a doubling function.*

Proof. Since the weak order and nontriviality assumptions involve only $\succsim$, we need not consider them.

Assume $\langle X, \succsim, \circ\rangle$ is a positive concatenation structure with θ the half element function. Since $\theta(x) \circ \theta(x) \sim x$, the monotonicity of $\circ$ implies the strict monotonicity of θ. We show the axioms of an intensive structure.

3. *Local definability*. Suppose $x * y$ is defined, $x \succsim w$, and $y \succsim z$. Then $x \circ y$ is defined and so by local definability of $\circ$, $w \circ z$ is defined, whence $w * z$ is defined.

4. *Monotonicity*. Suppose $x * z$ is defined, then $x \circ z$ is defined and by the monotonicity of $\circ$, $x \succsim y$ iff $x \circ z \succsim y \circ z$. By the strict monotonicity of θ, this holds iff $x * z \succsim y * z$. The other case is similar.

5. *Intern*. Suppose $x \succ y$ and $x * y$ is defined. Since $x * y = \theta(x \circ y)$, we see $(x * y) \circ (x * y) \sim x \circ y$. By the monotonicity of $\circ$, $(x * y) \circ (x * y) \succ y \circ y$, whence $x * y \succ y$. Suppose $x * y \succsim x$, then $\theta(x \circ y) \sim x * y \succsim x \sim \theta(x \circ x)$, where by the monotonicity of θ and of $\circ$, $y \succsim x$, contrary to assumption. A similar argument holds for $y * x$.

If $x \sim y$, then since $\theta(x \circ x) \sim x$, $x * y \sim x$.

Next we show that θ^{-1} is a doubling function. Let A be the domain of θ^{-1}.

(i) The strict monotonicity of θ and hence of θ^{-1} was shown above.

(ii) Suppose x is in A and $x \succsim y$. Let $z = \theta^{-1}(x)$, so $z \sim x \circ x$. By local definability of $\circ$, $x \succsim y$ implies $y \circ y$ is defined, whence $y \circ y \sim \theta^{-1}(y)$.

(iii) Suppose $x \succ y$. By restricted solvability, there exists u in X such that $x \succ y \circ u \sim \theta^{-1}(y * u)$.

(iv) Suppose $x * y$ is in A and that $\theta^{-1}(x * y) \succ x, y$ is false. If $x \succsim \theta^{-1}(x * y)$, then by the strict monotonicity of θ, $\theta(x) \succsim x * y \sim \theta(x \circ y)$, whence $x \succsim x \circ y$, contrary to the positivity of $\circ$.

(v) Suppose $y \succsim x_n$, $n = 1, 2, \ldots$, where x_n is in A, $x_1 = x$, and

$$x_n = \theta^{-1}(x_{n-1}) * x = \theta[\theta^{-1}(x_{n-1}) \circ x] .$$

A contradiction will be shown. Since $\theta^{-1}(x_n) \sim (n+1)x$ [where $1x = x$ and for each $m \in I^+$, $(m+1)x = (mx) \circ x$], *Archimedean* (Definition 2.1) is contradicted.

Conversely, suppose $\langle X, \succsim, * \rangle$ is an intensive concatenation structure and θ^{-1} is a doubling function.

3. *Local definability.* Suppose $x \circ y$ is defined, $x \succsim w$, $y \succsim z$. Since $x * y \sim \theta(x \circ y)$ is defined, by local definability of $*$, $w * z$ is defined, so $\theta^{-1}(w * z) = w \circ z$ is defined.

4. *Monotonicity.* Assume $x \circ z$ is defined,

$x \succsim y$ iff $x * z \succsim y * z$ (monotonicity of $*$)

iff $\theta^{-1}(x * z) \succsim \theta^{-1}(y * z)$ (strict monotonicity of θ^{-1} and property (ii) of Def. 3.2)

iff $x \circ z \succsim y \circ z$.

The other case is similar.

5. *Restricted solvability.* Suppose $x \succ y$. By property (ii) of Definition 3.2, $x \succ \theta^{-1}(y * u) = y \circ u$.

6. *Positivity.* Suppose $x \circ y$ is defined, then by property (iv) of Definition 3.2, $x \circ y \sim \theta^{-1}(x * y) \succ x, y$.

7. *Archimedean.* Consider the sequence $nx = ((n-1)x) \circ x$. Let

$$\begin{aligned} x_n &\sim \theta(nx) \sim \theta[(n-1)x \circ x] \\ &\sim (n-1)x * x \sim \theta^{-1}[\theta((n-1)x)] * x \\ &\sim \theta^{-1}(x_{n-1}) * x . \end{aligned}$$

So for some n either x_n is not defined or $x_n \succsim \theta(y)$, whence $nx \succsim y$.

8. *Half elements.* Since $\theta(x) = \theta(x) * \theta(x)$,

$$x \sim \theta^{-1}[\theta(x) * \theta(x)] = \theta(x) \circ \theta(x). \quad \square$$

Corollary 3.1. *Under the assumptions of Theorem 3.1, the operation $\circ$ is bisymmetric (i.e., $(x \circ y) \circ (z \circ w) \sim (x \circ z) \circ (y \circ w)$) iff the operation $*$ is bisymmetric.*

Proof. $(x \circ y) \circ (z \circ w) \sim (x \circ z) \circ (y \circ w)$

iff $[(x * y) \circ (x * y)] \circ [(z * w) \circ (z * w)] \sim [(x * z) \circ (x * z)] \circ [(y * w) \circ (y * w)]$

(since $x \circ y \sim (x * y) \circ (x * y)$)

iff $[(x * y) \circ (z * w)] \circ [(x * y) \circ (z * w)] \sim [(x * z) \circ (y * w)] \circ [(x * z) \circ (y * w)]$

(bisymmetry)

iff $(x * y) \circ (z * w) \sim (x * z) \circ (y * w)$ (monotonicity of $\circ$)

iff $(x * y) * (z * w) \sim (x * z) * (y * w)$ (monotonicity of θ and $\theta(x \circ y) = x * y$). □

Definition 3.3. Suppose $\mathcal{X} = \langle X, \succsim, * \rangle$ is an intensive structure with a doubling function δ. φ is said to be a $\circledcirc$*-representation* for $\mathcal{X}$ if and only if φ is a function from X into Re^+ and $\circledcirc$ is a partial binary operation on Re^+ with half elements (let h denote the $\circledcirc$-half element function) such that the following three conditions are true for each x, y in X:

(i) $x \succsim y$ iff $\varphi(x) \geq \varphi(y)$;
(ii) $\varphi(x * y) = h[\varphi(x) \circledcirc \varphi(y)]$;
(iii) $\varphi(x) = h\varphi\delta(x)$ if x is in the domain of δ. □

Theorem 3.2. *Let $\mathcal{X} = \langle X, \succsim, * \rangle$ be an intensive structure with doubling function δ. Then there exist φ and $\circledcirc$ such that φ is a $\circledcirc$-representation. Moreover, if ψ is another $\circledcirc$-representation such that for some u in X $\psi(u) = \varphi(u)$, then $\psi = \varphi$.*

Proof. By Theorem 3.1, $x \circ y = \delta(x * y)$ defines a positive concatenation structure $\langle X, \succsim, \circ \rangle$ and by Theorem 2.1 there is a numerical operation $\circledcirc$ and a function φ that is a $\circledcirc$-representation of $\langle X, \succsim, \circ \rangle$. We show this is also a $\circledcirc$-representation of the intensive structure by proving (i)–(iii) of Definition 3.3. (i) holds in both structures. (ii) Since

$$x \circ y \sim \delta^{-1}(x \circ y) \circ \delta^{-1}(x \circ y) \sim (x * y) \circ (x * y),$$

then

$$\varphi(x * y) \circledcirc \varphi(x * y) = \varphi(x \circ y) = \varphi(x) \circledcirc \varphi(y),$$

whence

$$\varphi(x * y) = h[\varphi(x) \circledcirc \varphi(y)].$$

(iii) Since

$$\varphi\delta(x) = \varphi\delta(x * x) = \varphi(x \circ x) = \varphi(x) \circledcirc \varphi(x),$$

so

$$h\varphi\delta(x) = \varphi(x).$$

If φ and ψ are two such functions with $\varphi(u) = \psi(u)$, then using properties (ii) and (iii),

$$\varphi(x \circ y) = \varphi\delta(x * y) = \varphi(x * y) \circledcirc \varphi(x * y)$$
$$= h[\varphi(x) \circledcirc \varphi(y)] \circledcirc h[\varphi(x) \circledcirc \varphi(y)] = \varphi(x) \circledcirc \varphi(y).$$

Similarly, $\psi(x \circ y) = \psi(x) \circledcirc \psi(y)$. Thus, φ and ψ are both $\circledcirc$-representations of $\langle X, \succsim, \circ \rangle$ and so, by Theorem 2.2, $\varphi = \psi$. □

We next turn to the question, which is not fully answered, about the relation between doubling functions of the same intensive structure.

Theorem 3.3. *Suppose* $\langle X, \succsim, * \rangle$ *is an intensive concatenation structure with a doubling function* δ *and there exists* f *from* X *onto* X *such that*

(i) f *is strictly increasing;*

(ii) *if* $x * y$ *is defined, then* $f(x) * f(y)$ *is defined and*

$$f(x * y) = f(x) * f(y);$$

then $f^{-1}\delta f$ *is also a doubling function.*

Proof. We show $\delta' = f^{-1}\delta f$ is a doubling function: Let A, A' be the domains of δ, δ'.

(i) δ' is monotonic because f, f^{-1}, and δ are.

(ii) Suppose $x \succsim y$ and x is in A'. Since $f\delta' = \delta f$, we see $f(x)$ is in A, whence $f(y)$ is A and so y is in A'.

(iii) If $x \succ y$, then $f(x) \succ f(y)$, and so there is u such that $f(x) \succ \delta[f(x) * f(u)] = \delta[f(x * u)]$. Taking inverses $x \succ \delta'(x * u)$.

(iv) $\delta'(x * y) = f^{-1}\delta f(x * y) = f^{-1}\delta[f(x) * f(y)] \succ f^{-1}f(x), f^{-1}f(y) = x, y$.

(v) If x_n is a standard sequence of δ', we show $f(x_n)$ is one of δ. $x_n = \delta'(x_{n-1}) * x$. So

$$f(x_n) = f[\delta'(x_{n-1}) * x] = f\delta'(x_{n-1}) * f(x)$$

$$= ff^{-1}\delta f(x_{n-1}) * f(x) = \delta f(x_{n-1}) * f(x).$$

And so property (v) of Definition 3.2 holds for δ' because it does for δ. □

Theorem 3.3 fails to characterize the non-uniqueness of the doubling functions. We conjecture that the necessary and sufficient conditions for δ and δ' both to be doubling functions of the same intensive structure is the existence of an automorphism f of that structure such that $\delta' = f^{-1}\delta f$. This conjecture can be recast as a conjecture about either the relation between the two induced positive concatenation structures or the existence of a solution to a functional equation arising from the numerical representations of these positive concatenation structures.

First, let $\circ$ and $\circ'$ be the concatenation operations induced by $*$ through δ and δ', respectively. We observe that $\circ$ and $\circ'$ are constrained by the following important property: if all of the following concatenations are defined, then for all x, y, u, v in X,

$$x \circ y \sim u \circ v \quad \text{iff} \quad x \circ' y \sim u \circ' v. \tag{3.1}$$

This follows immediately from the fact $x \circ y \sim \delta(x * y)$ and $x \circ' y \sim \delta'(x * y)$. Moreover, if the original conjecture is correct that there is an automorphism f such that $\delta' = f^{-1}\delta f$, then it is easy to see that for all x, y in X for which the concatenations are defined,

$$f(x \circ' y) = f(x) \circ f(y). \tag{3.2}$$

So the question can be cast as: suppose $\langle X, \succsim, \circ \rangle$ and $\langle X, \succsim, \circ' \rangle$ are two positive concatenation structures satisfying eq. (3.1), does there then exist a function f such that eq. (3.2) holds?

Second, let φ, $\circledcirc$ and φ', $\circledcirc'$ be the numerical representations of Theorem 2.1 corresponding to $\circ$ and $\circ'$, respectively. Define $\circledcirc''$ by

$$\alpha \circledcirc'' \beta = \varphi'\varphi^{-1}(\alpha) \circledcirc' \varphi'\varphi^{-1}(\beta).$$

Then, as is easily shown, eq. (3.1) reduces to the assertion that $\circledcirc$ and $\circledcirc''$ have the same indifference curves. And eq. (3.2) translates into the existence of a numerical function $g = (\varphi' f \varphi^{-1})$ such that

$$g(\alpha \circledcirc \beta) = g(\alpha) \circledcirc' g(\beta).$$

We are not aware of any analysis of this functional equation except when $\circledcirc$ and $\circledcirc'$ are associative.

4. Local conjoint structures

The literature on conjoint structures has to date been concerned with weak orders on Cartesian products. Krantz et al. [4, p.275] noted that in practice a somewhat less restrictive concept is needed. The one given below attempts to capture that a preference ordering on the Cartesian product need only hold for pairs of elements that are comparable with the minimal element.

Definition 4.1. $\mathcal{C} = \langle X \times P, \succsim, ab \rangle$ is said to be a *local conjoint structure* (*with an identity element ab*) if and only if $\succsim$ is a binary relation on $X \times P$, $ab \in X \times P$, and the following eight axioms hold for all x, y, z in X and all p, q, r in P:

1. *Transitivity*: if $xp \succsim yq$ and $yq \succsim zr$, then $xp \succsim zr$.
2. *Local connectivity*: either $xp \succsim yq$ or $yq \succsim xp$ if and only if $xp \succsim ab$ and $yq \succsim ab$.
3. *Independence*: (i) if $xp \succsim ab$ and, for some s, $xs \succsim ys$, then $xp \succsim yp$; and (ii) if $xp \succsim ab$ and, for some w, $wp \succsim wq$, then $xp \succsim xq$.
4. *Component definability*: $xb \succsim ab$ and $ap \succsim ab$.
5. *Nontriviality*: there exists w such that $wb \succ ab$.
6. *Partial solvability*: (i) if $yq \succsim ab$, then there exists w such that $wb \sim yq$; and (ii) there exists t such that $xb \sim at$.
7. *Density*: if $xb \succ yq$, then for some s, $xb \succ ys \succ yq$.
8. *Archimedean*: for some $n \in I^+$, either $(nx)b \succsim yb$ or nx is not defined, where mx is defined inductively as follows: $1x = x$, and if mx is defined and s, w are such that $xb \sim as$ and $(mx)s \sim wb$, then $(m+1)x$ is some element u of X such that $wb \sim ub$, and otherwise $(m+1)x$ is not defined.

$\mathcal{C}$ is said to be a *local conjoint structure with half elements* if $\mathcal{C}$ also satisfies the following axiom:

9. *Half elements*: for each x in X there exist w, s such that $ws \sim xb$ and $wb \sim as$. □

Definition 4.2. Let $\mathcal{C} = \langle X \times P, \succsim, ab\rangle$ be a local conjoint structure with an identity element ab. Define $\succsim_X$ on X and $\succsim_P$ on P as follows: for each x, y in X,

$$x \succsim_X y \quad \text{iff} \quad xb \succsim yb\,;$$

and for each q, r in P,

$$q \succsim_P r \quad \text{iff} \quad aq \succsim ar\,.$$

It is easy to show that $\succsim_X$ and $\succsim_P$ are weak orderings. By partial solvability, let ξ be a function on $D = \{xp \mid xp \succsim ab\}$ such that for each xp in D, $xp \sim \xi(xp)b$. By partial solvability and component definability, let σ be a function on X such that for each $x \in X$, $xb \sim a\sigma(x)$. Let $\oplus$ be the binary partial operation defined on X as follows: For each x, y in X,

(i) $x \oplus y$ is defined iff $x\sigma(y) \succsim ab$,

and

(ii) if $x \oplus y$ is defined, then $x \oplus y = \xi(x\sigma(y))$. □

Lemma 4.1. *Let $\mathcal{C} = \langle X \times P, \succsim, ab\rangle$ be a local conjoint structure. Then the following three statements are true for each x, y in X and each p, q in P:*

(1) $xp \succsim yq$ *iff* $\xi(xp) \succsim_X \xi(yq)$;

(2) $x \succsim_X y$ *iff* $\sigma(x) \succsim_P \sigma(y)$;

(3) $\sigma(\xi(ap)) \sim_P p$.

Proof. Left to reader. □

Definition 4.3. Let $\mathcal{C} = \langle X \times P, \succsim, ab\rangle$ be a local conjoint structure with an identity ab. $\mathcal{Y} = \langle X^+, \succsim', \oplus'\rangle$ is said to be *the partial operation structure induced by* $\mathcal{C}$ if and only if $X^+ = \{x \in X \mid x \succ_X a\}$, $\succsim'$ is the restriction of $\succsim_X$ to X^+, and $\oplus'$ is the restriction of $\oplus$ to $X^+ \times X^+$. If $\mathcal{Y}$ is a partial operation structure induced by $\mathcal{C}$ that is also a positive concatenation structure, then $\mathcal{Y}$ is said to be *the positive concatenation structure induced by* $\mathcal{C}$. □

Theorem 4.1. *Let $\mathcal{C} = \langle X \times P, \succsim, ab\rangle$ be a local conjoint structure and let $\mathcal{Y} = \langle X^+, \succsim', \oplus'\rangle$ be the partial operation structure induced by $\mathcal{C}$. Then $\mathcal{Y}$ is a positive concatenation structure. Furthermore, if $\mathcal{C}$ has half elements, then $\mathcal{Y}$ has half elements.*

Proof. We will show that axioms 1–7 of Definition 2.1 hold for $\mathcal{Y}$.

1. Since $\succsim_X$ is a weak ordering on X, $\succsim'$ is a weak ordering on X^+.

2. Since $\mathcal{C}$ is nontrivial, let x be such that $xb \succ ab$. By density, let t be such that $xb \succ at \succ ab$. Then

$$xb \succ at \sim \xi(at)b \succ ab\,,$$

and thus by the definition of $\succ'$, $x \succ' \xi(at) \succ_X a$. Therefore $\mathcal{Y}$ is nontrivial.

3. Suppose that $x \circledcirc' y$ is defined, $x \succsim' w$, and $y \succsim' z$. Then by Lemma 4.1,

$$(x \circledcirc' y)b \sim x\sigma(y) \succsim x\sigma(z) \succsim w\sigma(z)\,,$$

and so $w \circledcirc' z = \xi(w\sigma(z))$ is defined.

(4). (i) Suppose that $x \circledcirc' z$ and $y \circledcirc' z$ are defined. Then

$$x \succsim' y \quad \text{iff} \quad x \succsim_X y \quad \text{iff} \quad x\sigma(z) \succsim y\sigma(z)$$
$$\text{iff} \quad \xi(x\sigma(z)) \succsim_X \xi(y\sigma(z)) \quad \text{iff} \quad x \circledcirc' z \succsim' y \circledcirc' z.$$

(ii) Suppose that $z \circledcirc' x$ and $z \circledcirc' y$ are defined. Then

$$x \succsim' y \quad \text{iff} \quad x \succsim_X y \quad \text{iff} \quad \sigma(x) \succsim \sigma(y) \quad \text{iff} \quad z\sigma(x) \succsim z\sigma(y)$$
$$\text{iff} \quad \xi(z\sigma(x)) \succsim_X \xi(z\sigma(y)) \quad \text{iff} \quad z \circledcirc' x \succsim' z \circledcirc' y.$$

Thus $\mathcal{Y}$ satisfies monotonicity.

5. Suppose that $x \succ' y$. Then $x \succ_X y$, and thus $xb \succ yb$. By density, let t be such that $xb \succ yt \succ yb$. Then $xb \succ at$. Let $u = \xi(at)$. Then by Lemma 4.1, $\sigma(u) \sim_P t$, and so

$$x \sim' \xi(xb) \succ' \xi(yt) \sim' \xi(y\sigma(u)) \sim' y \circ' u\,.$$

Thus $\mathcal{Y}$ satisfies restricted solvability.

6. Suppose that $x \circledcirc' y$ is defined. Then $x\sigma(y) \succ xb$, and thus by Lemma 4.1,

$$x \circledcirc' y = \xi(x\sigma(y)) \succ_X \xi(xb) \sim x\,,$$

and similarly,

$$x \circledcirc' y \succ_X y\,.$$

Therefore $\mathcal{Y}$ satisfies positivity.

7. Since $\mathcal{C}$ is Archimedean, it follows immediately that $\mathcal{Y}$ is Archimedean.

Suppose that $\mathcal{C}$ has half elements and x is an element of X^+. Let w, t be such that $wt \sim xb$ and $wb \sim at$. From the latter, $\sigma(w) \sim_P t$, and so $w\sigma(w) \sim xb$. Therefore,

$$w \circledcirc' w = \xi(w\sigma(w)) \sim' \xi(wt) \sim' \xi(xb) \sim' x\,.$$

Thus $\mathcal{Y}$ has half elements. □

Definition 4.4. Let $\mathcal{C} = \langle X \times P, \succsim, ab\rangle$ be a local conjoint structure. Then $\langle \varphi, \psi\rangle$ is said to be a $\circledcirc$*-representation for* $\mathcal{C}$ if and only if $\circledcirc$ is a partial binary operation on Re, $\varphi\colon X \to \text{Re}$, $\psi\colon P \to \text{Re}$, and the following three conditions hold for all x, y in X and all p, q in P:

(1) $\varphi(a) \circledcirc \psi(p) = \psi(p)$.

(2) $\varphi(x) \circledcirc \psi(b) = \varphi(x)$.

(3) $xp \succsim yq$ iff $xp \succsim ab$, $yq \succsim ab$, and $\varphi(x) \circledcirc \psi(p) \geq \varphi(y) \circledcirc \psi(q)$. □

Theorem 4.2. *Let* $\mathcal{C} = \langle X \times P, \succsim, ab\rangle$ *be a local conjoint structure. Then for some* $\circledcirc$, *there exists a $\circledcirc$-representation for $\mathcal{C}$. Furthermore, for each $\circledcirc$, $\circledcirc'$, φ, ψ, ψ', if $\langle\varphi, \psi\rangle$ is a $\circledcirc$-representation for $\mathcal{C}$ and $\langle\varphi, \psi'\rangle$ is a $\circledcirc'$-representation for $\mathcal{C}$, then $\psi = \psi'$ and for all $xp \succsim ab$, $\varphi(x) \circledcirc \psi(p) = \varphi(x) \circledcirc' \psi'(p)$.*

Proof. By Lemma 2.2 and Theorem 4.1, $\langle X, \succsim_X\rangle$ has a countable dense subset. Thus by a well-known theorem of Cantor (see Theorem 2.2 of Krantz et al. [4]), let φ be a function from X into Re such that for each x, y in X, $\varphi(x) \geq \varphi(y)$ iff $x \succsim_X y$. For each $p \in P$, let $\psi(p) = \varphi(\xi(ap))$. Let $\circledcirc$ be the partial binary operation on Re defined by

$$r \circledcirc s = w \quad \text{iff for some } xp \succsim ab,\ r = \varphi(x),$$
$$s = \psi(p), \text{ and } w = \varphi(\xi(xp)).$$

Then for each x, y in X and each p, q in P,

$$\begin{aligned} xp \succsim yq \quad &\text{iff} \quad \xi(xp) \succsim_X \xi(yq) \\ &\text{iff} \quad \varphi(\xi(xp)) \geq \varphi(\xi(yq)) \\ &\text{iff} \quad \varphi(x) \circledcirc \psi(p) \geq \varphi(y) \circledcirc \psi(q). \end{aligned}$$

Thus $\langle\varphi, \psi\rangle$ is a $\circledcirc$-representation for $\mathcal{C}$.

Suppose that $\langle\varphi, \psi\rangle$ is a $\circledcirc$-representation for $\mathcal{C}$ and that $\langle\varphi, \psi'\rangle$ is a $\circledcirc'$-representation for $\mathcal{C}$. Let $p \in P$. Then $\xi(ap)b \sim ap$, and thus

$$\varphi(\xi(ap)) \circledcirc \psi(b) = \varphi(a) \circledcirc \psi(p),$$

that is,

$$\varphi(\xi(ap)) = \psi(p).$$

Similarly,

$$\varphi(\xi(ap)) = \psi'(p).$$

Since p is an arbitrary element of P, $\psi = \psi'$. Now suppose that $xq \succsim ab$. Then

$$\varphi(\xi(xq)) = \varphi(x) \circledcirc \psi(q) = \varphi(x) \circledcirc' \psi'(q) = \varphi(x) \circledcirc' \psi(q). \ \square$$

The following definition formulates a sufficient condition for $\circ$ to be associative. The proofs of Theorems 4.2 and 4.3 utilize concepts developed in Holman [3]. Theorem 4.3 is similar to theorems of Luce and Tukey [9] and Luce [6], but uses somewhat different assumptions; in particular, different solvability conditions are assumed and $\succsim$ need not be defined for large elements of $X \times P$, *i.e.*, $xy \succsim ab$ need not hold for all xy in $X \times P$.

Definition 4.5. A local conjoint structure $\mathcal{A} = \langle X \times P, \succsim, ab\rangle$ is said to be *additive* if and only if $\mathcal{A}$ is a local conjoint structure and the following two axioms hold:

The Thomsen condition: For each x, y, z in X and each p, q, r in P, if $xp \sim yq$ and $yr \sim zp$, then $xr \sim zq$.

Unboundedness: For each xp in $X \times P$, there exists yq such that $yq \succ xp$.

Theorem 4.3. *Let $\mathcal{A} = \langle X \times P, \succsim, ab \rangle$ be an additive local conjoint structure and $\mathcal{Y} = \langle X^+, \succsim', \oplus' \rangle$ be the positive concatenation structure induced by $\mathcal{A}$. Then for each x, y, z in X^+,*

(i) *if $x \oplus' y$ is defined, then $y \oplus' x$ is defined and $x \oplus' y \sim' y \oplus' x$ (commutativity) and*

(ii) *if $x \oplus' (y \oplus' z)$ is defined, then $(x \oplus' y)$ is defined and $x \oplus' (y \oplus' z) \sim' (x \oplus' y) \oplus' z$ (associativity).*

Proof. (i) Suppose that $x \oplus' y$ is defined. We will show that $x\sigma(y) \sim y\sigma(x)$ and then $x \circ' y \sim y \circ' x$. Since $a\sigma(y) \sim yb$ and $xb \sim a\sigma(x)$, from the Thomsen condition it follows that $x\sigma(y) \sim y\sigma(x)$. Thus

$$x \oplus' y = \xi(x\sigma(y)) \sim' \xi(y\sigma(x)) = y \oplus' x.$$

(ii) Suppose that $x \oplus' (y \oplus' z)$ is defined. Since $y \oplus' z \succ' y$, $x \oplus' y$ is defined. Since

$$y\sigma(x) \sim (x \oplus' y)b$$

and

$$(y \oplus' z)b \sim y\sigma(z),$$

it follows from the Thomsen condition that

$$(y \oplus' z)\,\sigma(x) \sim (x \oplus' y)\,\sigma(z).$$

Since by part (i) of this proof,

$$(y \oplus' z)\,\sigma(x) \sim x\sigma(y \oplus' z),$$

it follows that

$$x\sigma(y \oplus' z) \sim (x \oplus' y)\,\sigma(z),$$

and thus

$$x \oplus' (y \oplus' z) = \xi(x\sigma(y \oplus' z)) \sim' \xi((x \oplus' y)\,\sigma(z)) = (x \oplus' y) \oplus' z. \quad \square$$

Theorem 4.4. *Let $\mathcal{A} = \langle X \times P, \succsim, ab \rangle$ be an additive local conjoint structure. Then there exist real valued functions φ on X and ψ on P such that for each xp, yq in $X \times P$,*

(1) $\varphi(a) = \psi(b) = 0$,

(2) *if $xp \succsim yq$, then $\varphi(x) + \psi(p) \geq \varphi(y) + \psi(q)$,*

and

(3) *if $xp \succsim ab$, $yq \succsim ab$, and $\varphi(x) + \psi(p) \geq \varphi(y) + \psi(q)$, then $xp \succsim yq$.*

Furthermore, if φ', ψ' are another pair of real valued functions on X, P respectively such that (1), (2), and (3) above hold and such that for some $u \in X^+$, $\varphi(u) = \varphi'(u)$, then $\varphi = \varphi'$ and $\psi = \psi'$.

Proof. *Existence.* Let $\mathcal{Y} = \langle X^+, \succsim', \oplus' \rangle$ be the positive concatenation structure induced by $\mathcal{C}$. By Theorem 4.3, $\mathcal{Y}$ is associative. Since by assumption $\mathcal{C}$ is unbounded, $\mathcal{Y}$ is unbounded. Thus $\mathcal{Y}$ is an extensive structure, and by Theorem 2.3, let φ_1 be an additive representation for $\mathcal{Y}$. Extend φ_1 to X as follows: let φ: $X \to \mathrm{Re}$ be such for all $x \in X^+$, $\varphi(x) = \varphi_1(x)$ and for all $x \sim_X a$, $\varphi(x) = 0$. For each $p \in P$, let $\psi(p) = \varphi(\xi(ap))$. Suppose that $xp \succsim yq$. Let $z = \xi(ap)$ and $w = \xi(aq)$. Then by Lemma 4.1, $\sigma(z) \sim' p$ and $\sigma(w) \sim' q$. Thus

$$x\sigma(z) \sim xp \succsim yq \sim y\sigma(w),$$

and therefore,

$$x \oplus' z = \xi(x\sigma(z)) \sim' \xi(xp) \succsim' \xi(yq) \sim' \xi(y\sigma(w)) \sim' y \oplus' w.$$

Thus

$$\begin{aligned}\varphi(x \oplus' z) &= \varphi(x) + \varphi(z) = \varphi(x) + \varphi(\xi(ap)) = \varphi(x) + \psi(p)\\ &\geqslant \varphi(y \oplus' w) = \varphi(y) + \varphi(w) = \varphi(y) + \varphi(\xi(aq)) = \varphi(y) + \psi(q).\end{aligned}$$

Uniqueness. Suppose that φ, ψ and φ', ψ' are pairs of functions that satisfy (1), (2), and (3), and that $u \in A^+$ is such that $\varphi(u) = \varphi'(u)$. Then by Theorem 2.3, $\varphi = \varphi'$. Let r be an arbitrary element of P. Then $\xi(ar)b \sim ar$. Thus $\varphi(\xi(ar)) + \psi(b) = \varphi(a) + \psi(r)$. Since $\psi(b) = \varphi(a) = 0$, $\varphi(\xi(ar)) = \psi(r)$. Similarly, $\varphi'(\xi(ar)) = \psi'(r)$. Since $\varphi = \varphi'$ and r is an arbitrary element of P, $\psi = \psi'$. □

For later applications, it is convenient to have a form of additive conjoint structures that does not assume the existence of identity elements. To this end, a representation and uniqueness theorem of Luce and Tukey [9] will be stated. The proof of this theorem follows from Theorem 4.2.

Definition 4.6. $\mathcal{A} = \langle X \times P, \succsim\rangle$ is said to be a *solvable additive conjoint structure* if and only if $\succsim$ is a binary relation on $X \times P$ and the following six axioms hold for each x, y in X and each p, q in P:

1. *Weak ordering*: $\succsim$ is transitive and connected.
2. *Independence*: (i) if for some r, $xr \succsim yr$, then for each s in P, $xs \succsim ys$; and (ii) if for some w, $wp \succsim wq$ then for all z in X, $zp \succsim zq$.
3. *Nontriviality*: for some w, z, r, w_1, s, t, $wr \succ zr$ and $w_1 s \succ w_1 t$.
4. *Solvability*: given any three of x_1, y_1 in X and p_1, q_1 in P, the fourth exists such that $x_1 p_1 \sim y_1 q_1$.
5. *Density*: if $xp \succ yq$, then for some s, $xp \succ ys \succ yq$.
6. *Thomsen condition*: for each z in X and r in P, if $xp \sim yq$ and $yr \sim zp$ then $xr \sim zq$.
7. *Archimedean*: for each $x, x_1, x_2, \ldots$ in X, if $xp \succ xq$ and $x_i p \sim x_{i+1} q$ for each $i \in I^+$, then for some j, $x_j p \succ xp$. □

Theorem 4.5. *Suppose that* $\langle X \times P, \succsim\rangle$ *is a solvable additive conjoint structure. Then there exist functions* φ *on* X *and* ψ *on* P *into the reals such that for each* xp, yq *in*

$X \times P$,

$$xp \gtrsim yq \quad \text{iff} \quad \varphi(x) + \psi(p) \geq \varphi(y) + \psi(q)\,. \tag{4.1}$$

Furthermore, if φ' and ψ' are functions on X and P, respectively, that satisfy eq. (4.1), *then for some r in* Re^+ *and some s, t in* Re,

$$\varphi' = r\varphi + s \quad \text{and} \quad \psi' = r\psi + t\,.$$

5. Distributive structures

Given the concepts of conjoint and extensive structures, a natural question, one of considerable importance in physical measurement, is how they relate to one another. The problem was first discussed axiomatically by Luce [5] (for a more comprehensive discussion, see Krantz et al. [4, Chapter 10]) who showed that if $\langle X \times P, \gtrsim \rangle$ is an additive conjoint structure and there are extensive operations on two of the three sets X, P, and $X \times P$ which are related by what he called laws of similitude and/or exchange, then the conjoint representations are power functions of the extensive ones. Later Narens [11] showed, in a special context, that much weaker assumptions are sufficient for the same conclusion. In brief, only one extensive operation is needed provided it exhibits a property called distributivity and, surprisingly, it is not necessary to assume the conjoint structure is additive.

Because the results for distributive structures are important for measurement theory and because proofs of such results may yield insights into new types of measurement structures, we provide two proofs of the main result. The first assumes strong topological (Dedekind completeness) and algebraic conditions which permit a transparent proof using a well-known functional equation. (In section 7 we provide algebraic assumptions that allow measurement structures to be extended to Dedekind complete ones.) The second proof is similar to that used by Narens [11]; it rests heavily on the representation and uniqueness theorems for extensive structures.

Definition 5.1. Let $\gtrsim$ be a binary relation on $X \times P$ and $\circ_P$ a partial operation on P. $\langle X \times P, \gtrsim, \circ_P \rangle$ is a *P-distributive structure* if and only if the following four axioms hold for all $x, y \in X$, $p, q, r, s \in P$:

1. *Weak ordering*: $\gtrsim$ is transitive and connected.

2. *Independence*: (i) If for some $x \in X$, $xp \gtrsim xq$, then for all $y \in X$, $yp \gtrsim yq$; (ii) if for some $p \in P$, $xp \gtrsim yp$, then for all $q \in P$, $xq \gtrsim yq$.

3. $\langle P, \gtrsim_P, \circ_P \rangle$, where $\gtrsim_P$ is as defined in Definition 4.2, is a positive concatenation structure.

4. *Distributivity*: If $p \circ_P q$ and $r \circ_P s$ are defined, $xp \sim yr$, and $xq \sim ys$, then $x(p \circ_P q) \sim y(r \circ_P s)$.

The structure is *solvable* * if given any three of $x, y \in X$, $p, q \in P$, the fourth exists such that $xp \sim yq$.

There is, of course, an analogous definition for $\langle X \times P, \succsim, \circ_X \rangle$ to be X-distributive. □

Definition 5.2. Let $\langle X \times P, \succsim, \circ_P \rangle$ be a solvable P-distributive structure and let $p_0 \in P$. Define the partial operation $\circ_X$ on X by: for $x, y \in X$, if there exists $x_0 \in X$, $r, s \in P$ such that $xp_0 \sim x_0 r$, $yp_0 \sim x_0 s$, and $r \circ_P s$ is defined, then $x \circ_X y$ is a solution to $(x \circ_X y)p_0 \sim x_0(r \circ_P s)$. (Observe that by distributivity this is unique up to $\sim$ and independent of the choice of x_0, r, and s.)

For fixed $x_0 \in X$, $p_0 \in P$, define $\tau: P \to X$ as a solution to $\tau(p)p_0 \sim x_0 p$.

Define Π: $X \times P \to P$ as a solution to $x_0 \Pi(x, p) \sim xp$. □

Lemma 5.1. *Suppose $\langle X \times P, \succsim, \circ_P \rangle$ is a solvable P-distributive structure. Then $\mathcal{X} = \langle X, \succsim_X, \circ_X \rangle$ is an extensive structure if $\langle P, \succsim_P, \circ_P \rangle$ is an extensive structure.*

Proof. Left to the reader. □

Lemma 5.2. *Suppose $\langle X \times P, \succsim, \circ_P \rangle$ is a solvable P-distributive structure. Then for all $x, y \in X$, $p, q \in P$, if $p \circ_P q$ is defined and $x \circ_X y$ is defined, then the right sides of the following expressions are defined and*

(i) $\tau(p \circ_P q) \sim_X \tau(p) \circ_X \tau(q)$,

(ii) $\Pi(x, p \circ_P q) \sim_P \Pi(x, p) \circ_P \Pi(x, q)$,

(iii) $\Pi(x \circ_X y, p_0) \sim_P \Pi(x, p_0) \circ_P \Pi(y, p_0)$.

Proof. By definition of τ,

$$x_0 p \sim \tau(p)p_0 \quad \text{and} \quad x_0 q \sim \tau(q)p_0,$$

whence by the definition of $\circ_X$ and τ,

$$\tau(p \circ_P q)p_0 \sim x_0(p \circ_P q) \sim [\tau(p) \circ_X \tau(q)]p_0.$$

(i) follows by independence.

By the definition of Π,

$$xp \sim x_0 \Pi(x, p) \text{ and } xq \sim x_0 \Pi(x, q).$$

By distributivity and the definition of Π,

$$x_0 \Pi(x, p \circ_P q) \sim x(p \circ_P q) \sim x_0 [\Pi(x, p) \circ_P \Pi(x, q)] \; .$$

(ii) follows by independence.

Let $r \sim_P \Pi(x, p_0)$ and $s \sim_P \Pi(y, p_0)$, then by definition of Π and $\circ_X$, $\Pi(x \circ_X y, p_0) \sim_P r \circ_P s$. □

* Each of the two proofs use a weaker form of solvability; they will be stated explicitly below.

Corollary 5.1. *Define* $\oplus$ *on* P *by*: $p \oplus q = \Pi[\tau(p), q]$. *Then for* $p, q, r \in P$,

$$p \oplus (q \circ_P r) \sim_P (p \oplus q) \circ_P (p \oplus r).$$

Proof. Definition of $\oplus$ and part (ii) of Lemma 5.2. □

Theorem 5.1. *Suppose* $\langle X \times P, \succsim, \circ_P \rangle$ *is a solvable P-distributive structure for which* $\langle P, \succsim_P, \circ_P \rangle$ *is an extensive structure. If* φ_P *is an additive representation of* $\langle P, \succsim_P, \circ_P \rangle$ *and* φ_X *an additive one of* $\langle X, \succsim_X, \circ_X \rangle$ *(see Lemma 5.1), then* $\varphi_X \varphi_P$ *is a multiplicative representation of* $\langle X \times P, \succsim \rangle$.

Proof for the case $\circ_P$ is closed and φ_P is onto the positive reals. In this case it is sufficient to postulate solvability for x_0 and p_0 only.

Observe that by part (iii) of Lemma 5.2, $\varphi_X = \varphi_P \Pi(\ \ , p_0)$ is an additive representation of $\langle X, \succsim_X, \circ_X \rangle$.

Define G on $X \times P$ by

$$G(x, p) = \varphi_P \Pi(x, p).$$

G is order preserving since

$$\begin{aligned} xp \succsim yq \quad &\text{iff} \quad \Pi(x,p) \succsim_P \Pi(y,q) \\ &\text{iff} \quad \varphi_P \Pi(x,p) \geq \varphi_P \Pi(y,q). \end{aligned}$$

By part (ii) of Lemma 5.2,

$$G(x, p \circ_P q) = G(x, p) + G(x, q).$$

Define G' on $\mathrm{Re}^+ \times \mathrm{Re}^+$ by:

$$G'(\alpha, \beta) = G(x, p)$$

if

$$\alpha = \varphi_P \Pi(x, p_0) = \varphi_X(x),$$

and

$$\beta = \varphi_P(p).$$

G' is well defined since if $\alpha = \varphi_X(x')$ and $\beta = \varphi_P(p')$, then $x \sim_X x'$ and $p \sim_P p'$, whence $xp \sim x'p'$. It is defined for all $\alpha, \beta > 0$ since φ_P is onto the positive reals.

It follows immediately that

$$G'(\alpha, \beta + \gamma) = G'(\alpha, \beta) + G'(\alpha, \gamma),$$

and as is well known [1], this means there is a positive function g such that

$$G'(\alpha, \beta) = g(\alpha)\beta,$$

and so

$$G(x, p) = g[\varphi_X(x)]\, \varphi_P(p).$$

This implies the Thompsen condition holds in the conjoint structure.

Finally, we show g is the identity function. Let r and s solve $xp_0 \sim x_0 r$ and $yp_0 \sim x_0 s$, so $(x \circ_X y)p_0 \sim x_0(r \circ_P s)$. Coupling $x_0 p \sim \tau(p)p_0$ with each of these and using the Thompsen condition,

$$xp \sim \tau(p)r, \quad yp \sim \tau(p)s, \quad (x \circ_X y)p \sim \tau(p)\,(r \circ_P s)\,.$$

Thus, by part (ii) of Lemma 5.2,

$$\begin{aligned}\Pi(x \circ_X y, p) &\sim_P \Pi[\tau(p), r \circ_P s]\\ &\sim_P \Pi[\tau(p), r] \circ_P \Pi[\tau(p), s]\\ &\sim_P \Pi(x, p) \circ_P \Pi(y, p)\,.\end{aligned}$$

From this

$$G(x \circ_X y, p) = G(x, p) + G(y, p)\,,$$

and the result follows from the same functional equation argument. □

General proof. Here it is sufficient to assume the following solvability condition: for each x, y, p, q; (i) if $xp \succsim yp$ then for some w, $xw \sim yp$; and (ii) if $xp \succsim xq$ then for some u, $up \sim xq$.

Let p_0, q_0, and r_0 in P be fixed and such that $p_0 = q_0 \circ_P r_0$, and let φ_P be the additive representation of $\langle P, \succsim_P, \circ_P\rangle$ for which $\varphi_P(p_0) = 1$.

For $w \in X$, let $X_w = \{x \mid x \in X \text{ and } w \succsim_X x\}$. Let $\succsim_w$ be the restriction of $\succsim_X$ to X_w, and define $\circ_w$ on X_w as follows: $x \circ_w y \sim_X z$ if $z \in X_w$ and for some $p, q \in P$, with $p_0 \succsim_P p, q$,

$$wp \sim xp_0, \quad wq \sim yp_0,$$

and

$$w(p \circ_P q) \sim zp_0\,.$$

The above form of solvability insures z exists whenever $p \circ_P q \precsim_P p_0$.

It is easy to verify $\langle X_w, \succsim_X, \circ_w\rangle$ is an extensive structure with a maximal element.

Define $\varphi_{X,w}$ as follows. To $x \in X_w$, let

$$\varphi_{X,w}(x) = \varphi_P(q)$$

where q is the solution to $wq \sim xp_0$. Let $R_{X,w}$ be the range of $\varphi_{X,w}$ and R_P that of φ_P. We show there is a partial numerical operation $\circledcirc_w$ on $R_{X,w} \times R_P$ such that

(1) $\alpha \circledcirc_w 1 = 1 \circledcirc_w \alpha = \alpha$,

(2) $xp \succsim yq$ iff $\varphi_{X,w}(x) \circledcirc_w \varphi_P(p) \geq \varphi_{X,w}(y) \circledcirc_w \varphi_P(q)$.

Suppose

$$\alpha \in R_{X,w}, \quad \beta \in R_P, \quad \varphi_{X,w}(x) = \alpha, \quad \varphi_P(p) = \beta.$$

Let $\eta(xp)$ solve $\eta(xp)p_0 \sim xp$, then let

$$\alpha \circledcirc_w \beta = \varphi_{X,w}\eta(x, p)\,.$$

Note that $\alpha = 1$ implies $x \sim_X w$ in which case $\eta(wp)p_0 \sim wp$, so $\varphi_{X,w}\eta(wp) = \varphi_p(p) = \beta$ and $1 \circledcirc_w \beta = \beta$. Similarly, $\alpha \circledcirc_w 1 = \alpha$. The order preserving property follows from

$$\begin{aligned} xp \succsim yp \quad &\text{iff} \quad \eta(xp) \succsim_X \eta(yq) \\ &\text{iff} \quad \varphi_{X,w}\eta(xp) \geq \varphi_{X,w}\eta(yq) \\ &\text{iff} \quad \varphi_{X,w}(x) \circledcirc_w \varphi_P(p) \geq \varphi_{X,w}(y) \circledcirc_w \varphi_P(q)\,. \end{aligned}$$

Now, $\varphi_{X,w}$ is an additive representation of $\langle X_w, \succsim_X, \circ_w \rangle$ because if $wp \sim xp_0$, $wq \sim yp_0$, and $w(p \circ q) \sim (x \circ y)p_0$, we see

$$\varphi_{X,w}(x \circ y) = \varphi_P(p \circ q) = \varphi_P(p) + \varphi_P(q) = \varphi_{X,w}(x) + \varphi_{X,w}(y).$$

However, $\circ_w$ is a subset of $\circ_X$ hence by the uniqueness of additive representations

$$\varphi_{X,w} = \frac{\varphi_X}{\varphi_X(w)}\,.$$

We use this to show that $\circledcirc_w$ is actually multiplication. Consider any z such that $w \succsim_X z$ and any q such that $p_0 \succsim_P q$. Let y be such that $yp_0 \sim zq$; note $z \succsim_X y$.

$$\begin{aligned} \varphi_{X,w}(z) \circledcirc_w \varphi_P(q) &= \varphi_{X,w}(y) \circledcirc_w \varphi_P(p_0) = \varphi_{X,w}(y) \\ &= \varphi_{X,w}(z)\,\varphi_{X,z}(y) \qquad \left(\text{since } \frac{\varphi_X(y)}{\varphi_X(w)} = \frac{\varphi_X(z)\,\varphi_X(y)}{\varphi_X(w)\,\varphi_X(z)}\right) \\ &= \varphi_{X,w}(z)\,[\varphi_{X,z}(y) \circledcirc_z \varphi_P(p_0)] \\ &= \varphi_{X,w}(z)\,[\varphi_{X,z}(z) \circledcirc_z \varphi_P(q)] \\ &= \varphi_{X,w}(z)\,\varphi_P(q)\,. \end{aligned}$$

Finally, we show $\varphi_X\varphi_P$ is order preserving. Suppose $x,y \in X$, $p,q \in P$. Let $w = \max(x,y)$, and p_0 be such that $p_0 \succsim_P \max(p,q)$ and $p_0 \sim r_0 \circ_P s_0$ for some r_0, s_0. By what we have just shown,

$$\begin{aligned} xp \succsim yq \quad &\text{iff} \quad \varphi_{X,w}(x)\varphi_P(p) \geq \varphi_{X,w}(y)\varphi_P(q) \\ &\text{iff} \quad \frac{\varphi_X(x)}{\varphi_X(w)}\varphi_P(p) \geq \frac{\varphi_X(y)}{\varphi_X(w)}\varphi_P(q) \\ &\text{iff} \quad \varphi_X(x)\varphi_P(p) \geq \varphi_X(y)\varphi_P(q)\,. \ \square \end{aligned}$$

It follows immediately from the construction used in the proof of Theorem 5.1 that representations for P-distributive structures have strong uniqueness conditions. . This is explicitly formulated in the following definition and theorem.

Definition 5.3. Let $\mathcal{D} = \langle X \times P, \succsim, \circ_P \rangle$ be a P-distributive structure for which $\langle P, \succsim_P, \circ_P \rangle$ is extensive. Then $\langle \varphi_X, \varphi_P, \circledcirc \rangle$ is said to be a *distributive representation for* $\mathcal{D}$ if and only if the following four conditions hold:

(i) $\varphi_X : X \to \mathrm{Re}^+$;

(ii) φ_P is an additive representation for the extensive structure $\langle P, \succsim_P, \circ_P \rangle$;

(iii) $\circledcirc$ is distributive over +, i.e., for each r, s, t in Re^+, if $r \circledcirc (s + t)$, $r \circledcirc s$, and $r \circledcirc t$ are defined, then $r \circledcirc (s + t) = (r \circledcirc s) + (r \circledcirc t)$;

(iv) for each x, y in X and each p, q in P, $xp \succsim yq$ iff $\varphi_X(x) \circledcirc \varphi_P(p)$ and $\varphi_X(y) \circledcirc \varphi_P(q)$ are defined and

$$\varphi_X(x) \circledcirc \varphi_P(p) \geq \varphi_X(y) \circledcirc \varphi_P(q) . \ \Box$$

Theorem 5.2. *Suppose that* $\mathcal{D} = \langle X \times P, \succsim, \circ_P \rangle$ *is a P-distributive structure for which* $\langle P, \succsim_P, \circ_P \rangle$ *is extensive and* $\langle \varphi_X, \varphi_P, \circledcirc \rangle$, $\langle \varphi'_X, \varphi'_P, \circledcirc' \rangle$ *are distributive representations for* $\mathcal{D}$. *Then there exist* $r, s, t \in \mathrm{Re}^+$ *such that for each* xp, yq *in* $X \times P$,

$$\varphi_X(x) \circledcirc \varphi_P(p) = r\varphi_X(x)\varphi_P(p) ,$$

$$\varphi'_X(x) = s\varphi_X(x) ,$$

and

$$\varphi'_P(p) = t\varphi_P(p) .$$

Proof. Left to reader. $\Box$

We now turn to structures in which the operation is on the cartesian product rather than on one of the components, and we show that it reduces readily to the previous cases.

Definition 5.4. Let $\succsim$ be a binary relation and $\circ$ a partial operation on $X \times P$. $\langle X \times P, \succsim, \circ \rangle$ is a *distributive structure* if and only if

1. It is a positive concatenation structure.
2. It satisfies independence (Axiom 2, Definition 5.1).
3. For all $x, y \in X$, $p, q, r \in P$, whenever the operations are defined,

$$(xp) \circ (xq) \sim xr \quad \text{iff} \quad (yp) \circ (yq) \sim yr .$$

Define $\circ_P$ on P by:

$$p \circ_P q = r \text{ if for some } x, \text{ hence for any } x, \ (xp) \circ (xq) \sim xr. \ \Box$$

Theorem 5.3. *If* $\langle X \times P, \succsim, \circ \rangle$ *is a solvable distributive structure, then* $\langle X \times P, \succsim, \circ_P \rangle$ *is a solvable P-distributive one; if the former is extensive, then* $\langle P, \succsim, \circ_P \rangle$ *is extensive.*

Proof. We leave it to the reader to prove that $\langle P, \succsim_P, \circ_P \rangle$ is a positive concatenation structure.

To show distributivity, suppose $xp \sim x'p'$ and $xq \sim x'q'$. If $p \circ_P q$ is defined, then by the monotonicity of $\circ$,

$$x(p \circ_P q) \sim (xp) \circ (xq) \sim (x'p') \circ (x'q') \sim x'(p' \circ_P q')$$

using solvability.

To complete the proof we must show that $\circ_P$ is associative when $\circ$ is. Let

$$s = (p \circ_P q) \circ_P r \quad \text{and} \quad s' = p \circ_P (q \circ_P r)\,.$$

Then,

$$xs \sim [x(p \circ_P q)] \circ (xr) \sim [(xp) \circ (xq)] \circ (xr)$$
$$\sim (xp) \circ [(xq) \circ (xr)] \sim (xp) \circ [x(q \circ_P r)] \sim xs'\,.$$

So, by independence $s \sim_P s'$. □

Finally, consider a structure $\langle X \times P, \succsim \rangle$ that has at least two of the following three operations: $\circ$ on $X \times P$ that is distributive, $\circ_X$ on X that is X-distributive, and $\circ_P$ on P that is P-distributive. According to the proof of Theorem 5.3, $\circ$ induces such operations on both X and P, so there is no loss in generality in assuming just $\circ_X$ and $\circ_P$. Assume that the hypotheses of Theorem 5.1 hold for both X and P. Then we know there exist additive representations of $\circ_X$ and $\circ_P$, φ_X and φ_P, and order preserving functions ψ_X and ψ_P such that both

$$\varphi_X \psi_P \quad \text{and} \quad \psi_X \varphi_P$$

preserve the order $\succsim$. By the uniqueness part of the additive conjoint representation (Theorem 4.4),

$$\psi_X = \alpha_X \varphi_X^{\beta} \quad \text{and} \quad \psi_P = \alpha_P \varphi_P^{1/\beta}.$$

Thus, the general form of the multiplicative representation must be

$$\alpha \varphi_X^{\beta_X} \varphi_P^{\beta_P}\,,$$

which is the structure of most measurement in classical physics. It is this that makes the units of all measures expressible as products of powers of a set of basic units of extensive measures.

Certain important cases are not, however, encompassed by these results. One, which we treat more fully in the next section, is relativistic velocity. If s, v, and t are the usual measures of distance, velocity, and time, they relate multiplicatively as $s = vt$. But v is not additive over the obvious concatenation $\circ_V$ of moving frames of reference; in fact,

$$v(x \circ_V y) = \frac{v(x) + v(y)}{1 + v(x)v(y)/v(c)^2}$$

where c denotes light. Thus if we let V be the set of velocities that are *less than* light, T the set of times, $\circ_T$ the usual concatenation operation on time, and $\succsim$ the usual ordering on distance, then $\langle V \times T, \succsim, \circ_T \rangle$ is T-distributive and $\langle V \times T, \succsim, \circ_V \rangle$ is not V-distributive.

6. Relativistic velocity

In this section, a simultaneous axiomatization of distance, time, and relativistic velocity is given. This axiomatization is a modification of Luce and Narens [8], and T-distributivity plays a major role.

In what follows, V denotes a set of (qualitative) velocities, T a set of (qualitative) times, and $V \times T$ a set of (qualitative) distances.

Definition 6.1. $\mathcal{V} = \langle V \times T, \succsim, \circ, \circ_V, \circ_T \rangle$ is said to be a *velocity structure* if and only if $\circ$, $\circ_V$, and $\circ_T$ are closed operations on $V \times T$, V, and T respectively, and the following three conditions hold:

1. $\langle V \times T, \succsim, \circ_T \rangle$ is a solvable T-distributive structure for which $\langle T, \succsim_T, \circ_T \rangle$ is an extensive structure.
2. $\langle V \times T, \succsim, \circ \rangle$ is an extensive structure.
3. For each v in V and each t, t' in T,

$$v(t \circ_T t') \sim (vt) \circ (vt'). \quad \square$$

Convention. Throughout the rest of this section let $\mathcal{V} = \langle V \times T, \succsim, \circ, \circ_V, \circ_T \rangle$ be a velocity structure and $\succsim_V$ and $\succsim_T$ be the weak orderings induced by $\succsim$ on V and T respectively. By Theorem 5.1 let φ_V and φ_T be functions on V and T respectively such that φ_T is an additive representation for $\langle T, \succsim_T, \circ_T \rangle$ and for each vt, $v't'$ in $V \times T$, $vt \succsim v't'$ iff $\varphi_V(v)\varphi_T(t) \geq \varphi_V(v')\varphi_T(t')$. Let $\varphi = \varphi_V \cdot \varphi_T$.

Lemma 6.1. *φ is an additive representation for $\langle V \times T, \succsim, \circ \rangle$.*

Proof. Suppose that vt, $v_1 t_1$ are arbitrary elements of $V \times T$. By Theorem 5.1,

$$vt \succsim v_1 t_1 \quad \text{iff} \quad \varphi_V(v)\varphi_T(t) \geq \varphi_V(v_1)\varphi_T(t_1).$$

Let t' be such that $v_1 t_1 \sim vt'$. Then

$$(vt) \circ (v_1 t_1) \sim (vt) \circ (vt') \sim v(t \circ_T t'),$$

and thus

$$\begin{aligned}\varphi((vt) \circ (v_1 t_1)) &= \varphi(v(t \circ_T t')) = \varphi_V(v)\varphi_T(t \circ_T t') \\ &= \varphi_V(v)(\varphi_T(t) + \varphi_T(t')) = \varphi_V(v)\varphi_T(t) + \varphi_V(v)\varphi_T(t') \\ &= \varphi(vt) + \varphi(vt') = \varphi(vt) + \varphi(v_1 t_1). \quad \square\end{aligned}$$

Definition 6.2. Let c be an element of V. For all v in V and t in T, define $\tau_c(v, t)$ to be a solution to

(6.1) $c\tau_c(v, t) \sim vt$.

If c is interpreted as light, then $\tau_c(v, t)$ is the time required for light to transverse the distance that the velocity v does in time t.] For all u, v in V and t in T, define $\tau(u, v, t)$ to be a solution to

(6.2) $(u \circ_V v)\tau(u, v, t) \sim (ut) \circ (vt)$.

[$\tau(u, v, t)$ is the time it takes the velocity $u \circ_V v$ to travel the distance which is the concatenation of the distance that u travels in time t with the distance that v travels in time t.] □

Lemma 6.2. *For all c, u, v in V and t in T,*

$$\tau_c(u, \tau_c(v, t)) \sim_T \tau_c(v, \tau_c(u, t)) .$$

Proof. Since

$$\varphi_V(u)\varphi_T(t) = \varphi_V(c)\varphi_T(\tau_c(u, t))$$

and

$$\varphi_V(c)\varphi_T(\tau_c(v, t)) = \varphi_V(v)\varphi_T(t) ,$$

it follows that

$$\varphi_V(u)\varphi_T(\tau_c(v, t)) = \varphi_V(v)\varphi_T(\tau_c(u, t)) ,$$

and thus

$$u\tau_c(v, t) \sim v\tau_c(u, t) .$$

Therefore

$$c\tau_c(u, \tau_c(v, t)) \sim u\tau_c(v, t) \sim v\tau_c(u, t) \sim c\tau_c(v, \tau_c(u, t)). \ \square$$

Definition 6.3. $\mathcal{V}$ is said to be *classical* if and only if for each u, v in V and t in T,

$$\tau(u, v, t) \sim_T t.$$

$\mathcal{V}$ is said to be *relativistic with respect to* c in V, if and only if for all u, v in V and t in T,

(6.3) $\tau(u, v, t) \sim_T \tau_c(u, \tau_c(v, t)) \circ_T t$. □

The following theorem is immediate:

Theorem 6.1. *$\mathcal{V}$ is classical if and only if for each u, v in V,*

$$\varphi_V(u \circ_V v) = \varphi_V(u) + \varphi_V(v) .$$

Theorem 6.2. *For c in V, $\mathcal{V}$ is relativistic with respect to c if and only if for all u, v* *in V*,

$$\varphi_V(u \circ_V v) = \frac{\varphi_V(u) + \varphi_V(v)}{1 + \varphi_V(u)\varphi_V(v)/\varphi_V(c)^2}. \tag{6.4}$$

Proof. Eq. (6.1) is equivalent to

$$\varphi_V(c)\varphi_T(\tau_c(v, t)) = \varphi_V(v)\varphi_T(t),$$

and eq. (6.2) to

$$\begin{aligned}\varphi_V(u \circ_V v)\varphi_T[\tau(u, v, t)] &= \varphi[(ut) \circ (vt)] \\ &= \varphi(ut) + \varphi(vt) \\ &= \varphi_V(u)\varphi_T(t) + \varphi_V(v)\varphi_T(t).\end{aligned}$$

Thus

$$\varphi_V(u \circ_V v) = \frac{[\varphi_V(u) + \varphi_V(v)]\varphi_T(t)}{\varphi_T[\tau(u, v, t)]}.$$

But eq. (6.3) is equivalent to

$$\begin{aligned}\varphi_T[\tau(u, v, t)] &= \varphi_T[\tau_c(u, \tau_c(v, t)) \circ_T t] \\ &= \varphi_T[\tau_c(u, \tau_c(v, t))] + \varphi_T(t) \\ &= \frac{\varphi_V(u)\varphi_T[\tau_c(v, t)]}{\varphi_V(c)} + \varphi_T(t) \\ &= \frac{\varphi_V(u)\varphi_V(v)\varphi_T(t)}{\varphi_V(c)^2} + \varphi_T(t),\end{aligned}$$

and thus eq. (6.4) is equivalent to eq. (6.3). □

Let $\mathcal{V}$ be relativistic. Suppose that ψ_V and ψ_T are functions from V and T respectively into Re^+, ψ_T is additive over $\circ_T$ (i.e., for all t, t' in T, $\psi_T(t \circ_T t') = \psi_T(t) + \psi_T(t')$), and $\odot$ is such that for all x, y, z in Re^+ and all $vt, v't'$ in $V \times T$,

$$x \odot (y + z) = (x \odot y) + (x \odot z),$$

and

$$vt \succsim v't' \quad \text{iff} \quad \psi_V(v) \odot \psi_T(t) \geq \psi_V(v') \odot \psi_T(t').$$

Then by Theorem 5.2, there exists r in Re^+ such that $\psi_V = r\varphi_V$. Thus for each u, v in V,

$$\psi_V(u \circ_V v) = \frac{\psi_V(u) + \psi_V(v)}{1 + \psi_V(u)\psi_V(v)/\psi_V(c)^2}.$$

7. Dedekind complete structures

Quite often measurement theorists include topological assumptions in their axiomatizations of empirical settings. In these axiomatizations, the assumptions can be divided into two types: (1) relational (algebraic, first-order) axioms and (2) topological axioms. The topological axioms are usually equivalent to Dedekind completeness. Several other measurement theorists have insisted on only using algebraic assumptions. These axiomatizations can also be divided into two types of assumptions: (1) relational (algebraic, first-order) axioms and (2) Archimedean axioms. These Archimedean axioms are usually similar to our formulation of the notion (Definitions 2.1 and 4.1) but may vary in their formulation from situation to situation. (For a discussion of what an "Archimedean axiom" is see Narens [11].) Topological axiomatizations usually yield briefer and more transparent proofs than their algebraic counterparts, which is only natural since topological axioms are more powerful assumptions than are Archimedean axioms: *in all known relevant cases, the topological axioms imply the corresponding Archimedean axioms, but the Archimedean axioms do not imply the topological axioms.* It should also be noted that the topological axiomatizations usually assume the relevant operations are closed. In this case, it is often quite easy to reformulate the measurement situation as a problem in functional equations and bring the vast functional equation literature (e.g., Aczél [1]) to bear on the production of the appropriate representation. (This is the approach of Pfangazl [13] and others.) Because of various measurement considerations, several measurement theorists go to great lengths to avoid the assumption that arbitrary concatenations can be formed. It should also be noted a closed operation together with Dedekind completeness allow all sorts of strong solvability conditions to be derived.

Since measurement deals with the assignment of numerical quantities to empirical objects, philosophical reservations about the nature of the characterization of the empirical structure are in order. Although it would be nice to avoid the use of infinity entirely in measurement theory, it is usually a necessary assumption for uniqueness of representations. However, *algebraic axiomatizations are satisfied by denumerable models whereas topological axiomatizations require models of the cardinality of the continuum.* Philosophically, one might accept a denumerable model as an idealization of a large finite model; it is much harder to accept a nondenumerable model as an idealization of any finite process.

The Archimedean and topological assumptions are used in part to guarantee the existence of numerical representations. However, in some measurement situations, Archimedean and therefore topological axioms seem to be inappropriate. The techniques developed in algebraic approaches often allow these situations to be dealt with by giving representations into some richer structure (e.g., the nonstandard reals in Narens [11,12] and vector space-like lexicographic representation in Narens [11]). We are not aware of any comparable results for topological axioms.

Finally, the algebraic techniques that apply to finite empirical structures can often be used to generate representations for infinite structures thus providing a link between the finite and the infinite. Narens [12] has exploited this link to show that in certain cases the unique numerical representation of an infinite structure is approximated by selecting any of the comparatively nonunique numerical representations for each of a sequence of increasingly large, finite substructures.

For a strongly expressed view supporting the introduction of topological axioms into measurement theory, see Ramsey [14].

In this section we will investigate conditions under which a positive concatenation structure can be Dedekind completed. We will basically follow Dedekind's procedure for completing the reals from the rationals. But since we assume neither a closed nor associative operation, the proofs are more subtle. In lieu of closure, we introduce a property called *tightness*, which is satisfied by a closed operation but is much weaker. And as a qualitative condition corresponding to continuity of the operation we introduce *interval solvability*. A tight, Dedekind complete, positive concatenation structure that satisfies interval solvability has half elements (Lemma 7.2) and satisfies a new relational condition called *regularity* (Theorem 7.3). The major significance of the latter two properties is that in a positive concatenation structure they are sufficient to construct a Dedekind completion (Theorem 7.4). We do not know, however, if tightness of the structure implies tightness of the completion, but closedness of the operation is transmitted. Thus, for a closed structure satisfying interval solvability, regularity is necessary and sufficient for the existence of a Dedekind completion. So, in most topological measurement situations, the topological axioms are replaceable by the relational axioms of interval solvability and regularity plus an Archimedean axiom. Finally, the section ends with several unresolved problems.

Definition 7.1. A positive concatenation structure $\langle X, \succsim, \circ\rangle$ satisfies *interval solvability* if and only if for all x, y, z in X, if $x \succ y \succ z$, then there exist u, v in X such that $u \circ z$, $z \circ v$ are defined and $x \succ u \circ z$, $z \circ v \succ y$. □

Theorem 7.1. *Suppose $\mathcal{X} = \langle X, \succsim, \circ\rangle$ is a Dedekind complete, positive concatenation structure without a maximal element. Then there exists a monotonic $\circledcirc$-representation φ of $\mathcal{X}$ that is onto* Re^+. *Interval solvability holds if $\circledcirc$ is continuous.*

Proof. Since $\mathcal{X}$ is unbounded from both above and below (Lemma 2.1), has countable dense subset (Lemma 2.2), and is Dedekind complete, by a well known theorem of set theory, there is an order homomorphism of $\langle X, \succsim\rangle$ onto $\langle \mathrm{Re}^+, \geq\rangle$. For each $r \in \mathrm{Re}^+$, let $\varphi^{-1}(r)$ be an element x in X such that $\varphi(x) = r$. Define the partial binary operation $\circledcirc$ on Re^+ as follows: for each r, s in Re^+, $r \circledcirc s$ is defined if and only if $\varphi^{-1}(r) \circ \varphi^{-1}(s)$ is defined, and if $r \circledcirc s$ is defined then

$$r \circledcirc s = \varphi(\varphi^{-1}(r) \circ \varphi^{-1}(s)) .$$

Then it is easy to show that φ is a $\circledcirc$-representation for $\mathcal{X}$.

Suppose that $r > r'$ and $r \circledcirc s$ is defined. Then by the monotonicity of $\circ$,

$$\begin{aligned} r > r' \quad &\text{iff} \quad \varphi^{-1}(r) \succ \varphi^{-1}(r') \\ &\text{iff} \quad \varphi^{-1}(r) \circ \varphi^{-1}(s) \succ \varphi^{-1}(r') \circ \varphi^{-1}(s) \\ &\text{iff} \quad \varphi(\varphi^{-1}(r) \circ \varphi^{-1}(s)) > \varphi(\varphi^{-1}(r') \circ \varphi^{-1}(s)) \\ &\text{iff} \quad r \circledcirc s > r' \circledcirc s. \end{aligned}$$

Thus $\circledcirc$ is monotonic in the first argument.

Suppose interval solvability holds and $\circledcirc$ is not continuous in the first argument. Then there is a gap such that for some s, t_0, t_1, with $t_0 < t_1$, and for all r for which $r \circledcirc s$ is defined, either $r \circledcirc s \leqslant t_0$ or $r \circledcirc s \geqslant t_1$, and neither set is empty. By positivity, $t_1 > t_0 > s$, and so by interval solvability there is an r such that $t_1 > r \circledcirc s > t_0$, which is a contradiction.

Conversely, suppose $\circledcirc$ is continuous in the first argument and $x \succ y \succ z$. By tightness, there exists u such that $u \circ z \succ x$, so by local definability, for all positive reals $\alpha \leqslant \varphi(u)$, $\alpha \circledcirc \varphi(z)$ exists. By continuity, for some α, $\varphi(x) > \alpha \circledcirc \varphi(z) > \varphi(y)$, and since φ is onto Re^+, interval solvability holds on the left.

The proof for the second argument is similar. □

Definition 7.2. Let $\mathfrak{X} = \langle X, \succsim, \circ \rangle$ be a positive concatenation structure. $\mathfrak{X}$ is said to be *tight* if and only if for all x, y in X if $x \succ y$, then there exist u, v in X such that $u \circ y$ and $y \circ v$ are defined and $u \circ y, y \circ v \succ x$. □

It should be noted that each positive concatenation structure with a closed operation is tight.

Lemma 7.1. *Let $\mathfrak{X} = \langle X, \succsim, \circ \rangle$ be a tight, Dedekind complete, positive concatenation structure, x, z elements of X, and Y a nonempty subset of X with* l.u.b. *$\bar{y}$ and such that for y in Y, $z \succ x \circ y$ ($z \succ y \circ x$). Then $x \circ \bar{y}$ ($\bar{y} \circ x$) is defined.*

Proof. If $\bar{y}$ is in Y, the lemma is immediate. So, assume $\bar{y}$ is not in Y. By positivity, $z \succ x$. By tightness, there is v such that $x \circ v$ exists and $x \circ v \succ z$. Suppose $\bar{y} \succ v$, then there exists y in Y such that $\bar{y} \succ y \succ v$, whence $z \succ x \circ y \succ x \circ v$, which is impossible. Thus, $v \succsim \bar{y}$, whence by local definability, $x \circ \bar{y}$ exists. □

Lemma 7.2. *Suppose $\mathfrak{X} = \langle X, \succsim, \circ \rangle$ is a tight, Dedekind complete, positive concatenation structure that satisfies interval solvability. Then $\mathfrak{X}$ has half elements.*

Proof. We first note tightness implies there is no maximal element.

Next, we show:

(i) if $x \succ y \circ y$, then there exists z in X such that $z \succ y$ and $x \succ z \circ z$. Let φ be a continuous and monotonic $\circ$-representation onto Re^+ and let $r = \varphi(x)$ and $s = \varphi(y)$.

Then

$$r = \varphi(x) > \varphi(y \circ y) = \varphi(\varphi^{-1}(s) \circ \varphi^{-1}(s)) = s \circledcirc s.$$

Choose $\epsilon > 0$ so that $r - s \circledcirc s > \epsilon$. By continuity of the first argument, select $u > s$ so that $u \circledcirc s - s \circledcirc s < \epsilon/2$. By continuity of the second argument, select $v > s$ so that $u \circledcirc v - u \circledcirc s < \epsilon/2$. Let t be the smaller of u and v. Then

$$\begin{aligned} t \circledcirc t - s \circledcirc s &\leq u \circledcirc v - s \circledcirc s \\ &= (u \circledcirc v - u \circledcirc s) + (u \circledcirc s - s \circledcirc s) \\ &\leq \epsilon/2 + \epsilon/2 < r - s \circledcirc s . \end{aligned}$$

Thus $r > t \circledcirc t$. Let $z = \varphi^{-1}(t)$. Then $x \succ z \circ z$.

A similar proof establishes that

(ii) if $y \circ y \succ x$, then there exists z in X such that $y \succ z$ and $z \circ z \succ x$.

For x in X, define $Y_x = \{y \mid x \succsim y \circ y\}$. By Lemma 2.1, $Y_x \neq \emptyset$ and by positivity it is bounded by x. Thus $\theta(x) = \text{a l.u.b. } Y_x$ exists by Dedekind completeness. By Lemma 7.1, $\theta(x) \circ \theta(x)$ exists. Suppose $x \succ \theta(x) \circ \theta(x)$. By part (i) there exists $z \succ \theta(x)$ and $x \succ z \circ z$. So z is in Y_x, and so $\theta(x)$ is not a l.u.b. Y_x, contrary to choice. Similarly, part (ii) renders $\theta(x) \circ \theta(x) \succ x$ impossible. Since $\succsim$ is a weak order, $x \sim \theta(x) \circ \theta(x)$. □

Theorem 7.2. *Suppose $\mathfrak{X}$ is a tight, Dedekind complete, positive concatenation structure. If φ and ψ are two continuous and monotonic $\circledcirc$-representations that are onto* Re^+ *and, for some x in X, $\psi(x) = \varphi(x)$, then $\psi \equiv \varphi$.*

Proof. Lemma 7.2 and Theorem 2.2. □

Definition 7.3. A positive concatenation structure $\langle X, \succsim, \circ\rangle$ satisfies *interval solvability* if and only if for all x, y, z in X, if $x \succ y \succ z$, then there exist u, v in X such that $u \circ z$, $z \circ v$ are defined and $x \succ u \circ z$, $z \circ v \succ y$. □

Definition 7.4. A positive concatenation structure $\langle X, \succsim, \circ\rangle$ satisfies *regularity* if and only if for all x, y, z in X for which $x \succ y$ and $x \circ z$ is defined, there exists v in X such that for all u in X, if $u \succsim z$, then $x \circ u \succ y \circ (u \circ v)$.

Theorem 7.3. *A Dedekind complete, positive concatenation structure that satisfies interval solvability also satisfies regularity.*

Proof. To establish regularity, consider $x \succ y$ and z for which $x \circ z$ is defined. For $u \precsim z$, let

$$V_u = \{w \mid x \circ u \succ y \circ (u \circ w)\}.$$

First, $V_u \neq \emptyset$. For by monotonicity and local definability, $x \circ u$ exists and $x \circ z \succsim x \circ u \succ y \circ u$. By positivity and interval solvability, there is p in X such that

$$x \circ u \succ y \circ p \succ y \circ u .$$

By monotonicity and restricted solvability, there is w such that $p \succ u \circ w$, whence

$$x \circ u \succ y \circ p \succ y \circ (u \circ w) ,$$

and so $V_u \neq \emptyset$.

Since for $u \precsim z$, V_u is bounded by $x \circ z$, by Dedekind completeness V_u has a least upper bound. Let $v(u)$ be one. Next we show that

$$x \circ u \sim y \circ [u \circ v(u)] ,$$

where by Lemma 7.1 $y \circ [u \circ v(u)]$ is defined. We consider two cases.

Case 1. $x \circ u \succ y \circ [u \circ v(u)]$.
By positivity and interval solvability, let q be such that

$$x \circ u \succ y \circ q \succ y \circ [u \circ v(u)] .$$

Then $q \succ u \circ v(u)$ and thus, by interval solvability, there is r such that

$$q \succ u \circ r \succ u \circ v(u) .$$

Then $r \succ v(u)$ and $x \circ u \succ y \circ (u \circ r)$, which contradicts that $v(u)$ is a l.u.b. of V_u.

Case 2. Suppose that

$$y \circ [u \circ v(u)] \succ x \circ u .$$

Since $x \circ u \succ y \circ u$, by interval solvability there is q such that

$$y \circ [u \circ v(u)] \succ y \circ q \succ x \circ u .$$

By monotonicity $u \circ v(u) \succ q$, and since $x \succ y$, $q \succ u$. Thus by interval solvability let r be such that

$$u \circ v(u) \succ u \circ r \succ q .$$

Then

$$y \circ [u \circ v(u)] \succ y \circ (u \circ r) \succ x \circ u .$$

Thus $v(u) \succ r$. Since $v(u)$ is a l.u.b. of V_u, we conclude that for some v in V_u,

$$y \circ (u \circ v) \succ x \circ u ,$$

which is contrary to the definition of V_u.

Since these two cases are impossible and $\succsim$ is a weak ordering, it follows that for each $u \precsim z$,

$$x \circ u \sim y \circ [u \circ v(u)] .$$

Now if for some v in X, $v \precsim v(u)$ for all $u \precsim z$, then regularity holds. Thus we need only show that the following is impossible: for each p in X there exists $u \precsim z$ such that $v(u) \prec p$. Assume on the contrary that the last statement is true. By Lemma 2.1 we can find a sequence w_i such that $w_i \precsim z$ and $v(w_i)$ becomes arbitrarily small for all sufficiently large i. If w_i also becomes arbitrarily small for all sufficiently large i, then j can be found so that

$$x \succ y \circ [w_j \circ v(w_j)] \ ,$$

and this is impossible. Thus there exists q in X such that $w_i \succsim q$ for infinitely many i. Therefore lim sup w_i and lim sup$[w_i \circ v(w_i)]$ exist. Since $v(w_i)$ becomes arbitrarily small for sufficiently large i,

$$\limsup w_i \sim \limsup [w_i \circ v(w_i)] \ .$$

Let $\bar{w} = \limsup w_i$. Since

$$x \circ w_i \sim y \circ [w_i \circ v(w_i)] \ ,$$

it follows that

$$x \circ \bar{w} \sim \limsup(x \circ w_i) \sim \limsup\{y \circ [w_i \circ v(w_i)]\}$$
$$\sim \limsup(y \circ w_i) \sim y \circ \bar{w} \ .$$

By monotonicity, $x \sim y$, and this is impossible. □

Theorem 7.4. *Suppose $\mathcal{X} = \langle X, \succsim, \circ \rangle$ is a positive concatenation structure that satisfies interval solvability and regularity. Then there exists a structure $\boldsymbol{\mathcal{X}} = \langle \mathbf{X}, \boldsymbol{\succsim}, \mathbf{o} \rangle$ and a subset X^* of $\mathbf{X}$ such that*

(i) *$\boldsymbol{\mathcal{X}}$ is a Dedekind complete, positive concatenation structure and $\boldsymbol{\succsim}$ is a linear ordering;*
(ii) *X^* is an order dense subset of $\mathbf{X}$;*
(iii) *$\mathcal{X}$ is homomorphic to the restriction of $\boldsymbol{\mathcal{X}}$ to X^*;*
(iv) *if $\mathcal{X}$ has no maximal element, $\boldsymbol{\mathcal{X}}$ has no maximal element;*
(v) *if $\circ$ is a closed operation, $\mathbf{o}$ is a closed operation.*

Proof. Let $\mathbf{X}$ consist of all subsets Y of X for which the following three conditions hold:

1. Y and $X - Y$ are nonempty.
2. For x, y in Y, if $x \succsim y$ and x is in Y, then y is in Y.
3. Y does not have a maximal element.

Let X^* consist of all sets of the form: for x in X,

$$\mathbf{x} = \{y \mid y \text{ in } X \text{ and } x \succ y\} \ .$$

Note that $X^* \subseteq \mathbf{X}$.

Define $\succsim$ on $\mathbf{X}$ by: for each Y and Z in $\mathbf{X}$

$$Y \succsim Z \quad \text{iff} \quad Y \supseteq Z\,.$$

Define $\mathbf{o}$ on $\mathbf{X}$ by: for each Y and Z in $\mathbf{X}$, $Y \mathbf{o} Z$ is defined if there exist u, v in X such that u is not in Y, v is not in Z, and $u \circ v$ is defined. In this case

$$Y \mathbf{o} Z = \{x \mid x \text{ in } X \text{ and there exist } y \text{ in } Y,\ z \text{ in } Z \text{ such that } y \circ z \succ x\}\,.$$

We break the proof up into a series of lemmas. The hypothesis in each case is that of the theorem; however, in some cases weaker hypotheses would do.

Lemma 7.3. (i) $\succsim$ *is a linear ordering of* $\mathbf{X}$.
(ii) $x \succsim y$ *iff* $\mathbf{x} \sim \mathbf{y}$.
(iii) *For* Y, Z *in* $\mathbf{X}$, *if* $Y \succ Z$, *then there exist* y, z *in* $Y - Z$ *such that* $\mathbf{y} \succ \mathbf{z} \succ Z$.
(iv) X^* *is order dense in* X.
(v) $\langle \mathbf{X}, \succsim \rangle$ *is Dedekind complete.*

Proof. (i) $\succsim$ is transitive and asymmetric because $\supseteq$ is. Suppose it is not connected. Then there exist y in $Y-Z$ and z in $Z-Y$. Without loss of generality, suppose $y \succsim z$. Then by definition of $\mathbf{X}$, z is in Y, which is impossible.
(ii) $x \succsim y$ iff $\mathbf{x} \supseteq \mathbf{y}$ iff $\mathbf{x} \succsim \mathbf{y}$.
(iii) Select x, y in $Y - Z$ with $x \succ y$. They exist because $Y \succ Z$ and Y has no maximal element. Thus $\mathbf{x} \supset \mathbf{y} \supset Z$, and so $\mathbf{x} \succ \mathbf{y} \succ Z$.
(iv) Suppose $Y \succ Z$. By part (iii), there exist y, z in Y such that $\mathbf{y} \succ \mathbf{z} \succ Z$. Clearly, $Y \succsim \mathbf{y}$.
(v) Let α be a nonempty, bounded subset of $\mathcal{X}$.
Define

$$Y_\alpha = \{x \mid x \text{ in } Y \text{ for some } Y \text{ in } \alpha\}\,.$$

Y_α is in $\mathbf{X}$ because:
1. $Y_\alpha \neq \emptyset$ since $\alpha \neq \emptyset$; $X - Y_\alpha \neq \emptyset$ since α is bounded.
2. Suppose x is in Y_α and $x \succsim y$. Let x be in Y of α. Then $x \succsim y$ implies y in Y, so y is in Y_α.
3. Suppose x is a maximal element in Y_α. Since x is in some Y of α, x is also a maximal element of Y, contrary to Y in $\mathbf{X}$.

By choice, Y_α is a bound on α since each Y of α is a subset of Y_α. We show it is a least upper bound. Suppose on the contrary, there is a bound Z of α for which $Y_\alpha \supset Z$. Let x be in $Y_\alpha - Z$, so there exists Y in α with $x \in Y$, whence $Y \supset Z$ and so Z is not a bound.

Lemma 7.4. $\mathbf{o}$ *is a partial operation for which local definability holds.*

Proof. To show $\mathbf{o}$ is a partial operation, we must show that when $Y \mathbf{o} Z$ is defined, $Y \mathbf{o} Z$ is in $\mathbf{X}$.

1. $X - Y \circ Z \neq \emptyset$ because when $Y \circ Z$ is defined it is bounded by $u \circ v$. $Y \circ Z \neq \emptyset$ since the existence of $u \circ v$ implies by local definability that $y \circ z$ is defined for y in Y, z in Z. So by Lemma 2.1, there is $w \prec y \circ z$.

2. Suppose w is in $Y \circ Z$ and $w \succsim u$. There exist y in Y, z in Z such that $y \circ z \succ w \succsim u$, so u is in $Y \circ Z$.

3. Suppose w in $Y \circ Z$ is a maximal element. There are y in Y and z in Z such that $y \circ z \succ w$. By restricted solvability, there is p in X such that $y \circ z \succ w \circ p$, and so $w \circ p$ is in $Y \circ Z$. Since by positivity, $w \circ p \succ w$, w is not maximal.

To show local definability, suppose $Y \circ Z$ is defined, $Y \succsim V$, and $Z \succsim W$. Since $Z \supseteq W$, the bounds u, v that insure $Y \circ Z$ is defined, also insure $Y \circ W$ is defined. And

$$\begin{aligned} Y \circ W &= \{x \mid x \text{ in } X \text{ and there exist } y \text{ in } Y,\ w \text{ in } W \text{ such that } y \circ w \succ x\} \\ &\subseteq \{x \mid x \text{ in } X \text{ and there exist } y \text{ in } Y,\ z \text{ in } Z \text{ such that } y \circ z \succ x\} \\ &= Y \circ Z. \end{aligned}$$

Similarly, $V \circ W \subseteq Y \circ W$. Thus, $V \circ W \subseteq Y \circ Z$. □

The following is the only place in the proof that regularity is used.

Lemma 7.5. *The following two statements are true for each Y, Z, W in $\mathbf{X}$:*
(i) *$Y \circ W \succsim Z \circ W$ iff $Y \succsim Z$ and $Y \circ W$, $Z \circ W$ are defined.*
(ii) *$W \circ Y \succsim W \circ Z$ iff $Y \succsim Z$ and $W \circ Y$, $W \circ Z$ are defined.*

Proof. If $Y \succsim Z$ and $Y \circ W$, $Z \circ W$ are defined, then it immediately follows from the definition of $\circ$ that $Y \circ W \succsim Z \circ W$. Conversely, suppose that $Y \circ W \succsim Z \circ W$, then we show $Y \succsim Z$ by contradiction. Suppose that $Z \succ Y$. By Lemma 7.3(iii) let x, y be elements of Z such that $\mathrm{x} \succ \mathrm{y} \succ Y$. Then

$$Z \succsim \mathrm{x} \succ \mathrm{y} \succ Y,$$

and thus

$$Z \circ W \succsim \mathrm{x} \circ W \succsim \mathrm{y} \circ W \succsim Y \circ W.$$

Since by assumption $Y \circ W \succsim Z \circ W$, it follows that $\mathrm{x} \circ W = \mathrm{y} \circ W$. By Lemma 2.2, let w_i be a sequence of elements of W such that $w_{i+1} \succ w_i$ and for each w in W there exists j such that $w_j \succ w$. Since $\mathrm{x} \circ W = \mathrm{y} \circ W$, a subsequence u_i of the sequence w_i can be found so that

$$y \circ u_{i+1} \succ x \circ u_i. \tag{7.1}$$

Since W is in $\mathbf{X}$, let t be a bound of W, i.e., $\mathrm{t} \succ W$. Since $x \succ y$, by regularity there exists v such that for all positive integers i,

$$u \circ u_i \succ y \circ (u_i \circ v). \tag{7.2}$$

Now for some positive integer j

$$u_j \circ v \succ u_{j+1}, \tag{7.3}$$

for if not, then

$$t \succ u_2 \succ u_1 \circ v \succ v ,$$

$$t \succ u_3 \succ u_2 \circ v \succ v \circ v = 2v ,$$

$$t \succ u_4 \succ u_3 \circ v \succ (2v) \circ v = 3v ,$$

etc.,

and this contradicts the Archimedean assumption. Combining eqs. (7.1), (7.2) and (7.3), for some k

$$y \circ u_{k+1} \succ x \circ u_k \succ y \circ (u_k \circ v) \succ y \circ u_{k+1} ,$$

and this is a contradiction. □

Lemma 7.6. *Let* Y *and* Z *be in* $\mathbf{X}$.

(i) *If* $Y \text{ o } Z$ *is defined,* $Y \text{ o } Z \succ Y, Z$.

(ii) *There exists* n *in* I^+ *such that either* nZ *is not defined or* $nZ \succsim Y$.

Proof. (i) By Lemma 7.3(v) Y has a least upper bound $\bar{Y}$ in $\mathbf{X}$. If y is in $\bar{Y}$, then we show there is an x in Y such that $x \succsim y$. For suppose $y \succ x$ for all x in Y, then $\bar{Y} \succ \mathrm{y} \succsim Y$, and $\bar{Y}$ is not a least upper bound of Y. For z in Z and this x and y, we have $x \circ z \succ x \succsim y$, whence $Y \text{ o } Z \supseteq \bar{Y}$. Thus, $Y \text{ o } Z \succsim \bar{Y} \succ Y$. The other case is similar.

(ii) Observe that if $Z \text{ o } Z$ is defined, there exist u, v such that $u \circ v$ is defined, $u, v \succ z$ for z in Z. Thus by local definability, $z \circ z$ is defined. By induction, if nZ is defined, so is nz for z in Z. Thus, a failure of the Archimedean axiom in $\mathfrak{X}$ implies a failure in $\mathfrak{X}$.

The following two lemmas make use of interval solvability.

Lemma 7.7. *Let* Y *and* Z *be in* $\mathbf{X}$. *If* $Y \succ Z$, *there exists* V *in* $\mathbf{X}$ *such that* $Y \succ Z \text{ o } V$.

Proof. By Lemma 7.3(iii) there exist u, w in Y such that $Y \succsim \mathbf{u} \succ \mathbf{w} \succ Z$. By restricted solvability, there exists p such that $u \succ w \circ p$. Define

$$V = \{v \mid p \succ v \text{ and for some } z \text{ in } Z,\ u \succ z \circ v\}.$$

First, we show V is in $\mathbf{X}$. By restricted solvability, $V \neq \emptyset$ and $X - V \neq \emptyset$ because V is bounded by p. If v is in V and $v \succsim s$, then s is in V by the monotonicity of $\circ$. Suppose $\bar{v}$ in V is maximal. Then for some z in Z, $u \succ w \circ p \succ z \circ \bar{v} \succ z$. By interval solvability, there is v such that $u \succ w \circ p \succ z \circ v \succ z \circ \bar{v}$. Then v is in V and $v \succ \bar{v}$, contrary to assumption.

$Z \text{ o } V$ is defined since $w \circ p$ is defined and $w \succ z$ for z in Z and $p \succ v$ for v in V:

$$Z \text{ o } V = \{w \mid z \text{ in } Z \text{ and } v \text{ in } V \text{ and } u \succ z \circ v \succ w\} \subset \mathbf{u} \subseteq Y.$$

Thus, $Y \succsim Z \text{ o } V$. □

Lemma 7.8. $z \sim x \circ y$ *if and only if* $\mathrm{z} = \mathrm{x} \text{ o } \mathrm{y}$.

Proof. We begin by proving that if $x \circ y \succ z$, then there exist u, v in X such that $x \succ u$, $y \succ v$, and $u \circ v \succ z$.

(1) If $x \succ z$, by Lemma 2.1 there is u such that $x \succsim u \succsim z$ and there is v such that $y \succ v$. Therefore $x \circ y \succ u \circ v \succ u \succsim z$.

(2) If $y \succ z$, the argument is similar.

(3) If $z \succsim x, y$, then since $x \circ y \succ z \succsim x$, interval solvability implies there is v such that $x \circ y \succ x \circ v \succ z$. By monotonicity, $y \succ v$. Applying interval solvability to $x \circ v \succ z \succsim y \succ v$, there is u such that $x \circ v \succ u \circ v \succ z$.

Now, suppose $z \sim x \circ y$. Since x is not in $\mathbf{x}$ and y is not in $\mathbf{y}$, this implies $\mathbf{x} \circ \mathbf{y}$ is defined. Clearly,

$$\begin{aligned} \mathbf{z} &= \{w \mid x \circ y = z \succ w\} \\ &\supseteq \{w \mid x' \text{ is in } \mathbf{x}, y' \text{ is in } \mathbf{y}, \text{ and } x' \circ y' \succ w\} \\ &= \mathbf{x} \circ \mathbf{y}\,. \end{aligned}$$

Suppose w is in $\mathbf{z}$, i.e., $w \prec x \circ y$. By what was shown above, there exist $u \prec x$, $v \prec y$ such that $w \prec u \circ v$. Thus, w is in $\mathbf{x} \circ \mathbf{y}$. So $\mathbf{z} = \mathbf{x} \circ \mathbf{y}$.

Conversely, suppose $\mathbf{z} = \mathbf{x} \circ \mathbf{y}$. Suppose $x \circ y \succ z$. Then by the above there exist $u \prec x$, $v \prec y$ and $u \circ v \succ z$, so $\mathbf{x} \circ \mathbf{y} \supset \mathbf{z}$, contrary to assumption. If $z \succ x \circ y$, then by Lemma 2.2 there exists u with $z \succ u \succ x \circ y$. So u is in $\mathbf{z}$ but not in $\mathbf{x} \circ \mathbf{y}$, contrary to assumption. So $z \sim x \circ y$. □

This concludes the proof of parts (i), (ii), and (iii) of Theorem 7.4. Part (iv) follows immediately from (ii) and (iii). Part (v) is immediate.

There are several unresolved problems concerning the conditions used for the Dedekind completion of a positive concatenation structure. Perhaps the most important general problem is to find methods of imbedding positive concatenation structures into ones with closed operations. (Luce and Marley [7] have done this for the associative case.) The specific instance of this problem that is most important for measurement theory is: *For each positive concatenation structure that is tight and satisfies interval solvability and regularity, does there exist a positive concatenation extension with a closed operation that satisfies interval solvability and regularity*? We have not worked out all of the logical connections between half elements, tightness, interval solvability, and regularity. It is easy to show that the axioms for positive concatenation structure with a closed operation do not imply half elements. (Take $\langle X, \succsim, + \rangle$ where X is the closure of the positive rationals and $\sqrt{2}$ with respect to +.) However, other implications seem more difficult. For example, if $\mathcal{P}$ stands for the axioms of a positive concatenation structure with a closed operation, does $\mathcal{P}$ and interval solvability imply regularity? Does $\mathcal{P}$ and half elements imply interval solvability?

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