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A QUALITATIVE EQUIVALENT TO THE RELATIVISTIC ADDITION LAW FOR VELOCITIES

Luce and Marley (1969) used relativistic velocity as a motivating example for their discussion of concatenation structures with maximal elements. They did not, however, characterize the relativistic law for the addition of velocities, namely,

$$u \oplus v = \frac{u+v}{1+\dfrac{uv}{c^2}}.$$

An attempt at such a characterization was made in Krantz, Luce, Suppes, and Tversky (1971) where by simultaneously axiomatizing length and velocity while requiring velocity to be distance divided by time it was shown that there existed a unique composition formula for velocities. However, Krantz *et al.* were not able qualitatively to characterize this formula as the relativistic law for the addition of velocities. The purpose of this note is finally to give such a characterization. This is accomplished by simultaneously dealing with the variables length, velocity, and time and using some of the results and concepts of Chapter 10 of Krantz *et al.* (1971). It is assumed that the reader is familiar with the basic concepts and results of the above mentioned chapter.

Let V and T be sets, $\circ$ be a binary operation on $V \times T$, $\circ_V$ be a binary operation on V, $\circ_T$ be a binary operation on T, and $\succsim$ be a binary relation on $V \times T$. $\langle V \times T, \succsim, \circ, \circ_V, \circ_T \rangle$ is assumed to have the following four properties:

(i) $\langle V \times T, \succsim \rangle$ is an additive conjoint structure with unrestricted solvability and has a multiplicative representation $\psi = \psi_V \psi_T$;

(ii) $\langle V \times T, \succsim, \circ \rangle$ is an extensive structure with an additive representation φ;

(iii) $\succsim_V$ and $\succsim_T$ are weak orderings on V and T respectively which are by (i) induced by $\succsim$, and $\langle T, \succsim_T, \circ_T \rangle$ is an extensive structure with additive representation φ_T;

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(iv) the law of similitude

$$(v, it) \sim i(v, t),$$

where i is a positive integer, v is in V, t is in T, $ix = x$ if $i = 1$, and $ix = ((i-1)\,x) \circ x$ if $i > 1$, and units and exponents of the representations are selected so that $\psi = \varphi$ and $\psi_T = \varphi_T$.

Let c be a fixed element of V. For all v in V and t in T, define $\tau_c(v, t)$ to be the solution to

$$(c, \tau_c(v, t)) \sim (v, t). \tag{1}$$

If c is interpreted as the speed of light, then $\tau_c(v, t)$ is the time required for light to transverse the distance that the velocity v does in time t (all of this being qualitative identifications and not measures).

And for all u, v in V and t in T, define $\tau(u, v, t)$ to be the solution to

$$(u \circ_V v, \tau(u, v, t)) \sim (u, t) \circ (v, t). \tag{2}$$

$\tau(u, v, t)$ is the time it takes the velocity $u \circ_V v$ to travel the distance which is the concatenation of the distance that u travels in time t with the distance that v travels in time t.

LEMMA. *For all u, v in V and t in T,*

$$\tau_c(u, \tau_c(v, t)) \sim_T \tau_c(v, \tau_c(u, t)).$$

Proof. By applying the Thomsen condition (assumption (*i*)) to

$$(u, t) \sim (c, \tau_c(u, t))$$
$$(c, \tau_c(v, t)) \sim (v, t),$$

we obtain

$$(u, \tau_c(v, t)) \sim (v, \tau_c(u, t)).$$

Thus

$$\begin{aligned}(c, \tau_c(u, \tau_c(v, t))) &\sim (u, \tau_c(v, t)) \\ &\sim (v, \tau_c(u, t)) \\ &\sim (c, \tau_c(v, \tau_c(u, t))).\end{aligned}$$

□

THEOREM 1. *For some c in C,*

$$\psi_V(u \circ_V v) = \frac{\psi_V(u) + \psi_V(v)}{1 + \dfrac{\psi_V(u)\,\psi_V(v)}{\psi_V(c)^2}} \tag{3}$$

if and only if

$$\tau(u, v, t) \sim_T \tau_c(u, \tau_c(v, t)) \circ_T t. \tag{4}$$

Proof. Using assumptions (i)–(iv) freely, Equation (1) is equivalent to

$$\psi_V(c)\,\psi_T[\tau_c(v, t)] = \psi_V(v)\,\psi_T(t)$$

and Equation (2) to

$$\begin{aligned}
\psi_V(u \circ_V v)\,\psi_T[\tau(u, v, t)] &= \psi\,[(u, t) \circ (v, t)] \\
&= \varphi\,[(u, t) \circ (v, t)] \\
&= \varphi(u, t) + \varphi(v, t) \\
&= \psi(u, t) + \psi(v, t) \\
&= \psi_V(u)\,\psi_T(t) + \psi_V(v)\,\psi_T(t).
\end{aligned}$$

Thus

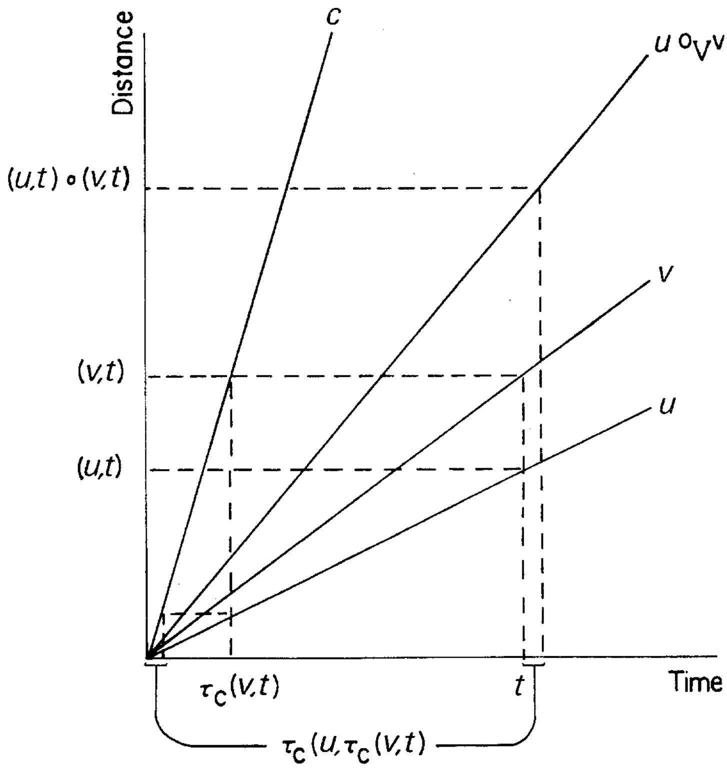
$$\psi_V(u \circ_V v) = \frac{[\psi_V(u) + \psi_V(v)]\,\psi_T(t)}{\psi_T[\tau(u, v, t)]}.$$

But Equation (4) is equivalent to

$$\begin{aligned}
\psi_T[\tau(u, v, t)] &= \psi_T[\tau_c(u, \tau_c(v, t)) \circ_T t] \\
&= \psi_T[\tau_c(u, \tau_c(v, t))] + \psi_T(t) \\
&= \frac{\psi_V(u)\,\psi_T[\tau_c(v, t)]}{\psi_V(c)} + \psi_T(t) \\
&= \frac{\psi_V(u)\,\psi_V(v)\,\psi_T(t)}{\psi_V(c)^2} + \psi_T(t),
\end{aligned}$$

and thus Equation (4) is equivalent to Equation (3). □

Figure 1 provides a graphic interpretation of the time relations formulated in Equation (4).



It is immediate that the addition law

$$\psi_V(u \circ_V v) = \psi_V(u) + \psi_V(v)$$

is equivalent to the qualitative condition

$$\tau(u, v, t) \sim_T t.$$

The following theorem shows how to derive two familiar properties of c from Equation (4).

THEOREM 2. *For all v in V,*

$$v \circ_V c \sim_V c$$

and

$$c \succsim_V v \quad \text{iff} \quad v \circ_V v \succsim_V v.$$

Proof.

$$
\begin{aligned}
(v \circ_V c, \tau(v, c, t)) &\sim (v, t) \circ (c, t) \\
&\sim (c, \tau_c(v, t)) \circ (c, t) \\
&\sim (c, \tau_c(v, t) \circ_T t) \\
&\sim (c, \tau_c(v, \tau_c(c, t)) \circ_T t \text{ [since } t = \tau_c(c, t)] \\
&\sim (c, \tau(v, c, t)),
\end{aligned}
$$

from which $v \circ_V c \sim_V c$ by the monotonicity of the conjoint structure.

$$
\begin{aligned}
c \gtrsim_V v &\text{ iff } \tau_c(v, t) \lesssim_T t && \text{[Equation (1)]} \\
&\text{ iff } \tau(v, v, t) \sim_T \tau_c(v, \tau_c(v, t)) \circ_T t \lesssim_T t \circ_T t && \text{[Equation (4)]} \\
&\text{ iff } (v \circ_V v, t \circ_T t) \gtrsim (v \circ_V v, \tau(v, v, t)) \\
&\qquad \sim (v, t) \circ (v, t) && \text{[Equation (2)]} \\
&\qquad \sim (v, t \circ_T t) && \text{[law of similitude]} \\
&\text{ iff } v \circ_V v \gtrsim_V v. && \square
\end{aligned}
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